Stochastic Calculus for Finance I: The Binomial Asset Pricing Model
Solution of Exercise Problems

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Abstract

This is a solution manual for Shreve [6]. If you find any typos/errors or have any comments, please email me at zypublic@hotmail.edu.

Contents

1 The Binomial No-Arbitrage Pricing Model

★ **Comments**:

1) Example 1.1.1 illustrates the essence of arbitrage: *buy low, sell high.* Since concrete numbers often obscure the nature of things, we review Example 1.1.1 in abstract symbols.

First, the possibility of replicating the payoff of a call option, $(S_1 - K)^+$, and its reverse, $-(S_1 - K)^+$.

To replicate the payoff $(S_1 - K)^+$ of a call option at time 1, we at time 0 construct a portfolio $(X_0 - \Delta_0 S_0, \Delta_0 S_0)$. At the operational level, we borrow X_0, buy Δ_0 shares of stock, and invest the residual amount $(X_0 - \Delta_0 S_0)$ into money market account. The result is a net cash flow of X_0 into the portfolio. The replication requirement at time 1 is

$$(1+r)(X_0 - \Delta_0 S_0) + \Delta_0 S_1 = (S_1 - K)^+.$$

Plug in $S_1(H) = uS_0$ and $S_1(T) = dS_0$, we obtain a system of two linear equations for two unknowns (X_0 and Δ_0) and it has a unique solution as long as $u \neq d$. This is how we obtain the magic number $X_0 = 1.20$ and $\Delta_0 = \frac{1}{2}$ in Example 1.1.1.

To replicate the reverse payoff $-(S_1 - K)^+$ of a call option at time 1, we at time 0 construct a portfolio $(-X_0 + \Delta_0 S_0, -\Delta_0 S_0)$. At the operational level, we short sell Δ_0 shares of stock, invest the income $\Delta_0 S_0$ into money market account, and withdraw a cash amount of X_0 from the money market account. The result is a net cash flow of X_0 out of the portfolio.

Second, the realization of arbitrage opportunity through *buy low, sell high.*

If the market price C_0 of the call option is greater than X_0, we just sell the call option for C_0 (sell high), spend X_0 on constructing the synthetic call option (buy low), and take the residual amount $(C_0 - X_0)$ as arbitrage profit. At time 1, the payoff of short position in the call option will cancel out with the payoff of the synthetic call option.

If the market price C_0 of the call option is less than X_0, we buy the call option for C_0 (buy low) and set up the portfolio $(-X_0 + \Delta_0 S_0, -\Delta_0 S_0)$ at time 0, which allows us to withdraw a cash amount of X_0 (sell high). The net cash flow $(X_0 - C_0)$ at time 0 is taken as arbitrage profit. At time 1, the payoff of long position in the call option will cancel out with the payoff of the portfolio $(-X_0 + \Delta_0 S_1, -\Delta_0 S_1)$.

2) The essence of Definition 1.2.3 is that we can find a replicating portfolio $(\Delta_0, \cdots, \Delta_{N-1})$ and "define" the portfolio's value at time n as the price of the derivative security at time n. The rationale is that if $P(\omega_1\omega_2\cdots\omega_N) > 0$ and the price of the derivative security at time n does not agree with the portfolio's value, we can make an arbitrage on sample path $\omega_1\omega_2\cdots\omega_N$, starting from time n. For a formal presentation of this argument, we refer to Delbaen and Schachermayer [3], Chapter 2.

Also note the definition needs the uniqueness of the replicating portfolio. This is guaranteed in the binomial model as seen from the uniqueness of solution of equation (1.1.3)-(1.1.4).

Finally, we note the wealth equation (1.2.14) can be written as

$$\frac{X_{n+1}}{(1+r)^{n+1}} = \frac{X_n}{(1+r)^n} + \Delta_n \left[\frac{S_{n+1}}{(1+r)^{n+1}} - \frac{S_n}{(1+r)^n}\right]$$

This leads to a representation by discrete stochastic integral:

$$\widetilde{X}_T = X_0 + (\Delta \cdot \widetilde{S})_T,$$

where $\widetilde{X}_n = \frac{X_n}{(1+r)^n}$ and $\widetilde{S}_n = \frac{S_n}{(1+r)^n}$, $n = 1, 2, \cdots, N$.

► **Exercise 1.1.** Assume the one-period binomail market of Section 1.1 that both H and T have positive probability of occurring. Show that condition (1.1.2) precludes arbitrage. In other words, show that if $X_0 = 0$ and

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0),$$

then we cannot have X_1 strictly positive with positive probability unless X_1 is strictly negative with positive probability as well, and this is the case regardless of the choice of the number Δ_0.

Proof. Note the random stock price has only two states at time 1, H and T. So "we cannot have X_1 strictly positive with positive probability unless X_1 is strictly negative with positive probability as well" can be succinctly summarized as

$$\text{“}X_1(H) > 0 \Rightarrow X_1(T) < 0 \text{ and } X_1(T) > 0 \Rightarrow X_1(H) < 0\text{”}.$$

We prove a slightly more general version of the problem to expose the nature of no-arbitrage:

$$\text{“}X_1(H) > (1+r)X_0 \Rightarrow X_1(T) < (1+r)X_0 \text{ and } X_1(T) > (1+r)X_0 \Rightarrow X_1(H) < (1+r)X_0\text{”}.$$

Note this is indeed a generalization since the original problem assumes $X_0 = 0$.

For a formal proof, we write X_1 as

$$X_1 = \Delta_0 S_0 \left(\frac{S_1 - S_0}{S_0} - r \right) + (1+r)X_0.$$

Then

$$X_1(H) - (1+r)X_0 = \Delta_0 S_0 [u - (1+r)]$$

and

$$X_1(T) - (1+r)X_0 = \Delta_0 S_0 [d - (1+r)].$$

Given the condition $d < 1 + r < u$, the factor $S_0[u - (1+r)]$ is positive and the factor $S_0[d - (1+r)]$ is negative. So

$$X_1(H) > (1+r)X_0 \Rightarrow \Delta_0 > 0 \Rightarrow X_1(T) < (1+r)X_0$$

and

$$X_1(T) > (1+r)X_0 \Rightarrow \Delta_0 < 0 \Rightarrow X_1(H) < (1+r)X_0.$$

This concludes our proof. □

Remark 1.1. *The textbook (page 2-3) has shown "negation of $d < 1 + r < u \Rightarrow$ arbitrage", or equivalently, "no arbitrage $\Rightarrow d < 1 + r < u$". This exercise problem asks us to prove "$d < 1 + r < u \Rightarrow$ no arbitrage".*

Remark 1.2. *In the equation $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$, the first term $\Delta_0 S_1$ is the value of the stock position at time 1 while the second term $(1+r)(X_0 - \Delta_0 S_0)$ is the value of the money market account at time 1.*

Remark 1.3. *The condition $X_0 = 0$ in the original problem formulation is not really essential, as far as a proper definition of arbitrage can be given. Indeed, for the one-period binomial model, we can define arbitrage as a trading strategy such that $P(X_1 \geq X_0(1+r)) = 1$ and $P(X_1 > X_0(1+r)) > 0$. That is, arbitrage is a trading strategy whose return beats the risk-free rate. See Shreve [7, page 254] Exercise 5.7 for this more general definition.*

Remark 1.4. *Note the condition $d < 1 + r < u$ is just $\frac{S_1(T) - S_0}{S_0} < r < \frac{S_1(H) - S_0}{S_0}$: the return of the stock investment is not guaranteed to be greater or less than the risk-free rate. This agrees with the intuition that "arbitrage is a trading strategy that is guaranteed to beat the risk-free investment".*

► **Exercise 1.2.** Suppose in the situation of Example 1.1.1 that the option sells for 1.20 at time zero. Consider an agent who begins with wealth $X_0 = 0$ and at time zero buys Δ_0 shares of stock and Γ_0 options. The numbers Δ_0 and Γ_0 can be either positive or negative or zero. This leaves the agent with a cash position of $-4\Delta_0 - 1.20\Gamma_0$. If this is positive, it is invested in the money market; if it is negative, it represents money borrowed from the money market. At time one, the value of the agent's portfolio of stock, option, and money market assets is

$$X_1 = \Delta_0 S_1 + \Gamma_0 (S_1 - 5)^+ - \frac{5}{4}(4\Delta_0 + 1.20\Gamma_0).$$

Assume that both H and T have positive probability of occurring. Show that if there is a positive probability that X_1 is positive, then there is a positive probability that X_1 is negative. In other words, one cannot find an arbitrage when the time-zero price of the option is 1.20.

Proof. $X_1(u) = \Delta_0 \times 8 + \Gamma_0 \times 3 - \frac{5}{4}(4\Delta_0 + 1.20\Gamma_0) = 3\Delta_0 + 1.5\Gamma_0$, and $X_1(d) = \Delta_0 \times 2 - \frac{5}{4}(4\Delta_0 + 1.20\Gamma_0) = -3\Delta_0 - 1.5\Gamma_0$. That is, $X_1(u) = -X_1(d)$. So if there is a positive probability that X_1 is positive, then there is a positive probability that X_1 is negative. This finishes the proof of the original problem.

Since the use of numbers often obscures the nature of a problem, and since the notation of this exercise problem is rather confusing (given the notation in Example 1.1.1), we shall re-state the problem in different notation and rewrite the proof in abstract symbols.

First, a re-statement of the problem in different notation: "If the option is priced at the replication cost C_0 (which is denoted by X_0 in Example 1.1.1), we cannot find arbitrage by investing in tradable securities (stock, option, and money market account). Formally, suppose an investor borrows money to buy α shares of stock and β options at time 0, the portfolio thus constructed is $(-\alpha S_0 - \beta C_0, \alpha S_0 + \beta C_0)$. Its value at time 1 becomes

$$X_1 = \alpha S_1 + \beta V_1 - (1+r)(\alpha S_0 + \beta C_0)$$

where V_1 is the option payoff $(S_1 - K)^+$. Prove $P(X_1 > 0) > 0 \Rightarrow P(X_1 < 0) > 0$."

Second, our proof in abstract symbols. Recall for the replication cost C_0, there exists some $\delta > 0$ such that

$$(1+r)(C_0 - \delta S_0) + \delta S_1 \equiv V_1.$$

Plug this into the expression of X_1, we have

$$\begin{aligned} X_1 &= \alpha S_1 + \beta V_1 - (1+r)(\alpha S_0 + \beta C_0) \\ &= \alpha S_1 + \beta[(1+r)(C_0 - \delta S_0) + \delta S_1] - (1+r)(\alpha S_0 + \beta C_0) \\ &= \alpha S_1 + \beta(1+r)C_0 - \beta\delta S_0(1+r) + \delta\beta S_1 - (1+r)\alpha S_0 - (1+r)\beta C_0 \\ &= (\alpha + \delta\beta)S_1 - S_0(1+r)(\alpha + \beta\delta) \\ &= (\alpha + \beta\delta)S_0\left[\frac{S_1}{S_0} - (1+r)\right]. \end{aligned}$$

Since $S_1/S_0 = u$ or d and $d < 1+r < u$, we conclude $X_1(H)$ and $X_1(T)$ have opposite signs. This concludes our proof. □

Remark 1.5. *Example 1.1.1 has shown that "no arbitrage ⇒ the time-zero price of the option is 1.20" by proving "the time-zero price of the option is not 1.20 ⇒ there exists arbitrage via linear combination of investments in stock, option, and money market account".*

This exercise problem asks us to prove "the time-zero price of the option is 1.20 ⇒ there exists no arbitrage via the linear combination of investments in stock, option and money market account". Although logically, "linear combination of tradable securities" is only a special way of constructing portfolios, in practice it is the only way. So it's all right to use this special form of arbitrage to stand for general arbitrage.

► **Exercise 1.3.** In the one-period binomial model of Section 1.1, suppose we want to determine the price at time zero of the derivative security $V_1 = S_1$ (i.e., the derivative security pays off the stock price.) (This can be regarded as a European call with strike price $K = 0$). What is the time-zero price V_0 given by the risk-neutral pricing formula (1.1.10)?

Solution. $V_0 = \frac{1}{1+r}\left[\frac{1+r-d}{u-d}S_1(H) + \frac{u-1-r}{u-d}S_1(T)\right] = \frac{S_0}{1+r}\left(\frac{1+r-d}{u-d}u + \frac{u-1-r}{u-d}d\right) = S_0$. This is not surprising, since this is exactly the cost of replicating S_1, via the buy-and-hold strategy. □

► **Exercise 1.4.** In the proof of Theorem 1.2.2, show under the induction hypothesis that

$$X_{n+1}(\omega_1\omega_2\cdots\omega_n T) = V_{n+1}(\omega_1\omega_2\cdots\omega_n T).$$

Proof.

$$\begin{aligned} X_{n+1}(T) &= \Delta_n d S_n + (1+r)(X_n - \Delta_n S_n) \\ &= \Delta_n S_n (d-1-r) + (1+r)V_n \\ &= \frac{V_{n+1}(H) - V_{n+1}(T)}{u-d}(d-1-r) + (1+r)\frac{\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)}{1+r} \\ &= \tilde{p}[V_{n+1}(T) - V_{n+1}(H)] + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) \\ &= \tilde{p}V_{n+1}(T) + \tilde{q}V_{n+1}(T) \\ &= V_{n+1}(T). \end{aligned}$$

□

▶ **Exercise 1.5.** In Example 1.2.4, we considered an agent who sold the look-back option for $V_0 = 1.376$ and bought $\Delta_0 = 0.1733$ shares of stock at time zero. At time one, if the stock goes up, she has a portfolio valued at $V_1(H) = 2.24$. Assume that she now takes a position of $\Delta_1(H) = \frac{V_2(HH)-V_2(HT)}{S_2(HH)-S_2(HT)}$ in the stock. Show that, at time two, if the stock goes up again, she will have a portfolio valued at $V_2(HH) = 3.20$, whereas if the stock goes down, her portfolio will be worth $V_2(HT) = 2.40$. Finally, under the assumption that the stock goes up in the first period and down in the second period, assume the agent takes a position of $\Delta_2(HT) = \frac{V_3(HTH)-V_3(HTT)}{S_3(HTH)-S_3(HTT)}$ in the stock. Show that, at time three, if the stock goes up in the third period, she will have a portfolio valued at $V_3(HTH) = 0$, whereas if the stock goes down, her portfolio will be worth $V_3(HTT) = 6$. In other words, she has hedged her short position in the option.

Proof. First, on the path $\omega_1 = H$, the investor's portfolio is worth of

$$X_1(H) = (1+r)(X_0 - \Delta_0 S_0) + \Delta_0 S_1(H) = (1+0.25)(1.376 - 0.1733 \cdot 4) + 0.1733 \cdot 8 = 2.24 = V_1(H).$$

If on the path $\omega_1 = H$, the investor takes a position of

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{3.20 - 2.40}{16 - 4} = 0.0667$$

in the stock, then on the path $\omega_1\omega_2 = HH$, the investor's portfolio is worth of

$$\begin{aligned} X_2(HH) &= (1+r)[X_1(H) - \Delta_1(H)S_1(H)] + \Delta_1(H)S_2(HH) \\ &= (1+0.25)(2.24 - 0.0667 \cdot 8) + 0.0667 \cdot 16 \\ &= 3.2 \\ &= V_2(HH), \end{aligned}$$

while on the path $\omega_1\omega_2 = HT$, the investor's portfolio is worth of

$$\begin{aligned} X_2(HT) &= (1+r)[X_1(H) - \Delta_1(H)S_1(H)] + \Delta_1(H)S_2(HT) \\ &= (1+0.25)(2.24 - 0.0667 \cdot 8) + 0.0667 \cdot 4 \\ &= 2.4 \\ &= V_2(HT). \end{aligned}$$

Second, if on the path $\omega_1\omega_2 = HT$, the agent takes a position

$$\Delta_2(HT) = \frac{V_3(HTH) - V_3(HTT)}{S_3(HTH) - S_3(HTT)} = \frac{0-6}{8-2} = -1$$

in the stock, then on the path $\omega_1\omega_2\omega_3 = HTH$, the investor's portfolio is worth of

$$\begin{aligned} X_3(HTH) &= (1+r)[X_2(HT) - \Delta_2(HT)S_2(HT)] + \Delta_2(HT)S_3(HTH) \\ &= (1+0.25)[2.4 - (-1) \cdot 4] + (-1) \cdot 8 \\ &= 0 \\ &= V_3(HTH), \end{aligned}$$

while on the path $\omega_1\omega_2\omega_3 = HTT$, the investor's portfolio is worth of

$$\begin{aligned} X_3(HTT) &= (1+r)[X_2(HT) - \Delta_2(HT)S_2(HT)] + \Delta_2(HT)S_3(HTT) \\ &= (1+0.25)[2.4 - (-1)\cdot 4] + (-1)\cdot 2 \\ &= 6 \\ &= V_3(HTT). \end{aligned}$$

□

► **Exercise 1.6. (Hedging a long position-one period).** Consider a bank that has a long position in the European call written on the stock price in Figure 1.1.2. The call expires at time one and has strike price $K = 5$. In Section 1.1, we determined the time-zero price of this call to be $V_0 = 1.20$. At time zero, the bank owns this option, while ties up capital $V_0 = 1.20$. The bank wants to earn the interest rate 25% on this capital until time one (i.e., without investing any more money, and regardless of how the coin tossing turns out, the bank wants to have

$$\frac{5}{4}\cdot 1.20 = 1.50$$

at time one, after collecting the payoff from the option (if any) at time one). Specify how the bank's trader should invest in the stock and money markets to accomplish this.

Solution. The bank's trader should set up a replicating portfolio whose payoff is the opposite of the option's payoff. More precisely, we solve the equation

$$(1+r)(X_0 - \Delta_0 S_0) + \Delta_0 S_1 = -(S_1 - K)^+.$$

Then $X_0 = -1.20$ and $\Delta_0 = -\frac{1}{2}$ since this equation is a linear equation of X_0 and Δ_0. The solution means the trader should sell short 0.5 share of stock, put the income 2 into a money market account, and then transfer 1.20 into a separate money market account. At time one, the portfolio consisting of a short position in stock and $0.8(1+r)$ in money market account will cancel out with the option's payoff. In the end, we end up with $1.20(1+r)$ in the separate money market account. □

Remark 1.6. *This problem illustrates why we are interested in hedging a long position. In case the stock price goes down at time one, the option will expire worthless. The initial amount of money 1.20 paid at time zero will be wasted. By hedging, we convert the option back into liquid assets (cash and stock) which guarantees a sure payoff at time one. As to why we hedge a short position (as a writer), see Wilmott [8, page 11-13], 1.4 What are Options For?*

► **Exercise 1.7. (Hedging a long position-multiple periods).** Consider a bank that has a long position in the lookback option of Example 1.2.4. The bank intends to hold this option until expiration and receive the payoff V_3. At time zero, the bank has capital $V_0 = 1.376$ tied up in the option and wants to earn the interest rate of 25% on this capital until time three (i.e., without investing any more money, and regardless of how the coin tossing turns out, the bank wants to have

$$\left(\frac{5}{4}\right)^3 \cdot 1.376 = 2.6875$$

at time three, after collecting the payoff from the lookback option at time three). Specify how the bank's trader should invest in the stock and the money market account to accomplish this.

Solution. The idea is the same as Exercise 1.6. The bank's trader only needs to set up the reverse of the replicating trading strategy described in Example 1.2.4. More precisely, he should short sell 0.1733 share of stock, invest the income 0.6933 into money market account, and transfer 1.376 into a separate money market account. The portfolio consisting a short position in stock and 0.6933-1.376 in money market account will replicate the opposite of the option's payoff. After they cancel out, we end up with $1.376(1+r)^3$ in the separate money market account. □

▶ **Exercise 1.8. (Asian option).** Consider the three-period model of Example 1.3.1,[1] with $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$, and take the interest rate $r = \frac{1}{4}$, so that $\tilde{p} = \tilde{q} = \frac{1}{2}$. For $n = 0, 1, 2, 3$, define $Y_n = \sum_{k=0}^{n} S_k$ to be the sum of the stock prices between times zero and n. Consider an *Asian call option* that expires at time three and has strike $K = 4$ (i.e., whose payoff at time three is $\left(\frac{1}{4}Y_3 - 4\right)^+$). This is like a European call, except the payoff of the option is based on the average stock price rather than the final stock price. Let $v_n(s, y)$ denote the price of this option at time n if $S_n = s$ and $Y_n = y$. In particular, $v_3(s, y) = \left(\frac{1}{4}y - 4\right)^+$.

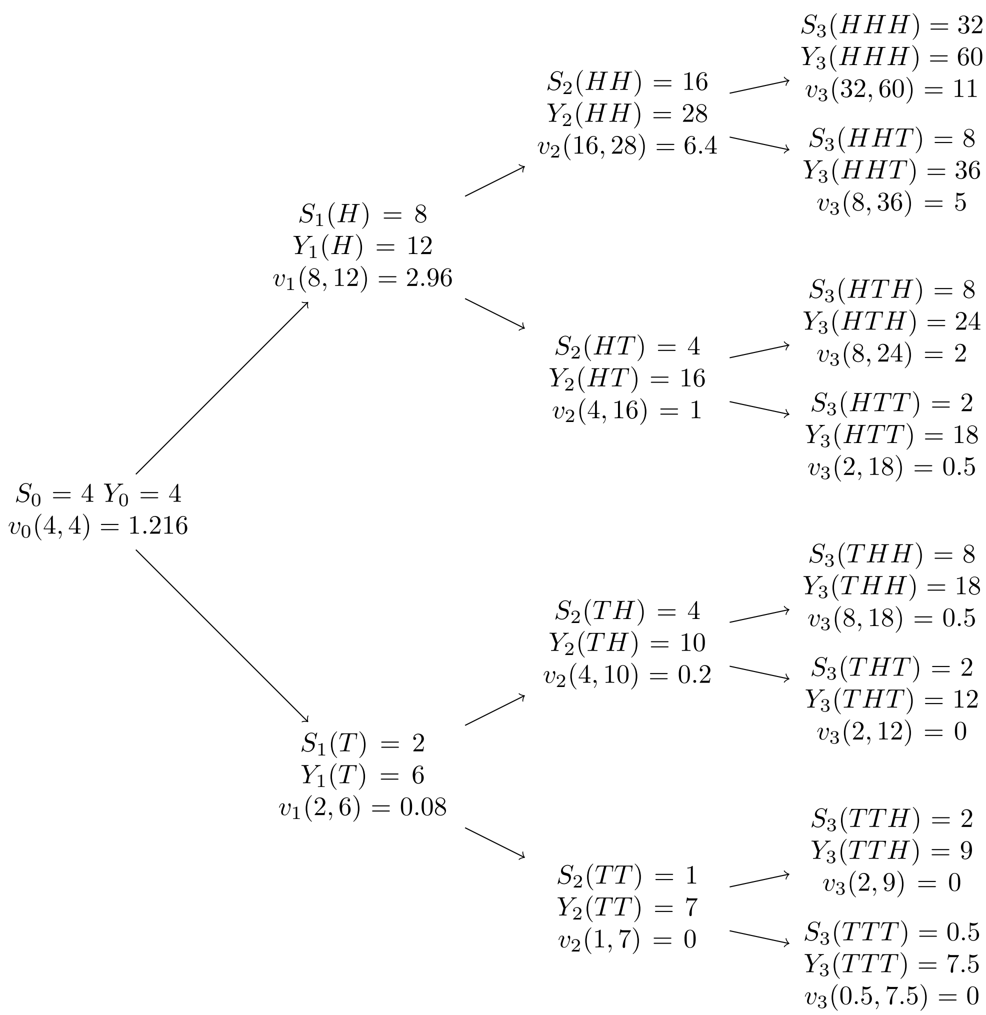
(i) Develop an algorithm for computing v_n recursively. In particular, write a formula for v_n in terms of v_{n+1}.

Solution. By risk-neutral pricing formula,

$$v_n(s, y) = \frac{1}{1+r}\left[\tilde{p}v_{n+1}(us, y + us) + \tilde{q}v_{n+1}\left(ds, y + ds\right)\right] = \frac{2}{5}\left[v_{n+1}(2s, y + 2s) + v_{n+1}\left(\frac{s}{2}, y + \frac{s}{2}\right)\right].$$

□

(ii) Apply the algorithm developed in (i) to compute $v_0(4, 4)$, the price of the Asian option at time zero.



Exercise 1.8. Asian option.

Solution. □

[1] The textbook said "Example 1.2.1" by mistake.

(iii) Provide a formula for $\delta_n(s, y)$, the number of shares of stock that should be held by the replicating portfolio at time n if $S_n = s$ and $Y_n = y$.

Solution.

$$\delta_n(s, y) = \frac{v_{n+1}(us, y + us) - v_{n+1}(ds, y + ds)}{(u - d)s}.$$

□

▶ **Exercise 1.9. (Stochastic volatility, random interest rate).** Consider a binomial pricing model, but at each time $n \geq 1$, the "up factor" $u_n(\omega_1\omega_2 \cdots \omega_n)$, the "down factor" $d_n(\omega_1\omega_2 \cdots \omega_n)$, and the interest rate $r_n(\omega_1\omega_2 \cdots \omega_n)$ are allowed to depend on n and on the first n coin tosses $\omega_1\omega_2 \cdots \omega_n$. The initial up factor u_0, the initial down factor d_0, and the initial interest rate r_0 are not random. More specifically, the stock price at time one is given by

$$S_1(\omega_1) = \begin{cases} u_0 S_0 & \text{if } \omega_1 = H, \\ d_0 S_0 & \text{if } \omega_1 = T, \end{cases}$$

and, for $n \geq 1$, the stock price at time $n + 1$ is given by

$$S_{n+1}(\omega_1\omega_2 \cdots \omega_n\omega_{n+1}) = \begin{cases} u_n(\omega_1\omega_2 \cdots \omega_n)S_n(\omega_1\omega_2 \cdots \omega_n) & \text{if } \omega_{n+1} = H, \\ d_n(\omega_1\omega_2 \cdots \omega_n)S_n(\omega_1\omega_2 \cdots \omega_n) & \text{if } \omega_{n+1} = T. \end{cases}$$

One dollar invested in or borrowed from the money market at time zero grows to an investment or debt of $1 + r_0$ at time one, and, for $n \geq 1$, one dollar invested in or borrowed from the money market at time n grows to an investment or debt of $1 + r_n(\omega_1\omega_2 \cdots \omega_n)$ at time $n + 1$. We assume that for each n and for all $\omega_1\omega_2 \cdots \omega_n$, the no-arbitrage condition

$$0 < d_n(\omega_1\omega_2 \cdots \omega_n) < 1 + r_n(\omega_1\omega_2 \cdots \omega_n) < u_n(\omega_1\omega_2 \cdots \omega_n)$$

holds. We also assume that $0 < d_0 < 1 + r_0 < u_0$.

(i) Let N be a positive integer. In the model just described, provide an algorithm for determining the price at time zero for a derivative security that at time N pays off a random amount V_N depending on the result of the first N coin tosses.

Solution. Similar to Theorem 1.2.2, but replace r, u and d everywhere with r_n, u_n and d_n. More precisely, set $\tilde{p}_n = \frac{1+r_n-d_n}{u_n-d_n}$ and $\tilde{q}_n = 1 - \tilde{p}_n$. Then

$$V_n = \frac{\tilde{p}_n V_{n+1}(H) + \tilde{q}_n V_{n+1}(T)}{1 + r_n}.$$

□

(ii) Provide a formula for the number of shares of stock that should be held at each time n $(0 \leq n \leq N - 1)$ by a portfolio that replicates the derivatives security V_N.

Solution. $\Delta_n = \frac{V_{n+1}(H)-V_{n+1}(T)}{S_{n+1}(H)-S_{n+1}(T)} = \frac{V_{n+1}(H)-V_{n+1}(T)}{(u_n-d_n)S_n}$. □

(iii) Suppose the initial stock price is $S_0 = 80$, with each head the stock price increases by 10, and with each tail the stock price decreases by 10. In other words, $S_1(H) = 90$, $S_1(T) = 70$, $S_2(HH) = 100$, etc. Assume the interest rate is always zero. Consider a European call with strike price 80, expiring at time five. What is the price of this call at time zero?

Solution. $u_n = \frac{S_{n+1}(H)}{S_n} = \frac{S_n+10}{S_n} = 1 + \frac{10}{S_n}$ and $d_n = \frac{S_{n+1}(T)}{S_n} = \frac{S_n-10}{S_n} = 1 - \frac{10}{S_n}$. So the risk-neutral probabilities at time n are $\tilde{p}_n = \frac{1-d_n}{u_n-d_n} = \frac{1}{2}$ and $\tilde{q}_n = \frac{1}{2}$.

To price the European call, we only need to focus on the nodes at time 5 of the binomial tree since $\tilde{p}_n$ and $\tilde{q}_n$ are constants. In order to have non-zero payoffs, a node at time 5 must have at least 3 H's. For such a node, we have

number of H's	option payoff	risk-neutral probability of paths
3	$(80+30-20)-80=10$	$\frac{1}{2^5}\cdot\frac{5!}{3!2!}=\frac{10}{32}$
4	$(80+40-10)-80=30$	$\frac{1}{2^5}\cdot 5=\frac{5}{32}$
5	$(80+50)-80=50$	$\frac{1}{2^5}=\frac{1}{32}$

So the price of the call at time zero is

$$\frac{1}{32}\cdot 50+\frac{5}{32}\cdot 30+\frac{10}{32}\cdot 10=\frac{300}{32}=9.375.$$

□

2 Probability Theory on Coin Toss Space

★ **Comments**:

1) The second proof of Theorem 2.4.4 also works for the random interest rate model of Exercise 1.9, as far as $\frac{S_{n+1}}{(1+r)^{n+1}}$ is replaced by $\frac{S_{n+1}}{(1+r_0)\cdots(1+r_n)}$. The requirement on the risk-neutral probability $\widetilde{\mathbb{P}}$ is that

$$\widetilde{\mathbb{P}}(w_{n+1}=H|\omega_1,\cdots,\omega_n):=\tilde{p}_n=\frac{1+r_n-d_n}{u_n-d_n}$$

and

$$\widetilde{\mathbb{P}}(w_{n+1}=T|\omega_1,\cdots,\omega_n):=1-\tilde{p}_n=\tilde{q}_n.$$

Results in measure theory guarantee the existence and uniqueness of such a probability measure $\widetilde{\mathbb{P}}$ on the sample space $\Omega=\{w:(w_1,\cdots,w_N)\}$ for a given family of conditional probabilities $(\tilde{p}_n,\tilde{q}_n)_{n=1}^N$.[2] With this setup, we have

$$\widetilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{S_n}\right]=u_n\tilde{p}_n+d_n\tilde{q}_n=(1+r_n).$$

In other words, the stock price process, when discounted by the random interest rates, is a martingale under $\widetilde{\mathbb{P}}$. When r_n's are deterministic, we are back to the case where ω_n's are independent of each other. This unifies the deterministic and random interest rate models.

2) On Theorem 2.4.8 (Cash flow valuation): by Theorem 2.4.7, it is natural to "conjecture" that the no-arbitrage price process of the derivative security is given by the risk-neutral pricing formula (2.4.13). However, pricing always needs to be justified by hedging/replication. We need to find a portfolio process which replicates the cash flows while equals to $(V_n)_{n=0}^N$ in value. The key insight to the construction of such a replicating portfolio is the wealth equation (2.4.16)

$$X_{n+1}=\Delta_nS_{n+1}+(1+r)(X_n-C_n-\Delta_nS_n),$$

which clearly indicates that the wealth process $(X_n)_{n=1}^N$ produces a cash outflow of C_n at step n.

Note the presentation of Theorem 2.4.8 follows the following flow of logic:

define V_n's $\Rightarrow$ define Δ_n's $\Rightarrow$ define X_n's $\Rightarrow$ prove $X_n=V_n$.

We could have taken a different, yet more natural path of logic, namely

define X_n's and Δ_n's simultaneously
$\Rightarrow$ prove formula (2.4.14) holds with V_n's replaced by X_n's
$\Rightarrow$ prove formula (2.4.13) holds with V_n's replaced by X_n's
$\Rightarrow$ define $V_n=X_n$.

[2] See, for example, Shiryaev [5, page 249], Theorem 2 (Ionescu Tulcea's Theorem on Extending a Measure and the Existence of a Random Sequence).

Indeed, we define $X_N = C_N$ and solve for X_{N-1} and Δ_{N-1} in the following equation

$$X_N = \Delta_{N-1} S_N + (1+r)(X_{N-1} - C_{N-1} - \Delta_{N-1} S_{N-1}).$$

By imitating the trick of (1.1.3)-(1.1.8) (page 6), we obtain

$$\Delta_{N-1}(\omega_1 \cdots \omega_{N-1}) = \frac{C_N(\omega_1 \cdots \omega_{N-1} H) - C_N(\omega_1 \cdots \omega_{N-1} T)}{S_N(\omega_1 \cdots \omega_{N-1} H) - S_N(\omega_1 \cdots \omega_{N-1} T)}$$

$$\begin{aligned} X_{N-1} &= C_{N-1} + \frac{1}{1+r}\left[\tilde{p} X_N(\omega_1 \cdots \omega_{N-1} H) + \tilde{q} X_N(\omega_1 \cdots \omega_{N-1} T)\right] \\ &= C_{N-1} + \widetilde{\mathbb{E}}_n\left[\frac{X_N}{1+r}\right] \\ &= \widetilde{\mathbb{E}}_n\left[\sum_{k=N-1}^{N} \frac{C_k}{(1+r)^{k-(N-1)}}\right]. \end{aligned}$$

Working backward by induction, we can provide definitions for each X_n and Δ_n, and prove for each n

$$X_n = C_n + \widetilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{1+r}\right] = \widetilde{\mathbb{E}}_n\left[\sum_{k=n}^{N} \frac{C_k}{(1+r)^{k-n}}\right].$$

Finally, we comment that Theorem 1.2.2 is a special case of Theorem 2.4.8, with $C_0 = C_1 = \cdots = C_{N-1} = 0$.

3) The textbook gives readers the impression that Markov property is preserved under the risk-neutral probability, at least in the setting of binomial model. A general result in this regard is recently announced in Schmock [4]:

Theorem 1 (Uwe Schmock and Ismail Cetin Gülüm)**.** *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0,\cdots,T}, \mathbb{P})$ be a (general) filtered probability space and let $S = \{S_t\}_{t\in\{0,\cdots,T\}}$ be an adapted, $\mathbb{R}^d$-valued discounted asset price process, which has the k-multiple Markov property w.r.t. $\mathbb{P}$. Then the following properties are equivalent:*
(a) The financial market model is free of arbitrage.
(b) There exists a probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

- *$\mathbb{P}^* \sim \mathbb{P}$ with $\varrho := d\mathbb{P}^*/d\mathbb{P} \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$.*
- *Integrability: $S_t \in L^1(\Omega, \mathcal{F}_t, \mathbb{P}^*)$ for all $t \in \{0, \cdots, T\}$.*
- *Martingale property w.r.t. $\mathbb{P}^*$:*

$$\mathbb{E}_{\mathbb{P}^*}[S_t|\mathcal{F}_{t-1}] \overset{a.s.}{=} S_{t-1} \text{ for all } t \in \{1, \cdots, T\}.$$

- *k-multiple Markov property: For all $B \in \mathcal{E}$ and $t \in \{k, \cdots, T\}$*

$$\mathbb{P}^*(S_t \in B|\mathcal{F}_{t-1}) \overset{a.s.}{=} \mathbb{P}^*(S_t \in B|S_{t-1}, S_{t-2}, \cdots, S_{t-k}).$$

For continuous time financial models, it is well-known that for Lipschitz continuous f, g, the stochastic differential equation of K. Itô

$$X_t = X_0 + \int_0^t f(s, X_s)dW_s + \int_0^t g(s, X_s)ds$$

has a unique solution which is a Markov process with continuous paths. Moreover if f and g satisfy $f(t,x) = f(x)$, $g(t,x) = g(x)$, then X is a time homogenous strong Markov process. These results combined with Girsanov's Theorem (Shreve [7] Chapter 5) will preserve Markov property of discounted asset process under the risk-neutral measure.

► **Exercise 2.1.** Using Definition 2.1.1, show the following.
(i) If A is an event and A^c denotes its complement, then $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

Proof. $\mathbb{P}(A^c) + \mathbb{P}(A) = \sum_{\omega \in A^c} \mathbb{P}(\omega) + \sum_{\omega \in A} \mathbb{P}(\omega) = \sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$. ☐

(ii) If $A_1, A_2, \cdots, A_N$ is a finite set of events, then

$$\mathbb{P}(\cup_{n=1}^N A_n) \leq \sum_{n=1}^N \mathbb{P}(A_n). \tag{2.8.1}$$

If the events $A_1, A_2, \cdots, A_N$ are disjoint, then equality holds in (2.8.1).

Proof. By induction, it suffices to work on the case $N = 2$. When A_1 and A_2 are disjoint, $\mathbb{P}(A_1 \cup A_2) = \sum_{\omega \in A_1 \cup A_2} \mathbb{P}(\omega) = \sum_{\omega \in A_1} \mathbb{P}(\omega) + \sum_{\omega \in A_2} \mathbb{P}(\omega) = \mathbb{P}(A_1) + \mathbb{P}(A_2)$. When A_1 and A_2 are arbitrary, using the result when they are disjoint, we have $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}((A_1 - A_2) \cup A_2) = \mathbb{P}(A_1 - A_2) + \mathbb{P}(A_2) \leq P(A_1) + \mathbb{P}(A_2)$. ☐

▶ **Exercise 2.2.** Consider the stock price S_3 in Figure 2.3.1.
(i) What is the distribution of S_3 under the risk-neutral probabilities $\tilde{p} = \frac{1}{2}$, $\tilde{q} = \frac{1}{2}$.

Solution. Under the risk-neutral probability, the distribution of ω_{n+1} conditioning on $\omega_1, \cdots, \omega_n$ is deterministic

$$\widetilde{\mathbb{P}}(\omega_{n+1} = H | \omega_1 \omega_2 \cdots \omega_n) := \tilde{p} = \frac{1+r-d}{u-d}, \ \ \widetilde{\mathbb{P}}(\omega_{n+1} = T | \omega_1 \omega_2 \cdots \omega_n) := \tilde{q} = \frac{u-1-r}{u-d}.$$

So ω_n's are independent of each other under $\widetilde{\mathbb{P}}$ and we have

$$\widetilde{\mathbb{P}}(S_3 = 32) = \tilde{p}^3 = \frac{1}{8}, \ \ \widetilde{\mathbb{P}}(S_3 = 8) = 3\tilde{p}^2\tilde{q} = \frac{3}{8}, \ \ \widetilde{\mathbb{P}}(S_3 = 2) = 3\tilde{p}\tilde{q}^2 = \frac{3}{8}, \ \ \widetilde{\mathbb{P}}(S_3 = 0.5) = \tilde{q}^3 = \frac{1}{8}.$$

☐

(ii) Compute $\widetilde{\mathbb{E}}S_1$, $\widetilde{\mathbb{E}}S_2$, and $\widetilde{\mathbb{E}}S_3$. What is the average rate of growth of the stock price under $\widetilde{\mathbb{P}}$?

Solution.

$$\begin{cases} \widetilde{\mathbb{E}}[S_1] = 8\widetilde{\mathbb{P}}(S_1 = 8) + 2\widetilde{\mathbb{P}}(S_1 = 2) = 8\tilde{p} + 2\tilde{q} = 5 \\ \widetilde{\mathbb{E}}[S_2] = 16\tilde{p}^2 + 4 \cdot 2\tilde{p}\tilde{q} + 1 \cdot \tilde{q}^2 = 6.25 \\ \widetilde{\mathbb{E}}[S_3] = 32 \cdot \frac{1}{8} + 8 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 0.5 \cdot \frac{1}{8} = 7.8125. \end{cases}$$

So the average rates of growth of the stock price under $\widetilde{\mathbb{P}}$ are, respectively:

$$\begin{cases} \tilde{r}_0 = \frac{\widetilde{\mathbb{E}}[S_1]}{S_0} - 1 = \frac{5}{4} - 1 = 0.25, \\ \tilde{r}_1 = \frac{\widetilde{\mathbb{E}}[S_2]}{\widetilde{\mathbb{E}}[S_1]} - 1 = \frac{6.25}{5} - 1 = 0.25, \\ \tilde{r}_2 = \frac{\widetilde{\mathbb{E}}[S_3]}{\widetilde{\mathbb{E}}[S_2]} - 1 = \frac{7.8125}{6.25} - 1 = 0.25. \end{cases}$$

☐

Remark 2.1. *An alternative solution is to use martingale property:*

$$\widetilde{\mathbb{E}}\left[\frac{S_n}{(1+r)^n}\right] = S_0 \Rightarrow \widetilde{\mathbb{E}}\left[S_n\right] = (1+r)^n S_0, \ \frac{\widetilde{\mathbb{E}}\left[S_n\right]}{\widetilde{\mathbb{E}}\left[S_{n-1}\right]} = (1+r).$$

(iii) Answer (i) and (ii) again under the actual probabilities $p = \frac{2}{3}$, $q = \frac{1}{3}$.

Solution. $\mathbb{P}(S_3 = 32) = (\frac{2}{3})^3 = \frac{8}{27}$, $\mathbb{P}(S_3 = 8) = 3 \cdot (\frac{2}{3})^2 \cdot \frac{1}{3} = \frac{4}{9}$, $\mathbb{P}(S_3 = 2) = 2 \cdot \frac{1}{9} = \frac{2}{9}$, and $\mathbb{P}(S_3 = 0.5) = \frac{1}{27}$.
Accordingly, $\mathbb{E}[S_1] = 6$, $\mathbb{E}[S_2] = 9$ and $\mathbb{E}[S_3] = 13.5$. So the average rates of growth of the stock price under $\mathbb{P}$ are, respectively: $r_0 = \frac{6}{4} - 1 = 0.5$, $r_1 = \frac{9}{6} - 1 = 0.5$, and $r_2 = \frac{13.5}{9} - 1 = 0.5$. ☐

Remark 2.2. *An alternative solution is to use Markov property:* $\mathbb{E}_n[S_{n+1}] = S_n\mathbb{E}_n[S_{n+1}/S_n] = S_n(pu+qd)$.

► **Exercise 2.3.** Show that a convex function of a martingale is a submartingale. In other words, let M_0, M_1, $\cdots$, M_N be a martingale and let φ be a convex function. Show that $\varphi(M_0)$, $\varphi(M_1)$, $\cdots$, $\varphi(M_N)$ is a submartingale.

Proof. Apply conditional Jensen's inequality, Theorem 2.3.2 (v). □

► **Exercise 2.4.** Toss a coin repeatedly. Assume the probability of head on each toss is $\frac{1}{2}$, as is the probability of tail. Let $X_j = 1$ if the jth toss results in a head and $X_j = -1$ if the jth toss results in a tail. Consider the stochastic process M_0, M_1, M_2, $\cdots$ defined by $M_0 = 0$ and

$$M_n = \sum_{j=1}^{n} X_j, n \geq 1.$$

This is called a *symmetric random walk*; with each head, it steps up one, and with each tail, it steps down one.

(i) Using the properties of Theorem 2.3.2, show that M_0, M_1, M_2, $\cdots$ is a martingale.

Proof. $\mathbb{E}_n[M_{n+1}] = M_n + \mathbb{E}_n[X_{n+1}] = M_n + \mathbb{E}[X_{n+1}] = M_n$. □

(ii) Let σ be a positive constant and, for $n \geq 0$, define

$$S_n = e^{\sigma M_n}\left(\frac{2}{e^{\sigma} + e^{-\sigma}}\right)^n.$$

Show that S_0, S_1, S_2, $\cdots$ is a martingale. Note that even though the symmetric random walk M_n has no tendency to grow, the "geometric symmetric random walk" $e^{\sigma M_n}$ does have a tendency to grow. This is the result of putting a martingale into the (convex) exponential function (see Exercise 2.3). In order to again have a martingale, we must "discount" the geometric symmetric random walk, using the term $\frac{2}{e^{\sigma}+e^{-\sigma}}$ as the discount rate. This term is strictly less than one unless $\sigma = 0$.

Proof.

$$\mathbb{E}_n\left[\frac{S_{n+1}}{S_n}\right] = \mathbb{E}_n\left[e^{\sigma X_{n+1}}\frac{2}{e^{\sigma} + e^{-\sigma}}\right] = \frac{2}{e^{\sigma} + e^{-\sigma}}\mathbb{E}\left[e^{\sigma X_{n+1}}\right] = 1.$$

□

► **Exercise 2.5.** Let M_0, M_1, M_2, $\cdots$ be the symmetric random walk of Exercise 2.4, and define $I_0 = 0$ and

$$I_n = \sum_{j=0}^{n-1} M_j(M_{j+1} - M_j),\ n = 1, 2, \cdots.$$

(i) Show that

$$I_n = \frac{1}{2}M_n^2 - \frac{n}{2}.$$

Proof.

$$\begin{aligned} 2I_n &= 2\sum_{j=0}^{n-1} M_j(M_{j+1} - M_j) = 2\sum_{j=0}^{n-1} M_jM_{j+1} + M_n^2 - \sum_{j=0}^{n-1} M_{j+1}^2 - \sum_{j=0}^{n-1} M_j^2 \\ &= M_n^2 - \sum_{j=0}^{n-1}(M_{j+1} - M_j)^2 = M_n^2 - \sum_{j=0}^{n-1} X_{j+1}^2 = M_n^2 - n. \end{aligned}$$

□

Remark 2.3. *This is the discrete version of the integration-by-parts formula for stochastic integral:*

$$M_T^2 - M_0^2 = 2\int_0^T M_t dM_t + [M, M]_T.$$

with $I_T = \int_0^T M_t dM_t$.

(ii) Let n be an arbitrary nonnegative integer, and let $f(i)$ be an arbitrary function of a variable i. In terms of n and f, define another function $g(i)$ satisfying

$$\mathbb{E}_n[f(I_{n+1})] = g(I_n).$$

Note that although the function $g(I_n)$ on the right-hand side of this equation may depend on n, the only random variable that may appear in its argument is I_n; the random variable M_n may not appear. You will need to use the formula in part (i). The conclusion of part (ii) is that the process I_0, I_1, I_2, $\cdots$ is a Markov process.

Solution.

$$\begin{aligned} & \mathbb{E}_n[f(I_{n+1})] \\ = & \ \mathbb{E}_n[f(I_n + M_n(M_{n+1} - M_n))] \\ = & \ \mathbb{E}_n[f(I_n + M_n X_{n+1})] \\ = & \ \frac{1}{2}[f(I_n + M_n) + f(I_n - M_n)] \\ = & \ g(I_n), \end{aligned}$$

where $g(x) = \frac{1}{2}[f(x + \sqrt{2x+n}) + f(x - \sqrt{2x+n})]$, since $\sqrt{2I_n + n} = |M_n|$. □

► **Exercise 2.6 (Discrete-time stochastic integral).** Suppose M_0, M_1, $\cdots$, M_N is a martingale, and let Δ_0, Δ_1, $\cdots$, Δ_{N-1} be an adapted process. Define the *discrete-time stochastic integral* (sometimes called a *martingale transform*) I_0, I_1, $\cdots$, I_N by setting $I_0 = 0$ and

$$I_n = \sum_{j=0}^{n-1} \Delta_j (M_{j+1} - M_j), \ n = 1, \cdots, N.$$

Show that I_0, I_1, $\cdots$, I_N is a martingale.

Proof. $\mathbb{E}_n[I_{n+1} - I_n] = \mathbb{E}_n[\Delta_n(M_{n+1} - M_n)] = \Delta_n \mathbb{E}_n[M_{n+1} - M_n] = 0$. □

► **Exercise 2.7.** In a binomial model, give an example of a stochastic process that is a martingale but is not Markov.

Solution. We denote by X_n the result of the nth coin toss:

$$\begin{cases} X_n = 1 & \text{if } \omega_n = H \\ X_n = -1 & \text{if } \omega_n = T \end{cases}$$

We also suppose $\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = \frac{1}{2}$. Define $S_1 = X_1$ and $S_{n+1} = S_n + b_n(X_1, \cdots, X_n)X_{n+1}$, where $b_n(\cdot)$ is a bounded function on $\{-1, 1\}^n$, to be determined later on. Clearly $(S_n)_{n\geq 1}$ is an adapted stochastic process, and we can show it is a martingale. Indeed,

$$\mathbb{E}_n[S_{n+1} - S_n] = b_n(X_1, \cdots, X_n)\mathbb{E}_n[X_{n+1}] = 0.$$

For any arbitrary function f, $\mathbb{E}_n[f(S_{n+1})] = \frac{1}{2}[f(S_n + b_n(X_1, \cdots, X_n)) + f(S_n - b_n(X_1, \cdots, X_n))]$. Then intuitively, $\mathbb{E}_n[f(S_{n+1}]$ cannot be solely dependent upon S_n when b_n's are properly chosen. Therefore in general, $(S_n)_{n\geq 1}$ cannot be a Markov process. □

Remark 2.4. *If X_n is regarded as the gain/loss of n-th bet in a gambling game, then S_n would be the wealth at time n. b_n is therefore the wager for the $(n+1)$th bet and is devised according to past gambling results.*

▶ **Exercise 2.8.** Consider an N-period binomial model.

(i) Let M_0, M_1, $\cdots$, M_N and M_0', M_1', $\cdots$, M_N' be martingales under the risk-neutral measure $\widetilde{\mathbb{P}}$. Show that if $M_N = M_N'$ (for every possible outcome of the sequence of coin tosses), then, for each n between 0 and N, we have $M_n = M_n'$ (for every possible outcome of the sequence of coin tosses).

Proof. Note $M_n = \mathbb{E}_n[M_N] = \mathbb{E}_n[M_N'] = M_n'$, $n = 0, 1, \cdots, N$. □

(ii) Let V_N be the payoff at time N of some derivative security. This is a random variable that can depend on all N coin tosses. Define recursively V_{N-1}, V_{N-2}, $\cdots$, V_0 by the algorithm (1.2.16) of Chapter 1. Show that

$$V_0, \frac{V_1}{1+r}, \cdots, \frac{V_{N-1}}{(1+r)^{N-1}}, \frac{V_N}{(1+r)^N}$$

is a martingale under $\widetilde{\mathbb{P}}$.

Proof. In the proof of Theorem 1.2.2, we proved by induction that $X_n = V_n$ where X_n is defined by (1.2.14) of Chapter 1: $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$. Since $\left\{\frac{X_n}{(1+r)^n}\right\}_{0\le n\le N}$ is a martingale under $\widetilde{\mathbb{P}}$ (Theorem 2.4.5), $\left\{\frac{V_n}{(1+r)^n}\right\}_{0\le n\le N}$ is also a martingale under $\widetilde{\mathbb{P}}$. □

(iii) Using the risk-neutral pricing formula (2.4.11) of this chapter, define

$$V_n' = \widetilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right], n = 0, 1, \cdots, N-1.$$

Show that

$$V_0', \frac{V_1'}{1+r}, \cdots, \frac{V_{N-1}'}{(1+r)^{N-1}}, \frac{V_N}{(1+r)^N}$$

is a martingale.

Proof. This is obvious by iterated conditioning. □

(iv) Conclude that $V_n = V_n'$ for every n (i.e., the algorithm (1.2.16) of Theorem 1.2.2 of Chapter 1 gives the same derivative security prices as the risk-neutral pricing formula (2.4.11) of Chapter 2).

Proof. Combine (ii) and (iii), then use (i). □

▶ **Exercise 2.9 (Stochastic volatility, random interest rate).** Consider a two-period stochastic volatility, random interest rate model of the type described in Exercise 1.9 of Chapter 1. The stock prices and interest rates are shown in Figure 2.8.1.

(i) Determine risk-neutral probabilities

$$\widetilde{\mathbb{P}}(HH), \widetilde{\mathbb{P}}(HT), \widetilde{\mathbb{P}}(TH), \widetilde{\mathbb{P}}(TT),$$

such that the time-zero value of an option that pays off V_2 at time two is given by the risk-neutral pricing formula

$$V_0 = \widetilde{\mathbb{E}}\left[\frac{V_2}{(1+r_0)(1+r_1)}\right].$$

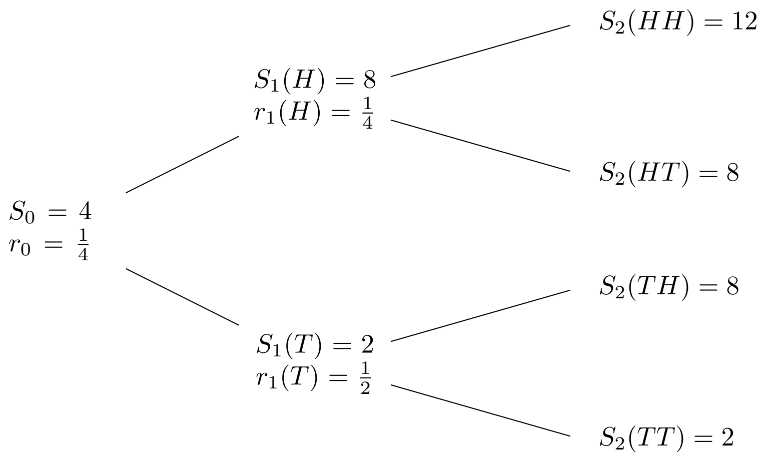


Fig. 2.8.1. A stochastic volatility, random interest rate model.

Solution.

$$\begin{aligned} u_0 &= \frac{S_1(H)}{S_0} = 2,\ d_0 = \frac{S_1(H)}{S_0} = \frac{1}{2}, \\ u_1(H) &= \frac{S_2(HH)}{S_1(H)} = 1.5,\ d_1(H) = \frac{S_2(HT)}{S_1(H)} = 1, \\ u_1(T) &= \frac{S_2(TH)}{S_1(T)} = 4,\ d_1(T) = \frac{S_2(TT)}{S_1(T)} = 1. \end{aligned}$$

Therefore

$$\begin{aligned} \tilde{p}_0 &= \frac{1+r_0-d_0}{u_0-d_0} = \frac{1}{2},\ \tilde{q}_0 = 1-\tilde{p}_0 = \frac{1}{2} \\ \tilde{p}_1(H) &= \frac{1+r_1(H)-d_1(H)}{u_1(H)-d_1(H)} = \frac{1}{2},\ \tilde{q}_1(H) = 1-\tilde{p}_1(H) = \frac{1}{2}, \\ \tilde{p}_1(T) &= \frac{1+r_1(T)-d_1(T)}{u_1(T)-d_1(T)} = \frac{1}{6},\ \tilde{q}_1(T) = 1-\tilde{p}_1(T) = \frac{5}{6}. \end{aligned}$$

and

$$\widetilde{\mathbb{P}}(HH) = \tilde{p}_0\tilde{p}_1(H) = \frac{1}{4},\ \widetilde{\mathbb{P}}(HT) = \tilde{p}_0\tilde{q}_1(H) = \frac{1}{4},\ \widetilde{\mathbb{P}}(TH) = \tilde{q}_0\tilde{p}_1(T) = \frac{1}{12},\ \widetilde{\mathbb{P}}(TT) = \tilde{q}_0\tilde{q}_1(T) = \frac{5}{12}.$$

The proofs of Theorem 2.4.4, Theorem 2.4.5 and Theorem 2.4.7 still work for the random interest rate model, with proper modifications (i.e. $\widetilde{\mathbb{P}}$ would be constructed according to the family of *conditional probabilities* $\widetilde{\mathbb{P}}(\omega_{n+1} = H|\omega_1, \cdots, \omega_n) := \tilde{p}_n$ and $\widetilde{\mathbb{P}}(\omega_{n+1} = T|\omega_1, \cdots, \omega_n) := \tilde{q}_n$. See Footnote 2 for comments.). So the time-zero value of an option that pays off V_2 at time two is given by the risk-neutral pricing formula $V_0 = \widetilde{\mathbb{E}}\left[\frac{V_2}{(1+r_0)(1+r_1)}\right]$. □

(ii) Let $V_2 = (S_2 - 7)^+$. Compute V_0, $V_1(H)$, and $V_1(T)$.

Solution. $V_2(HH) = 5$, $V_2(HT) = 1$, $V_2(TH) = 1$ and $V_2(TT) = 0$. So

$$\begin{aligned} V_1(H) &= \frac{\tilde{p}_1(H)V_2(HH) + \tilde{q}_1(H)V_2(HT)}{1+r_1(H)} = 2.4 \\ V_1(T) &= \frac{\tilde{p}_1(T)V_2(TH) + \tilde{q}_1(T)V_2(TT)}{1+r_1(T)} = \frac{1}{9} \\ V_0 &= \frac{\tilde{p}_0V_1(H) + \tilde{q}_0V_1(T)}{1+r_0} = 1.00444. \end{aligned}$$

□

(iii) Suppose an agent sells the option in (ii) for V_0 at time zero. compute the position Δ_0 she should take in the stock at time zero so that at time one, regardless of whether the first coin toss results in head or tail, the value of her portfolio is V_1.

Solution. $\Delta_0 = \frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)} = \frac{2.4-\frac{1}{9}}{8-2} = 0.4 - \frac{1}{54} \approx 0.3815.$ □

(iv) Suppose in (iii) that the first coin toss results in head. What position $\Delta_1(H)$ should the agent now take in the stock be sure that, regardless of whether the second coin toss results in head or tail, the value of her portfolio at time two will be $(S_2 - 7)^+$?

Solution. $\Delta_1(H) = \frac{V_2(HH)-V_2(HT)}{S_2(HH)-S_2(HT)} = \frac{5-1}{12-8} = 1.$ □

▶ **Exercise 2.10 (Dividend-paying stock).**[3] We consider a binomial asset pricing model as in Chapter 1, except that, after each movement in the stock price, a dividend is paid and the stock price is reduced accordingly. To describe this in equations, we define

$$Y_{n+1}(\omega_1 \cdots \omega_n \omega_{n+1}) = \begin{cases} u, & \text{if } \omega_{n+1} = H, \\ d, & \text{if } \omega_{n+1} = T. \end{cases}$$

Note that Y_{n+1} depends only on the $(n+1)$st coin toss. In the binomial model of Chapter 1, $Y_{n+1}S_n$ was the stock price at time $n+1$. In the dividend-paying model considered here, we have a random variable $A_{n+1}(\omega_1 \cdots \omega_n \omega_{n+1})$, taking values in $(0,1)$, and the dividend paid at time $n+1$ is $A_{n+1}Y_{n+1}S_n$. After the dividend is paid, the stock price at time $n+1$ is

$$S_{n+1} = (1 - A_{n+1})Y_{n+1}S_n.$$

An agent who begins with initial capital X_0 and at each time n takes a position of Δ_n shares of stock, where Δ_n depends only on the first n coin tosses, has a portfolio value governed by the wealth equation (see (2.4.6))[4]

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n) + \Delta_n A_{n+1} Y_{n+1} S_n = \Delta_n Y_{n+1} S_n + (1+r)(X_n - \Delta_n S_n). \quad (2.8.2)$$

(i) Show that the discounted wealth process is a martingale under the risk-neutral measure (i.e., Theorem 2.4.5 still holds for the wealth process (2.8.2)). As usual, the risk-neutral measure is still defined by the equations

$$\tilde{p} = \frac{1+r-d}{u-d}, \ \tilde{q} = \frac{u-1-r}{u-d}.$$

Proof.

$$\begin{aligned}
\widetilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] &= \widetilde{\mathbb{E}}_n\left[\frac{\Delta_n Y_{n+1} S_n}{(1+r)^{n+1}} + \frac{(1+r)(X_n - \Delta_n S_n)}{(1+r)^{n+1}}\right] \\
&= \frac{\Delta_n S_n}{(1+r)^{n+1}}\widetilde{\mathbb{E}}_n[Y_{n+1}] + \frac{X_n - \Delta_n S_n}{(1+r)^n} \\
&= \frac{\Delta_n S_n}{(1+r)^{n+1}}(u\tilde{p} + d\tilde{q}) + \frac{X_n - \Delta_n S_n}{(1+r)^n} \\
&= \frac{\Delta_n S_n + X_n - \Delta_n S_n}{(1+r)^n} = \frac{X_n}{(1+r)^n}.
\end{aligned}$$

□

[3]Compare this problem with §5.5 of Shreve [7].

[4]Note the assumption of the equation is that the dividends are reinvested. Also note the equation can be written as

$$X_{n+1} - X_n = \Delta_n(S_{n+1} - S_n) + r(X_n - \Delta_n S_n) + \Delta_n A_{n+1} Y_{n+1} S_n,$$

which gives three sources of wealth change: change of stock price, interest earned from money market account, and stock dividends.

(ii) Show that the risk-neutral pricing formula still applies (i.e., Theorem 2.4.7 holds for the dividend-paying model).

Proof. From (2.8.2), we have

$$\begin{cases}\Delta_n uS_n + (1+r)(X_n - \Delta_n S_n) = X_{n+1}(H)\\ \Delta_n dS_n + (1+r)(X_n - \Delta_n S_n) = X_{n+1}(T).\end{cases}$$

So

$$\Delta_n = \frac{X_{n+1}(H) - X_{n+1}(T)}{uS_n - dS_n},\ X_n = \widetilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{1+r}\right].$$

To make the portfolio replicate the payoff at time N, we must have $X_N = V_N$. So $X_n = \widetilde{\mathbb{E}}_n\left[\frac{X_N}{(1+r)^{N-n}}\right] = \widetilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right]$. Since $(X_n)_{0\leq n\leq N}$ is the value process of the unique replicating portfolio (uniqueness is guaranteed by the uniqueness of the solution to the above linear equations), the no-arbitrage price of V_N at time n is $V_n = X_n = \widetilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right]$. □

(iii) Show that the discounted stock price is not a martingale under the risk-neutral measure (i.e., Theorem 2.4.4 no longer holds). However, if A_{n+1} is a constant $a \in (0,1)$, regardless of the value of n and the outcome of the coin tossing $\omega_1\cdots\omega_{n+1}$, then $\frac{S_n}{(1-a)^n(1+r)^n}$ is a martingale under the risk-neutral measure.

Proof.

$$\begin{aligned}
& \widetilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] \\
= & \frac{1}{(1+r)^{n+1}}\widetilde{\mathbb{E}}_n[(1-A_{n+1})Y_{n+1}S_n] \\
= & \frac{S_n}{(1+r)^{n+1}}\{\tilde{p}[1-A_{n+1}(\omega_1\cdots\omega_n H)]u + \tilde{q}[1-A_{n+1}(\omega_1\cdots\omega_n T)]d\} \\
< & \frac{S_n}{(1+r)^{n+1}}[\tilde{p}u + \tilde{q}d] \\
= & \frac{S_n}{(1+r)^n}.
\end{aligned}$$

If A_{n+1} is a constant a, then

$$\widetilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] = \frac{S_n}{(1+r)^{n+1}}(1-a)(\tilde{p}u+\tilde{q}d) = \frac{S_n}{(1+r)^n}(1-a).$$

So $\widetilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}(1-a)^{n+1}}\right] = \frac{S_n}{(1+r)^n(1-a)^n}$, which implies $\frac{S_n}{(1-a)^n(1+r)^n}$ is a martingale under the risk-neutral measure. □

► **Exercise 2.11 (Put-call parity).** Consider a stock that pays no dividend in an N-period binomial model. A European call has payoff $C_N = (S_N - K)^+$ at time N. The price C_n of this call at earlier times is given by the risk-neutral pricing formula (2.4.11):

$$C_n = \widetilde{\mathbb{E}}_n\left[\frac{C_N}{(1+r)^{N-n}}\right],\ n = 0, 1, \cdots, N-1.$$

Consider also a put with payoff $P_N = (K - S_N)^+$ at time N, whose price at earlier time is

$$P_n = \widetilde{\mathbb{E}}_n\left[\frac{P_N}{(1+r)^{N-n}}\right],\ n = 0, 1, \cdots, N-1.$$

Finally, consider a *forward contract* to buy one share of stock at time N for K dollars. The price of this contract at time N is $F_N = S_N - K$, and its price at earlier times is

$$F_n = \widetilde{\mathbb{E}}_n\left[\frac{F_N}{(1+r)^{N-n}}\right],\ n = 0, 1, \cdots, N-1.$$

(Note that, unlike the call, the forward contract requires that the stock be purchased at time N for K dollars and has a negative payoff if $S_N < K$.)

(i) If at time zero you buy a forward contract and a put, and hold them until expiration, explain why the payoff you receive is the same as the payoff of call; i.e., explain why $C_N = F_N + P_N$.

Solution.

$$F_N + P_N = S_N - K + (K - S_N)^+ = \begin{cases} S_N - K + K - S_N & \text{if } K > S_N \\ S_N - K & \text{if } K \leq S_N \end{cases} = (S_N - K)^+ = C_N.$$

□

(ii) Using the risk-neutral pricing formulas given above for C_n, P_n, and F_n and the linearity of conditional expectations, show that $C_n = F_n + P_n$ for every n.

Proof. $C_n = \widetilde{\mathbb{E}}_n[\frac{C_N}{(1+r)^{N-n}}] = \widetilde{\mathbb{E}}_n[\frac{F_N}{(1+r)^{N-n}}] + \widetilde{\mathbb{E}}_n[\frac{P_N}{(1+r)^{N-n}}] = F_n + P_n$. □

(iii) Using the fact that the discounted stock price is a martingale under the risk-neutral measure, show that $F_0 = S_0 - \frac{K}{(1+r)^N}$.

Proof. $F_0 = \widetilde{\mathbb{E}}[\frac{F_N}{(1+r)^N}] = \frac{1}{(1+r)^N}\widetilde{\mathbb{E}}[S_N - K] = S_0 - \frac{K}{(1+r)^N}$. □

(iv) Suppose you begin at time zero with F_0, buy one share of stock, borrowing money as necessary to do that, and make no further trades. Show that at time N you have a portfolio valued at F_N. (This is called a *static replication* of the forward contract. If you sell the forward contract for F_0 at time zero, you can use this static replication to hedge your short position in the forward contract.)

Proof. At time zero, the trader has $F_0 - S_0$ in money market account and one share of stock. At time N, the trader has a wealth of $(F_0 - S_0)(1+r)^N + S_N = -K + S_N = F_N$. □

(v) The *forward price* of the stock at time zero is defined to be that value of K that causes the forward contract to have price zero at time zero. The forward price in this model is $(1+r)^N S_0$. Show that, at time zero, the price of a call struck at the forward price is the same as the price of a put struck at the forward price. This fact is called *put-call parity*.

Proof. By (iii), $F_0 = S_0 - \frac{(1+r)^N S_0}{(1+r)^N} = 0$. So by (ii), $C_0 = F_0 + P_0 = P_0$. □

Remark 2.5. *The forward price can be easily determined by the static replication: to hedge a short position in forward contract, we can buy and hold one share of stock until time N. The cost of this strategy is S_0 at time zero and $S_0(1+r)^N$ at time N. At time N, we receive K dollars for compensation. So we must have $K = S_0(1+r)^N$ to eliminate arbitrage.*

(vi) If we choose $K = (1+r)^N S_0$, we just saw in (v) that $C_0 = P_0$. Do we have $C_n = P_n$ for every n?

Proof. By (ii), $C_n = P_n$ if and only if $F_n = 0$. Note $F_n = \widetilde{\mathbb{E}}_n[\frac{S_N - K}{(1+r)^{N-n}}] = S_n - \frac{(1+r)^N S_0}{(1+r)^{N-n}} = S_n - S_0(1+r)^n$. So F_n is zero if and only if $n = 0$ or $r = 0$. □

▶ **Exercise 2.12 (Chooser option).** Let $1 \le m \le N-1$ and $K > 0$ be given. A *chooser option* is a contract sold at time zero that confers on its owner the right to receive either a call or a put at time m. The owner of the chooser may wait until time m before choosing. The call or put chosen expires at time N with strike K. Show that the time-zero price of a chooser option is the sum of the time-zero price of a put, expiring at time N and having strike price K, and a call, expiring at time m and having strike price $\frac{K}{(1+r)^{N-m}}$. (Hint: Use put-call parity (Exercise 2.11).)

Proof. First, the no-arbitrage price of the chooser option at time m must be $\max(C_m, P_m)$, where

$$C_m = \widetilde{\mathbb{E}}_m\left[\frac{(S_N-K)^+}{(1+r)^{N-m}}\right] \text{ and } P_m = \widetilde{\mathbb{E}}_m\left[\frac{(K-S_N)^+}{(1+r)^{N-m}}\right].$$

That is, C_m is the no-arbitrage price of a call option at time m and P_m is the no-arbitrage price of a put option at time m. Both of the call and the put have maturity date N and strike price K. Suppose the market is liquid, then the chooser option is equivalent to receiving a payoff of $\max(C_m, P_m)$ at time m. Therefore, its no-arbitrage price at time 0 should be $\widetilde{\mathbb{E}}[\frac{\max(C_m,P_m)}{(1+r)^m}]$.

By the put-call parity, $C_m = S_m - \frac{K}{(1+r)^{N-m}} + P_m$. So $\max(C_m, P_m) = P_m + \left[S_m - \frac{K}{(1+r)^{N-m}}\right]^+$. Therefore, the time-zero price of a chooser option is

$$\widetilde{\mathbb{E}}\left[\frac{P_m}{(1+r)^m}\right] + \widetilde{\mathbb{E}}\left[\frac{\left(S_m - \frac{K}{(1+r)^{N-m}}\right)^+}{(1+r)^m}\right] = \widetilde{\mathbb{E}}\left[\frac{(K-S_N)^+}{(1+r)^N}\right] + \widetilde{\mathbb{E}}\left[\frac{\left(S_m - \frac{K}{(1+r)^{N-m}}\right)^+}{(1+r)^m}\right].$$

The first term stands for the time-zero price of a put, expiring at time N and having strike price K, and the second term stands for the time-zero price of a call, expiring at time m and having strike price $\frac{K}{(1+r)^{N-m}}$. □

▶ **Exercise 2.13 (Asian option).** Consider an N-period binomial model. An *Asian option* has a payoff based on the average stock price, i.e.,

$$V_N = f\left(\frac{1}{N+1}\sum_{n=0}^{N} S_n\right),$$

where the function f is determined by the contractual details of the option.

(i) Define $Y_n = \sum_{k=0}^{n} S_k$ and use the Independence Lemma 2.5.3 to show that the two-dimensional process $\{S_n, Y_n\}$, $n = 0, 1, \cdots, N$ is Markov.

Proof. Note the conditional distribution

$$\widetilde{\mathbb{P}}(\omega_{n+1} = i|\omega_1, \cdots, \omega_n) = \begin{cases} \frac{1+r-d}{u-d} & \text{if } i = H \\ \frac{u-1-4}{u-d} & \text{if } i = T \end{cases}$$

is deterministic (i.e., independent of $\omega_1, \cdots, \omega_n$). So ω_n's are i.i.d. under $\widetilde{\mathbb{P}}$, as well as under $\mathbb{P}$. Hence, without loss of generality, we can work under $\mathbb{P}$. For any function g,

$$\begin{aligned} & \mathbb{E}_n[g(S_{n+1}, Y_{n+1})] \\ = \; & \mathbb{E}_n\left[g\left(\frac{S_{n+1}}{S_n}S_n, Y_n + \frac{S_{n+1}}{S_n}S_n\right)\right] \\ = \; & pg(uS_n, Y_n + uS_n) + qg(dS_n, Y_n + dS_n), \end{aligned}$$

which is a function of (S_n, Y_n). This shows $(S_n, Y_n)_{0 \le n \le N}$ is Markov under $\mathbb{P}$. □

(ii) According to Theorem 2.5.8, the price V_n of the Asian option at time n is some function v_n of S_n and Y_n; i.e.

$$V_n = v_n(S_n, Y_n), \; n = 0, 1, \cdots, N.$$

Give a formula for $v_N(s, y)$, and provide an algorithm for computing $v_n(s, y)$ in terms of v_{n+1}.

Proof. Set $v_N(s,y) = f(\frac{y}{N+1})$. Then $v_N(S_N, Y_N) = f\left(\frac{\sum_{n=0}^N S_n}{N+1}\right) = V_N$. Suppose v_{n+1} is given, then

$$\begin{aligned} V_n &= \widetilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{1+r}\right] \\ &= \widetilde{\mathbb{E}}_n\left[\frac{v_{n+1}(S_{n+1}, Y_{n+1})}{1+r}\right] \\ &= \frac{1}{1+r}[\tilde{p}v_{n+1}(uS_n, Y_n + uS_n) + \tilde{q}v_{n+1}(dS_n, Y_n + dS_n)]. \end{aligned}$$

So the formula for computing $v_n(s,y)$ in terms of v_{n+1} is

$$v_n(s,y) = \frac{\tilde{p}v_{n+1}(us, y+us) + \tilde{q}v_{n+1}(ds, y+ds)}{1+r}.$$

□

▶ **Exercise 2.14 (Asian option continued).** Consider an N-period binomial model, and let M be a fixed number between 0 and $N-1$. Consider an Asian option whose payoff at time N is

$$V_n = f\left(\frac{1}{N-M}\sum_{n=M+1}^{N} S_n\right),$$

where again the function f is determined by the contractual details of the option.

(i) Define

$$Y_n = \begin{cases} 0, & \text{if } 0 \le n \le M, \\ \sum_{k=M+1}^n S_k, & \text{if } M+1 \le n \le N. \end{cases}$$

Show that the two-dimensional process (S_n, Y_n), $n = 0, 1, \cdots, N$ is Markov (under the risk-neutral measure $\widetilde{\mathbb{P}}$).

Proof. For $n \le M$, $(S_n, Y_n) = (S_n, 0)$. Since coin tosses ω_n's are i.i.d. under $\widetilde{\mathbb{P}}$, $(S_n, Y_n)_{0\le n\le M}$ is Markov under $\widetilde{P}$. More precisely, for any function h, $\widetilde{\mathbb{E}}_n[h(S_{n+1})] = \tilde{p}h(uS_n) + \tilde{q}h(dS_n)$, for $n = 0, 1, \cdots, M-1$.

For any function g of two variables, we have

$$\widetilde{\mathbb{E}}_M[g(S_{M+1}, Y_{M+1})] = \widetilde{\mathbb{E}}_M[g(S_{M+1}, S_{M+1})] = \tilde{p}g(uS_M, uS_M) + \tilde{q}g(dS_M, dS_M).$$

And for $n \ge M+1$,

$$\widetilde{\mathbb{E}}_n[g(S_{n+1}, Y_{n+1})] = \widetilde{\mathbb{E}}_n\left[g\left(\frac{S_{n+1}}{S_n}S_n, Y_n + \frac{S_{n+1}}{S_n}S_n\right)\right] = \tilde{p}g(uS_n, Y_n + uS_n) + \tilde{q}g(dS_n, Y_n + dS_n).$$

So $(S_n, Y_n)_{0\le n\le N}$ is Markov under $\widetilde{\mathbb{P}}$. □

(ii) According to Theorem 2.5.8, the price V_n of the Asian option at time n is some function v_n of S_n and Y_n, i.e.

$$V_n = v_n(S_n, Y_n),\ n = 0, 1, \cdots, N.$$

Of course, when $n \le M$, Y_n is not random and does not need to be included in this function. Thus, for such n we should seek a function v_n of S_n alone and have

$$V_n = \begin{cases} v_n(S_n), & \text{if } 0 \le n \le M, \\ v_n(S_n, Y_n), & \text{if } M+1 \le n \le N. \end{cases}$$

Give a formula for $v_N(s,y)$, and provide an algorithm for computing v_n in terms of v_{n+1}. Note that the algorithm is different for $n < M$ and $n > M$, and there is a separate transition formula for $v_M(s)$ in terms of $v_{M+1}(\cdot,\cdot)$.

Proof. Set $v_N(s,y) = f(\frac{y}{N-M})$. Then $v_N(S_N, Y_N) = f\left(\frac{\sum_{K=M+1}^N S_k}{N-M}\right) = V_N$.

Suppose v_{n+1} is already given.

a) If $n > M$, then

$$\widetilde{\mathbb{E}}_n[v_{n+1}(S_{n+1}, Y_{n+1})] = \tilde{p}v_{n+1}(uS_n, Y_n + uS_n) + \tilde{q}v_{n+1}(dS_n, Y_n + dS_n).$$

So $v_n(s,y) = \tilde{p}v_{n+1}(us, y+us) + \tilde{q}v_{n+1}(ds, y+ds)$.

b) If $n = M$, then

$$\widetilde{\mathbb{E}}_M[v_{M+1}(S_{M+1}, Y_{M+1})] = \tilde{p}v_{M+1}(uS_M, uS_M) + \tilde{v}_{n+1}(dS_M, dS_M).$$

So $v_M(s) = \tilde{p}v_{M+1}(us, us) + \tilde{q}v_{M+1}(ds, ds)$.

c) If $n < M$, then

$$\widetilde{\mathbb{E}}_n[v_{n+1}(S_{n+1})] = \tilde{p}v_{n+1}(uS_n) + \tilde{q}v_{n+1}(dS_n).$$

So $v_n(s) = \tilde{p}v_{n+1}(us) + \tilde{q}v_{n+1}(ds)$. □

3 State Prices

★ **Comments**:

According to the *Fundamental Theorem of Asset Pricing* (FTAP), the no-arbitrage property is associated with the existence of a probability measure called *equivalent martingale measure* (EMM) and a positive process called *numéraire*, such that the price processes of tradable assets discounted by the numéraire are martingales under the given probability measure (hence the name *martingale measure*).

Theorem 2.4.4 shows that the risk-neutral probability $\mathbb{P}$ defined by

$$\tilde{p} = \frac{1+r-d}{u-d},\ \tilde{q} = \frac{u-1-r}{u-d}$$

and the value process of the money market account $\{(1+r)^n\}_{n=0}^N$ is such an EMM-numéraire pair. Theorem 3.2.7 shows that the actual probability $\mathbb{P}$ and the inverse of the state price density process $\left\{\frac{1}{\zeta_n}\right\}_{n=0}^N$ is such an EMM-numéraire pair. Note numéraire does not have to the price process of a tradable asset, as the inverse of the state price density process is just an abstract stochastic process.

For a survey of the various formulations of the two Fundamental Theorem of Asset Pricing, we refer to Zeng [9].

► **Exercise 3.1.** Under the conditions of Theorem 3.1.1, show the following analogues of properties (i)-(iii) of that theorem:

(i') $\widetilde{\mathbb{P}}\left(\frac{1}{Z} > 0\right) = 1$;

(ii') $\widetilde{\mathbb{E}}\frac{1}{Z} = 1$;

(iii') for any random variable Y,

$$\mathbb{E}Y = \widetilde{\mathbb{E}}\left[\frac{1}{Z}\cdot Y\right].$$

In other words, $\frac{1}{Z}$ facilitates the switch from $\widetilde{\mathbb{E}}$ to $\mathbb{E}$ in the same way Z facilitates the switch from $\mathbb{E}$ to $\widetilde{\mathbb{E}}$.

Proof. Define $\widetilde{Z}(\omega) := \frac{\mathbb{P}(\omega)}{\widetilde{\mathbb{P}}(\omega)} = \frac{1}{Z(\omega)}$. Apply Theorem 3.1.1 with $\mathbb{P}$, $\widetilde{\mathbb{P}}$, Z replaced by $\widetilde{\mathbb{P}}$, $\mathbb{P}$, $\widetilde{Z}$, we get the analogues of properties (i)-(iii) of Theorem 3.1.1. □

► **Exercise 3.2.** Let $\mathbb{P}$ be a probability measure on a finite probability space Ω. In this problem, we allow the possibility that $\mathbb{P}(\omega) = 0$ for some values of $\omega \in \Omega$. Let Z be a random variable on Ω with the property that $\mathbb{P}(Z \geq 0) = 1$ and $\mathbb{E}Z = 1$. For $\omega \in \Omega$, define $\widetilde{\mathbb{P}}(\omega) = Z(\omega)\mathbb{P}(\omega)$, and for events $A \subset \Omega$, define $\widetilde{\mathbb{P}}(A) = \sum_{\omega\in A}\widetilde{\mathbb{P}}(\omega)$. Show the following.

(i) $\widetilde{\mathbb{P}}$ is a probability measure; i.e., $\widetilde{\mathbb{P}}(\Omega) = 1$.

Proof. $\widetilde{\mathbb{P}}(\Omega) = \sum_{\omega\in\Omega} \widetilde{\mathbb{P}}(\omega) = \sum_{\omega\in\Omega} Z(\omega)\mathbb{P}(\omega) = E[Z] = 1.$ □

(ii) If Y is a random variable, then $\widetilde{\mathbb{E}}Y = \mathbb{E}[ZY]$.

Proof. $\widetilde{\mathbb{E}}[Y] = \sum_{\omega\in\Omega} Y(\omega)\widetilde{\mathbb{P}}(\omega) = \sum_{\omega\in\Omega} Y(\omega)Z(\omega)\mathbb{P}(\omega) = \mathbb{E}[YZ].$ □

(iii) If A is an event with $\mathbb{P}(A) = 0$, then $\widetilde{\mathbb{P}}(A) = 0$.

Proof. $\widetilde{\mathbb{P}}(A) = \sum_{\omega\in A} Z(\omega)\mathbb{P}(\omega)$. Since $\mathbb{P}(A) = 0$, $\mathbb{P}(\omega) = 0$ for any $\omega \in A$. So $\widetilde{\mathbb{P}}(A) = 0$. □

(iv) Assume that $\mathbb{P}(Z > 0) = 1$. Show that if A is an event with $\widetilde{\mathbb{P}}(A) = 0$, then $\mathbb{P}(A) = 0$.

Proof. If $\widetilde{\mathbb{P}}(A) = \sum_{\omega\in A} Z(\omega)\mathbb{P}(\omega) = 0$, by $\mathbb{P}(Z > 0) = 1$, we conclude $\mathbb{P}(\omega) = 0$ for any $\omega \in A$. So $\mathbb{P}(A) = \sum_{\omega\in A} \mathbb{P}(\omega) = 0$. □

When two probability measures agree which events have probability zero (i.e., $\mathbb{P}(A) = 0$ if and only if $\widetilde{\mathbb{P}}(A) = 0$), the measure are said to be *equivalent*. From (iii) and (iv) above, we see that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent under the assumption that $\mathbb{P}(Z > 0) = 1$.
(v) Show that if $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, then they agree which events have probability one (i.e., $\mathbb{P}(A) = 1$ if and only if $\widetilde{\mathbb{P}}(A) = 1$).

Proof. $\mathbb{P}(A) = 1 \iff \mathbb{P}(A^c) = 0 \iff \widetilde{\mathbb{P}}(A^c) = 0 \iff \widetilde{\mathbb{P}}(A) = 1.$ □

(vi) Construct an example in which we have only $\mathbb{P}(Z \geq 0) = 1$ and $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are not equivalent.

Solution. Pick ω_0 such that $1 > \mathbb{P}(\omega_0) > 0$. Define

$$Z(\omega) = \begin{cases} 0 & \text{if } \omega \neq \omega_0 \\ \frac{1}{\mathbb{P}(\omega_0)} & \text{if } \omega = \omega_0. \end{cases}$$

Then $\mathbb{P}(Z \geq 0) = 1$ and $\mathbb{E}[Z] = \frac{1}{\mathbb{P}(\omega_0)} \cdot \mathbb{P}(\omega_0) = 1$.

Clearly

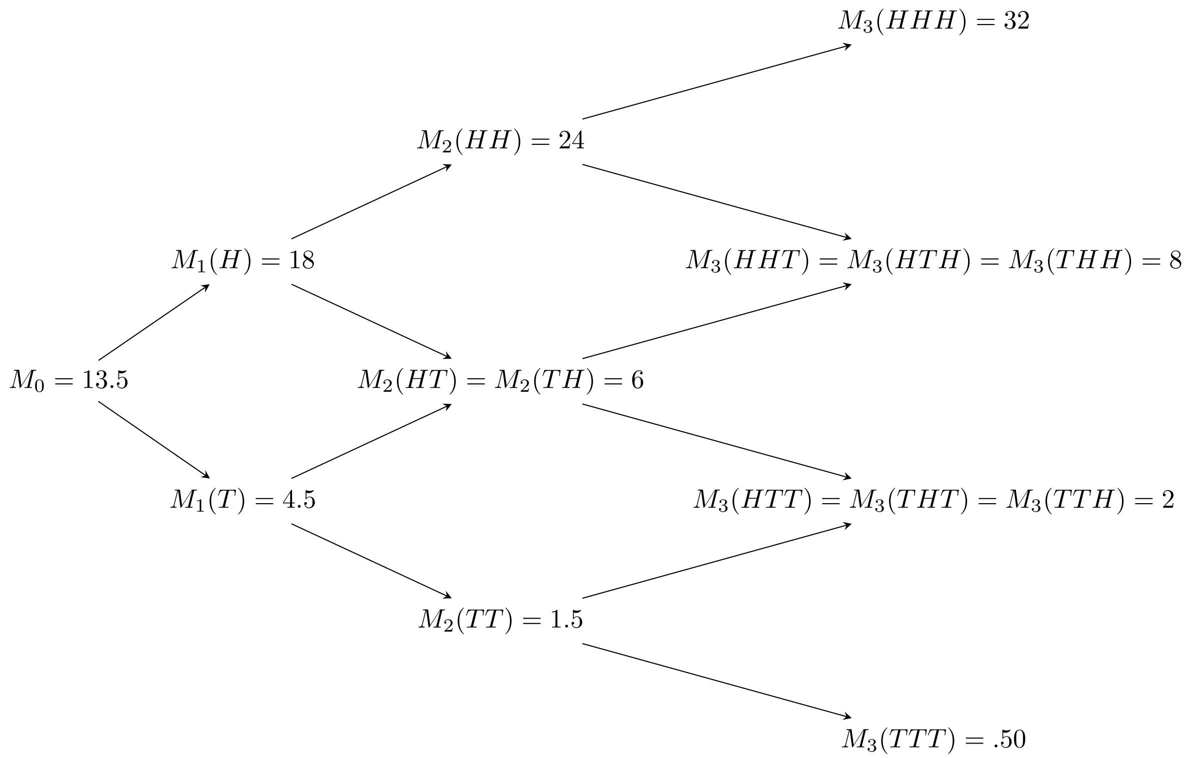
$$\widetilde{\mathbb{P}}(\Omega \setminus \{\omega_0\}) = \sum_{\omega\neq\omega_0} Z(\omega)\mathbb{P}(\omega) = 0.$$

But $\mathbb{P}(\Omega \setminus \{\omega_0\}) = 1 - \mathbb{P}(\omega_0) > 0$ since $\mathbb{P}(\omega_0) < 1$. Hence $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ cannot be equivalent. □

► **Exercise 3.3.** Using the stock price model of Figure 3.1.1 and the actual probabilities $p = \frac{2}{3}$, $q = \frac{1}{3}$, define the estimates of S_3 at various times by

$$M_n = \mathbb{E}_n[S_3],\ n = 0, 1, 2, 3.$$

Fill in the values of M_n in a tree like that of Figure 3.1.1. Verify that M_n, $n = 0, 1, 2, 3$, is a martingale.



Exercise 3.3.

Solution. $M_3 = S_3$,

$$
\begin{aligned}
M_2(HH) &= \mathbb{E}_2[S_3](HH) = pS_3(HHH) + qS_3(HHT) = \frac{2}{3}\cdot 32 + \frac{1}{3}\cdot 8 = 24 \\
M_2(HT) &= \mathbb{E}_2[S_3](HT) = pS_3(HTH) + qS_3(HTT) = \frac{2}{3}\cdot 8 + \frac{1}{3}\cdot 2 = 6 \\
M_2(TH) &= \mathbb{E}_2[S_3](TH) = pS_3(THH) + qS_3(THT) = \frac{2}{3}\cdot 8 + \frac{1}{3}\cdot 2 = 6 \\
M_2(TT) &= \mathbb{E}_2[S_3](TT) = pS_3(TTH) + qS_3(TTT) = \frac{2}{3}\cdot 2 + \frac{1}{3}\cdot 0.5 = 1.5 \\
M_1(H) &= \mathbb{E}_1[S_3](H) = p^2 S_3(HHH) + pqS_3(HHT) + qpS_3(HTH) + q^2 S_3(HTT) \\
&= \left(\frac{2}{3}\right)^2 \cdot 32 + \frac{2}{3}\cdot\frac{1}{3}\cdot(8+8) + \left(\frac{1}{3}\right)^2 \cdot 2 = 18 \\
M_1(T) &= \mathbb{E}_1[S_3](T) = p^2 S_3(THH) + pqS_3(THT) + qpS_3(TTH) + q^2 S_3(TTT) \\
&= \left(\frac{2}{3}\right)^2 \cdot 8 + \frac{2}{3}\cdot\frac{1}{3}\cdot(2+2) + \left(\frac{1}{3}\right)^2 \cdot 0.5 = 4.5 \\
M_0 &= \mathbb{E}_0[S_3] = \left(\frac{2}{3}\right)^3 \cdot 32 + \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3}\cdot(8+8+8) + \frac{2}{3}\cdot\left(\frac{1}{3}\right)^2 \cdot (2+2+2) + \left(\frac{1}{3}\right)^3 \cdot 0.5 = 13.5
\end{aligned}
$$

To verify $(M_n)_{n=0}^3$ is a martingale, we note $M_2 = \mathbb{E}_2[S_3] = \mathbb{E}_2[M_3]$, since $M_3 = S_3$. Moreover, we have

$$\begin{aligned}
\mathbb{E}_1[M_2](H) &= pM_2(HH) + qM_2(HT) = \frac{2}{3}\cdot 24 + \frac{1}{3}\cdot 6 = 18 = M_1(H) \\
\mathbb{E}_1[M_2](T) &= pM_2(TH) + qM_2(TT) = \frac{2}{3}\cdot 6 + \frac{1}{3}\cdot 1.5 = 4.5 = M_1(T) \\
\mathbb{E}_0[M_1](T) &= pM_1(H) + qM_1(T) = \frac{2}{3}\cdot 18 + \frac{1}{3}\cdot 4.5 = 13.5 = M_0.
\end{aligned}$$

Therefore, $(M_n)_{n=0}^3$ is a martingale. □

▶ **Exercise 3.4.** This problem refers to the model of Example 3.1.2, whose Radon-Nikodým process Z_n appears in Figure 3.2.1.

(i) Compute the state price densities

$$\begin{aligned}
&\zeta_3(HHH), \\
&\zeta_3(HHT) = \zeta_3(HTH) = \zeta_3(THH), \\
&\zeta_3(HTT) = \zeta_3(THT) = \zeta_3(TTH), \\
&\zeta_3(TTT)
\end{aligned}$$

explicitly.

Solution. By formula (3.1.5), we have

$$\begin{aligned}
&\zeta_3(HHH) = \frac{27}{64}\cdot\frac{1}{(1+0.25)^3} = 0.216 \\
&\zeta_3(HHT) = \zeta_3(HTH) = \zeta_3(THH) = \frac{27}{32}\cdot\frac{1}{(1+0.25)^3} = 0.432 \\
&\zeta_3(HTT) = \zeta_3(THT) = \zeta_3(TTH) = \frac{27}{16}\cdot\frac{1}{(1+0.25)^3} = 0.864 \\
&\zeta_3(TTT) = \frac{27}{8}\cdot\frac{1}{(1+0.25)^3} = 1.728.
\end{aligned}$$

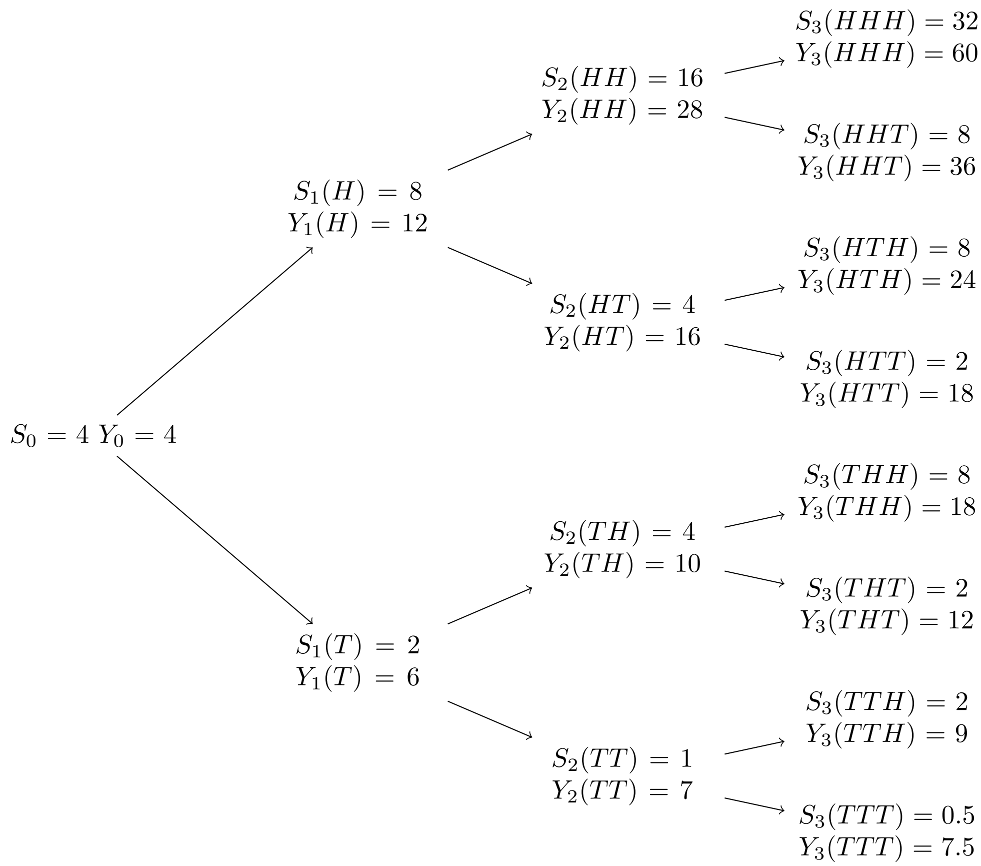
□

(ii) Use the numbers computed in (i) in formula (3.1.10) to find the time-zero price of the Asian option of Exercise 1.8 of Chapter 1. You should get $v_0(4,4)$ computed in part (ii) of that exercise.

Solution. Recall in Example 3.1.2, we have $p = \frac{2}{3}$ and $q = \frac{1}{3}$. So, by formula (3.1.10), the time-zero price of the Asian option of Exercise 1.8 is

$$\begin{aligned}
V_0 &= \mathbb{E}\left[\zeta_3\left(\frac{1}{4}Y_3 - 4\right)^+\right] = \sum_{\omega\in\Omega}\left(\frac{1}{4}Y_3(\omega) - 4\right)^+ \zeta(\omega)\mathbb{P}(\omega) \\
&= \left(\frac{1}{4}\cdot 60 - 4\right)^+ \cdot 0.216\cdot\left(\frac{2}{3}\right)^3 + \left(\frac{1}{4}\cdot 36 - 4\right)^+ \cdot 0.432\cdot\left(\frac{2}{3}\right)^2\cdot\left(\frac{1}{3}\right) + \\
&\quad \left(\frac{1}{4}\cdot 24 - 4\right)^+ \cdot 0.432\cdot\left(\frac{2}{3}\right)^2\cdot\left(\frac{1}{3}\right) + \left(\frac{1}{4}\cdot 18 - 4\right)^+ \cdot 0.864\cdot\left(\frac{1}{3}\right)^2\cdot\left(\frac{2}{3}\right) + \\
&\quad \left(\frac{1}{4}\cdot 18 - 4\right)^+ \cdot 0.432\cdot\left(\frac{2}{3}\right)^2\cdot\left(\frac{1}{3}\right) \\
&= 1.216.
\end{aligned}$$

□



Exercise 1.8. Asian option.

(iii) Compute also the state price densities $\zeta_2(HT) = \zeta_2(TH)$.

Solution. We note $\zeta_2 = \frac{Z_2}{(1+r)^2} = \mathbb{E}_2\left[\frac{Z}{(1+r)^3}\right] \cdot (1+r)$. By formula (3.1.5),

$$\zeta_2(HT) = \frac{1}{(1+r)^2}[pZ(HTH) + qZ(HTT)] = \frac{\frac{2}{3}\cdot\frac{27}{32} + \frac{1}{3}\cdot\frac{27}{16}}{1.25^2} = 0.72$$
$$\zeta_2(TH) = \frac{1}{(1+r)^2}[pZ(THH) + qZ(THT)] = \frac{1}{(1+r)^2}[pZ(HTH) + qZ(HTT)] = 0.72.$$

□

(iv) Use the risk-neutral pricing formula (3.2.6) in the form

$$V_2(HT) = \frac{1}{\zeta_2(HT)}\mathbb{E}_2[\zeta_3 V_3](HT),$$
$$V_2(TH) = \frac{1}{\zeta_2(TH)}\mathbb{E}_2[\zeta_3 V_3](TH)$$

to compute $V_2(HT)$ and $V_2(TH)$. You should get $V_2(HT) = v_2(4,16)$ and $V_2(TH) = v_2(4,10)$, where $v_2(s,y)$ was computed in part (ii) of Exercise 1.8 of Chapter 1. Note that $V_2(HT) \neq V_2(TH)$.

Solution. By Exercise 1.8 of Chapter 1, $V_2(HT) = v_2(4,16) = 1$, $V_2(TH) = v_2(4,10) = 0.2$. Meanwhile, we have

$$\begin{aligned}
V_2(HT) &= \frac{1}{\zeta_2(HT)}\mathbb{E}_2[\zeta_3 V_3](HT) = \frac{1}{\zeta_2(HT)}[p\zeta_3(HTH)V_3(HTH) + q\zeta_3(HTT)V_3(HTT)] \\
&= \frac{1}{0.72}\left[\frac{2}{3}\cdot 0.432 \cdot \left(\frac{1}{4}\cdot 24 - 4\right)^+ + \frac{1}{3}\cdot 0.864 \cdot \left(\frac{1}{4}\cdot 18 - 4\right)^+\right] \\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}
V_2(TH) &= \frac{1}{\zeta_2(TH)}\mathbb{E}_2[\zeta_3 V_3](TH) = \frac{1}{\zeta_2(TH)}[p\zeta_3(THH)V_3(THH) + q\zeta_3(THT)V_3(THT)] \\
&= \frac{1}{0.72}\left[\frac{2}{3}\cdot 0.432 \cdot \left(\frac{1}{4}\cdot 18 - 4\right)^+ + \frac{1}{3}\cdot 0.864 \cdot \left(\frac{1}{4}\cdot 12 - 4\right)^+\right] \\
&= 0.2.
\end{aligned}$$

□

► **Exercise 3.5 (Stochastic volatility, random interest rate).** Consider the model of Exercise 2.9 of Chapter 2. Assume that the actual probability measure is

$$\mathbb{P}(HH) = \frac{4}{9},\ \mathbb{P}(HT) = \frac{2}{9},\ \mathbb{P}(TH) = \frac{2}{9},\ \mathbb{P}(TT) = \frac{1}{9}.$$

The risk-neutral measure was computed in Exercise 2.9 of Chapter 2.
(i) Compute the Radon-Nikodým derivative $Z(HH)$, $Z(HT)$, $Z(TH)$, and $Z(TT)$ of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$.

Solution. In Exercise 2.9 of Chapter 2, we have

$$\widetilde{\mathbb{P}}(HH) = \frac{1}{4},\ \widetilde{\mathbb{P}}(HT) = \frac{1}{4},\ \widetilde{\mathbb{P}}(TH) = \frac{1}{12},\ \widetilde{\mathbb{P}}(TT) = \frac{5}{12}.$$

Therefore

$$\begin{aligned}
Z(HH) &= \frac{\widetilde{\mathbb{P}}(HH)}{\mathbb{P}(HH)} = \frac{9}{16} \\
Z(HT) &= \frac{\widetilde{\mathbb{P}}(HT)}{\mathbb{P}(HT)} = \frac{9}{8} \\
Z(TH) &= \frac{\widetilde{\mathbb{P}}(TH)}{\mathbb{P}(TH)} = \frac{3}{8} \\
Z(TT) &= \frac{\widetilde{\mathbb{P}}(TT)}{\mathbb{P}(TT)} = \frac{15}{4}.
\end{aligned}$$

□

(ii) The Radon-Nikodým derivative process Z_0, Z_1, Z_2 satisfies $Z_2 = Z$. Compute $Z_1(H)$, $Z_1(T)$, and Z_0. Note that $Z_0 = \mathbb{E}Z = 1$.

Solution. From the specification of $\mathbb{P}(HH)$, $\mathbb{P}(HT)$, $\mathbb{P}(TH)$, and $\mathbb{P}(TT)$, we can conclude ω_1 and ω_2 are i.i.d. under $\mathbb{P}$, with $p := \mathbb{P}(\omega_1 = H) = \mathbb{P}(\omega_2 = H) = \frac{2}{3}$ and $q := \mathbb{P}(\omega_1 = T) = \mathbb{P}(\omega_2 = T) = \frac{1}{3}$. Therefore,

$$\begin{aligned}
Z_1(H) &= \mathbb{E}_1[Z_2](H) = Z_2(HH)p + Z_2(HT)q = \frac{9}{16}\cdot\frac{2}{3} + \frac{9}{8}\cdot\frac{1}{3} = \frac{3}{4}, \\
Z_1(T) &= \mathbb{E}_1[Z_2](T) = Z_2(TH)p + Z_2(TT)q = \frac{3}{8}\cdot\frac{2}{3} + \frac{15}{4}\cdot\frac{1}{3} = \frac{3}{2}, \\
Z_0 &= \mathbb{E}[Z_1] = pZ_1(H) + qZ_1(T) = \frac{2}{3}\cdot\frac{3}{4} + \frac{1}{3}\cdot\frac{3}{2} = 1.
\end{aligned}$$

□

(iii) The version of the risk-neutral pricing formula (3.2.6) appropriate for this model, which does not use the risk-neutral measure, is

$$\begin{aligned}
V_1(H) &= \frac{1+r_0}{Z_1(H)}\mathbb{E}_1\left[\frac{Z_2}{(1+r_0)(1+r_1)}V_2\right](H) \\
&= \frac{1}{Z_1(H)(1+r_1(H))}\mathbb{E}_1[Z_2V_2](H), \\
V_1(T) &= \frac{1+r_0}{Z_1(H)}\mathbb{E}_1\left[\frac{Z_2}{(1+r_0)(1+r_1)}V_2\right](T) \\
&= \frac{1}{Z_1(H)(1+r_1(H))}\mathbb{E}_1[Z_2V_2](T), \\
V_0 &= \mathbb{E}\left[\frac{Z_2}{(1+r_0)(1+r_1)}V_2\right].
\end{aligned}$$

Use this formula to compute $V_1(H)$, $V_1(T)$, and V_0 when $V_2 = (S_2-7)^+$. Compare the result with your answers in Exercise 2.9(ii) of Chapter 2.[5]

Solution.

$$V_1(H) = \frac{Z_2(HH)V_2(HH)p + Z_2(HT)V_2(HT)q}{Z_1(H)(1+r_1(H))} = \frac{\frac{9}{16}\cdot(12-7)^+\cdot\frac{2}{3} + \frac{15}{4}\cdot(8-7)^+\cdot\frac{1}{3}}{\frac{3}{4}\cdot\left(1+\frac{1}{4}\right)} = 2.4,$$

$$V_1(T) = \frac{Z_2(TH)V_2(TH)p + Z_2(TT)V_2(TT)q}{Z_1(T)(1+r_1(T))} = \frac{\frac{3}{8}\cdot(8-7)^+\cdot\frac{2}{3} + \frac{15}{4}\cdot(2-7)^+\cdot\frac{1}{3}}{\frac{3}{2}\cdot\left(1+\frac{1}{2}\right)} = \frac{1}{9},$$

and

$$\begin{aligned}
V_0 &= \frac{Z_2(HH)V_2(HH)}{(1+r_0)(1+r_1(H))}\mathbb{P}(HH) + \frac{Z_2(HT)V_2(HT)}{(1+r_0)(1+r_1(H))}\mathbb{P}(HT) + \frac{Z_2(TH)V_2(TH)}{(1+r_0)(1+r_1(T))}\mathbb{P}(TH) + 0 \\
&= \frac{\frac{9}{16}\cdot(12-7)^+\cdot\frac{4}{9}}{(1+\frac{1}{4})(1+\frac{1}{4})} + \frac{\frac{9}{8}\cdot(8-7)^+\cdot\frac{2}{9}}{(1+\frac{1}{4})(1+\frac{1}{4})} + \frac{\frac{3}{8}\cdot(8-7)^+\cdot\frac{2}{9}}{(1+\frac{1}{4})(1+\frac{1}{2})} \\
&= 1.00444.
\end{aligned}$$

□

▶ **Exercise 3.6.** Consider Problem 3.3.1 in an N-period binomial model with the utility function $U(x) = \ln x$. Show that the optimal wealth process corresponding to the optimal portfolio process is given by $X_n = \frac{X_0}{\zeta_n}$, $n = 0, 1, \cdots, N$, where ζ_n is the state price density process defined in (3.2.7).

Proof. $U'(x) = \frac{1}{x}$, so $I(x) = \frac{1}{x}$. (3.3.26) gives $\mathbb{E}\left[\frac{Z}{(1+r)^N}\frac{(1+r)^N}{\lambda Z}\right] = X_0$. So $\lambda = \frac{1}{X_0}$. By (3.3.25), we have $X_N = \frac{(1+r)^N}{\lambda Z} = \frac{X_0}{Z}(1+r)^N$. Hence

$$X_n = \widetilde{\mathbb{E}}_n\left[\frac{X_N}{(1+r)^{N-n}}\right] = \widetilde{\mathbb{E}}_n\left[\frac{X_0(1+r)^n}{Z}\right] = X_0(1+r)^n\widetilde{\mathbb{E}}_n\left[\frac{1}{Z}\right] = X_0(1+r)^n\frac{1}{Z_n}\mathbb{E}_n\left[Z\cdot\frac{1}{Z}\right] = \frac{X_0}{\zeta_n},$$

where the second to last "=" comes from Lemma 3.2.6. □

▶ **Exercise 3.7.** Consider Problem 3.3.1 in an N-period binomial model with the utility function $U(x) = \frac{1}{p}x^p$, where $p < 1$, $p \neq 0$. Show that the optimal wealth at time N is

$$X_N = \frac{X_0(1+r)^N Z^{\frac{1}{p-1}}}{\mathbb{E}\left[Z^{\frac{p}{p-1}}\right]},$$

where Z is the Radon-Nikodým derivative of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$.

[5] The textbook said "Exercise 2.6" by mistake.

Proof. $U'(x) = x^{p-1}$, so $I(x) = x^{\frac{1}{p-1}}$. By (3.3.26), we have $\mathbb{E}\left[\frac{Z}{(1+r)^N}\left(\frac{\lambda Z}{(1+r)^N}\right)^{\frac{1}{p-1}}\right] = X_0$. Solve it for λ, we get

$$\lambda = \left(\frac{X_0}{\mathbb{E}\left[\frac{Z^{\frac{p}{p-1}}}{(1+r)^{\frac{Np}{p-1}}}\right]}\right)^{p-1} = \frac{X_0^{p-1}(1+r)^{Np}}{\left(\mathbb{E}\left[Z^{\frac{p}{p-1}}\right]\right)^{p-1}}.$$

So by (3.3.25),

$$X_N = \left[\frac{\lambda Z}{(1+r)^N}\right]^{\frac{1}{p-1}} = \frac{\lambda^{\frac{1}{p-1}} Z^{\frac{1}{p-1}}}{(1+r)^{\frac{N}{p-1}}} = \frac{X_0(1+r)^{\frac{Np}{p-1}}}{\mathbb{E}\left[Z^{\frac{p}{p-1}}\right]} \frac{Z^{\frac{1}{p-1}}}{(1+r)^{\frac{N}{p-1}}} = \frac{(1+r)^N X_0 Z^{\frac{1}{p-1}}}{\mathbb{E}\left[Z^{\frac{p}{p-1}}\right]}.$$

□

► **Exercise 3.8.** The Lagrange Multiplier Theorem used in the solution of Problem 3.3.5 has hypotheses that we did not verify in the solution of that problem. In particular, the theorem states that if the gradient of the constraint function, which in this case is the vector $(p_1\zeta_1, \cdots, p_m\zeta_m)$, is not the zero vector, then the optimal solution must satisfy the Lagrange multiplier equations (3.3.22). This gradient is not the zero vector, so this hypothesis is satisfied. However, even when this hypothesis is satisfied, the theorem does not guarantee that there is an optimal solution; the solution to the Lagrange multiplier equations may in fact minimize the expected utility. The solution could also be neither a maximizer nor a minimizer. Therefore, in this exercise, we outline a different method for verifying that the random variable X_N given by (3.3.25) maximizes the expected utility.

We begin by changing the notation, calling the random variable given by (3.3.25) X_N^* rather than X_N. In other words,

$$X_N^* = I\left(\frac{\lambda}{(1+r)^N} Z\right), \tag{3.6.1}$$

where λ is the solution of equation (3.3.26). This permits us to use the notation X_N for an arbitrary (not necessarily optimal) random variable satisfying (3.3.19). We must show that

$$\mathbb{E}U(X_N) \leq \mathbb{E}U(X_N^*). \tag{3.6.2}$$

(i) Fix $y > 0$, and show that the function of x given by $U(x) - yx$ is maximized by $x = I(y)$.[6] Conclude that

$$U(x) - yx \leq U(I(y)) - yI(y) \text{ for every } x. \tag{3.6.3}$$

Proof. $\frac{d}{dx}(U(x) - yx) = U'(x) - y$. So $x = I(y)$ is an extreme point of $U(x) - yx$. Because $\frac{d^2}{dx^2}(U(x) - yx) = U''(x) \leq 0$ (U is concave), $x = I(y)$ is a maximum point. Therefore $U(x) - yx \leq U(I(y)) - yI(y)$ for every x. □

(ii) In (3.6.3), replace the dummy variable x by the random variable X_N and replace the dummy variable y by the random variable $\frac{\lambda Z}{(1+r)^N}$. Take expectations of both sides and use (3.3.19) and (3.3.26) to conclude that (3.6.2) holds.

Proof. Following the hint of the problem, we have

$$\mathbb{E}[U(X_N)] - \mathbb{E}\left[X_N \frac{\lambda Z}{(1+r)^N}\right] \leq \mathbb{E}\left[U\left(I\left(\frac{\lambda Z}{(1+r)^N}\right)\right)\right] - \mathbb{E}\left[\frac{\lambda Z}{(1+r)^N} I\left(\frac{\lambda Z}{(1+r)^N}\right)\right],$$

i.e.

$$\begin{aligned} \mathbb{E}[U(X_N)] - \lambda X_0 &\overset{(3.3.19)}{=} \mathbb{E}[U(X_N)] - \lambda\widetilde{\mathbb{E}}\left[\frac{X_N}{(1+r)^N}\right] = \mathbb{E}[U(X_N)] - \mathbb{E}\left[X_N \frac{\lambda Z}{(1+r)^N}\right] \\ \leq \; & \mathbb{E}[U(X_N^*)] - \lambda\mathbb{E}\left[\frac{Z}{(1+r)^N} I\left(\frac{\lambda Z}{(1+r)^N}\right)\right] \overset{(3.3.26)}{=} \mathbb{E}[U(X_N^*)] - \lambda X_0. \end{aligned}$$

[6] The textbook said "$y = I(x)$" by mistake.

So $\mathbb{E}[U(X_N)] \leq \mathbb{E}[U(X_N^*)]$. □

▶ **Exercise 3.9 (Maximizing probability of reaching a goal).** (Kulldorf [30], Heath [19]) A wealthy investor provides a small amount of money X_0 for you to use to prove the effectiveness of your investment scheme over the next N periods. You are permitted to invest in the N-period binomial model, subject to the condition that the value of your portfolio is never allowed to be negative. If at time N the value of your portfolio X_N is at least γ, a positive constant specified by the investor, then you will be given a large amount of money to manage for her. Therefore, your problem is the following:
Maximize

$$\mathbb{P}(X_N \geq \gamma),$$

where X_N is generated by a portfolio process beginning with the initial wealth X_0 and where the value X_n of your portfolio satisfies

$$X_n \geq 0,\ n = 1, 2, \cdots, N.$$

In the way that Problem 3.3.1 was reformulated as Problem 3.3.3, this problem may be reformulated as
Maximize

$$\mathbb{P}(X_N \geq \gamma)$$

subject to

$$\widetilde{\mathbb{E}}\frac{X_N}{(1+r)^N} = X_0,\ X_n \geq 0,\ n = 1, 2, \cdots, N.$$

(i) Show that if $X_N \geq 0$, then $X_n \geq 0$ for all n.

Proof. By the martingale property, $X_n = \widetilde{\mathbb{E}}_n\left[\frac{X_N}{(1+r)^{N-n}}\right]$. So if $X_N \geq 0$, then $X_n \geq 0$ for all n. □

(ii) Consider the function

$$U(x) = \begin{cases} 0, & \text{if } 0 \leq x < \gamma, \\ 1, & \text{if } x \geq \gamma. \end{cases}$$

Show that for each fixed $y > 0$, we have

$$U(x) - yx \leq U(I(y)) - yI(y)\ \forall x \geq 0,$$

where

$$I(y) = \begin{cases} \gamma, & \text{if } 0 < y \leq \frac{1}{\gamma}, \\ 0, & \text{if } y > \frac{1}{\gamma}. \end{cases}$$

Proof. We consider the following four scenarios.

a) If $0 \leq x < \gamma$ and $0 < y \leq \frac{1}{\gamma}$, then $U(x) - yx = -yx \leq 0$ and $U(I(y)) - yI(y) = U(\gamma) - y\gamma = 1 - y\gamma \geq 0$. So $U(x) - yx \leq U(I(y)) - yI(y)$.

b) If $0 \leq x < \gamma$ and $y > \frac{1}{\gamma}$, then $U(x) - yx = -yx \leq 0$ and $U(I(y)) - yI(y) = U(0) - y \cdot 0 = 0$. So $U(x) - yx \leq U(I(y)) - yI(y)$.

c) If $x \geq \gamma$ and $0 < y \leq \frac{1}{\gamma}$, then $U(x) - yx = 1 - yx$ and $U(I(y)) - yI(y) = U(\gamma) - y\gamma = 1 - y\gamma \geq 1 - yx$. So $U(x) - yx \leq U(I(y)) - yI(y)$.

d) If $x \geq \gamma$ and $y > \frac{1}{\gamma}$, then $U(x) - yx = 1 - yx < 0$ and $U(I(y)) - yI(y) = U(0) - y \cdot 0 = 0$. So $U(x) - yx \leq U(I(y)) - yI(y)$.

□

(iii) Assume there is a solution λ to the equation

$$\mathbb{E}\left[\frac{Z}{(1+r)^N} I\left(\frac{\lambda Z}{(1+r)^N}\right)\right] = X_0. \tag{3.6.4}$$

Following the argument of Exercise 3.8, show that the optimal X_N is given by

$$X_N^* = I\left(\frac{\lambda Z}{(1+r)^N}\right).$$

Proof. Using (ii) and set $x = X_N$, $y = \frac{\lambda Z}{(1+r)^N}$, where X_N is a random variable satisfying $\widetilde{\mathbb{E}}\left[\frac{X_N}{(1+r)^N}\right] = X_0$, we have

$$\mathbb{E}[U(X_N)] - \mathbb{E}\left[\frac{\lambda Z}{(1+r)^N}X_N\right] \leq \mathbb{E}[U(X_N^*)] - \mathbb{E}\left[\frac{\lambda Z}{(1+r)^N}X_N^*\right].$$

That is, $\mathbb{E}[U(X_N)] - \lambda X_0 \leq \mathbb{E}[U(X_N^*)] - \lambda X_0$. So $\mathbb{E}[U(X_N)] \leq \mathbb{E}[U(X_N^*)]$. □

(iv) As we did to obtain Problem 3.3.5, let us list the $M = 2^N$ possible coin toss sequences, labeling them $\omega^1, \cdots, \omega^M$, and then define $\zeta_m = \zeta(\omega^m)$, $p_m = \mathbb{P}(\omega^m)$. However, here we list these sequences in ascending order of ζ_m i.e., we label the coin toss sequences so that

$$\zeta_1 \leq \zeta_2 \leq \cdots \leq \zeta_M.$$

Show that the assumption that there is a solution λ to (3.6.4) is equivalent to assuming that for some positive integer K we have $\zeta_K < \zeta_{K+1}$ and

$$\sum_{m=1}^{K} \zeta_m p_m = \frac{X_0}{\gamma}. \tag{3.6.5}$$

Proof. Plug p_m and ξ_m into (3.6.4), we have

$$X_0 = \sum_{m=1}^{2^N} p_m \xi_m I(\lambda \xi_m) = \sum_{m=1}^{2^N} p_m \xi_m \gamma 1_{\{\lambda\xi_m \leq \frac{1}{\gamma}\}}.$$

So $\frac{X_0}{\gamma} = \sum_{m=1}^{2^N} p_m \xi_m 1_{\{\lambda\xi_m \leq \frac{1}{\gamma}\}}$. Suppose there is a solution λ to (3.6.4), note $\frac{X_0}{\gamma} > 0$, we then can conclude $\{m : \lambda\xi_m \leq \frac{1}{\gamma}\} \neq \emptyset$. Let $K = \max\{m : \lambda\xi_m \leq \frac{1}{\gamma}\}$, then $\lambda\xi_K \leq \frac{1}{\gamma} < \lambda\xi_{K+1}$. So $\xi_K < \xi_{K+1}$ and $\frac{X_0}{\gamma} = \sum_{m=1}^{K} p_m \xi_m$ (Note, however, that K could be 2^N. In this case, ξ_{K+1} is interpreted as ∞. Also, note we are looking for positive solution $\lambda > 0$). Conversely, suppose there exists some K so that $\xi_K < \xi_{K+1}$ and $\sum_{m=1}^{K} \xi_m p_m = \frac{X_0}{\gamma}$. Then we can find $\lambda > 0$, such that $\xi_K < \frac{1}{\lambda\gamma} < \xi_{K+1}$. For such λ, we have

$$\mathbb{E}\left[\frac{Z}{(1+r)^N} I\left(\frac{\lambda Z}{(1+r)^N}\right)\right] = \sum_{m=1}^{2^N} p_m \xi_m 1_{\{\lambda\xi_m \leq \frac{1}{\gamma}\}} \gamma = \sum_{m=1}^{K} p_m \xi_m \gamma = X_0.$$

Hence (3.6.4) has a solution. □

(v) Show that X_N^* is given by

$$X_N(\omega^m) = \begin{cases} \gamma, & \text{if } m \leq K \\ 0, & \text{if } m \geq K+1. \end{cases}$$

Proof.

$$X_N^*(\omega^m) = I(\lambda\xi_m) = \gamma 1_{\{\lambda\xi_m \leq \frac{1}{\gamma}\}} = \begin{cases} \gamma, & \text{if } m \leq K, \\ 0, & \text{if } m \geq K+1. \end{cases}$$

□

4 American Derivative Securities

★ **Comments**:

1) Before proceeding to the exercise problems, we first give a brief summary of pricing American derivative securities as presented in the textbook. We shall use the notation of the book.

From the buyer's perspective. At time n, if the derivative security has not been exercised, then the buyer can choose a policy τ with $\tau \in \mathcal{S}_n$. The valuation formula for cash flow (Theorem 2.4.8) gives the fair price of the derivative security exercised according to τ (the cash flow sequence $C_n, \cdots, C_N$ at times $n, \cdots, N$ are defined by $C_n = 1_{\{\tau=n\}}G_n, \cdots, C_N = 1_{\{\tau=N\}}G_N$):

$$V_n(\tau) = \sum_{k=n}^{N} \widetilde{\mathbb{E}}_n \left[1_{\{\tau=k\}} \frac{G_k}{(1+r)^{k-n}} \right] = \widetilde{\mathbb{E}}_n \left[1_{\{\tau\le N\}} \frac{G_\tau}{(1+r)^{\tau-n}} \right].$$

The buyer wants to consider all the possible τ's, so that he can find the least upper bound of security value, which will be the maximum price of the derivative security acceptable to him. This is the price given by Definition 4.4.1: $V_n = \max_{\tau\in\mathcal{S}_n} \widetilde{\mathbb{E}}_n \left[1_{\{\tau\le N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau \right]$.

From the seller's perspective. A price process $(V_n)_{0\le n\le N}$ is acceptable to him if and only if at each time n, (i) $V_n \ge G_n$ (selling price is sufficient to cover potential obligation) and (ii) he needs no further investing into the portfolio as time goes by (no follow-up cost to maintain (i)).

Formally, the seller can find hedging process $(\Delta_n)_{0\le n\le N}$ and cash flow process $(C_n)_{0\le n\le N}$ such that $C_n \ge 0$ and $V_{n+1} = \Delta_n S_{n+1} + (1+r)(V_n - C_n - \Delta_n S_n) \ge G_{n+1}$. Since $\left\{ \frac{S_n}{(1+r)^n} \right\}_{0\le n\le N}$ is a martingale under the risk-neutral measure $\widetilde{\mathbb{P}}$, we conclude

$$\widetilde{\mathbb{E}}_n \left[\frac{V_{n+1}}{(1+r)^{n+1}} \right] - \frac{V_n}{(1+r)^n} = -\frac{C_n}{(1+r)^n} \le 0,$$

i.e. $\left\{ \frac{V_n}{(1+r)^n} \right\}_{0\le n\le N}$ is a supermartingale under $\widetilde{\mathbb{P}}$. This inspires us to check if the converse is also true. This is exactly the content of Theorem 4.4.4. So $(V_n)_{0\le n\le N}$ is the value process of a portfolio that needs no further investing if and only if $\left\{ \frac{V_n}{(1+r)^n} \right\}_{0\le n\le N}$ is a supermartingale under $\widetilde{\mathbb{P}}$ (note this is independent of the requirement $V_n \ge G_n$). In summary, a price process $(V_n)_{0\le n\le N}$ is acceptable to the seller if and only if (i) $V_n \ge G_n$; (ii) $\left\{ \frac{V_n}{(1+r)^n} \right\}_{0\le n\le N}$ is a supermartingale under $\widetilde{\mathbb{P}}$.

Theorem 4.4.2 shows the buyer's upper bound is the seller's lower bound. So it gives the price acceptable to both. Theorem 4.4.3 gives a specific algorithm for calculating the price, Theorem 4.4.4 establishes the one-to-one correspondence between super-replication and supermartingale property, and finally, Theorem 4.4.5 shows how to decide on the optimal exercise policy.

2) The definition of a stopping time typically seen in textbooks is

$$\{\tau = n\} \in \mathcal{F}_n := \sigma(\omega_1, \cdots, \omega_n),\ n = 0, 1, 2, \cdots, N.$$

To see Definition 4.3.1 agrees with this version of definition, note a typical element of $\mathcal{F}_n$ has the form $\{\omega : \omega_1 \in A_1, \cdots, \omega_n \in A_n\}$. So $\{\tau = n\}$ imposes conditions on $\omega_1, \omega_2, \cdots, \omega_n$ only.

3) Comment on Theorem 4.4.4 (Replication of path-dependent American derivatives): in the proof, the condition that $(V_n)_{n=0}^N$ comes from Definition 4.4.1 is only used for $V_n \ge G_n$. The proof applies to any $\widetilde{\mathbb{P}}$-supermartingale $\left\{ \frac{V_n}{(1+r)^n} \right\}_n$. This justifies the paragraph immediately after Theorem 4.4.2. Conversely, if $(V_n)_n$ can be replicated via formula (4.4.16) with $C_n \ge 0$, then $\left\{ \frac{V_n}{(1+r)^n} \right\}_n$ is a $\widetilde{\mathbb{P}}$-supermartingale. That is, if we can replicate a stochastic process $(V_n)_n$ by money market account and stock without injecting more cash into the portfolio, then $\left\{ \frac{V_n}{(1+r)^n} \right\}_n$ is a $\widetilde{\mathbb{P}}$-supermartingale.

► **Exercise 4.1.** In the three-period model of Figure 1.2.2 of Chapter 1, let the interest rate be $r = \frac{1}{4}$ so the risk-neutral probabilities are $\tilde{p} = \tilde{q} = \frac{1}{2}$.

(i) Determine the price at time zero, denoted V_0^P, of the American put that expires at time three and has intrinsic value $g_P(s) = (4 - s)^+$.

Solution. At time three, the payoff of the put is

$$\begin{aligned}
&V_3^P(HHH) = (4-32)^+ = 0\\
&V_3^P(HHT) = V_3^P(HTH) = V_3^P(THH) = (4-8)^+ = 0\\
&V_3^P(HTT) = V_3^P(THT) = V_3^P(TTH) = (4-2)^+ = 2\\
&V_3^P(TTT) = (4-0.5)^+ = 3.5.
\end{aligned}$$

At time two, the value of the put is calculated by

$$V_2^P(\cdot) = \max\left\{(4-S_2)^+, \widetilde{\mathbb{E}}_2\left[\frac{V_3^P}{1+r}\right]\right\} = \max\left\{(4-S_2)^+, \frac{\tilde{p}V_3^P(\cdot H) + \tilde{q}V_3^P(\cdot T)}{1+r}\right\}.$$

Therefore,

$$\begin{aligned}
V_2^P(HH) &= \max\left\{[4-S_2(HH)]^+, \frac{2}{5}[V_3^P(HHH)+V_3^P(HHT)]\right\} = \max\left\{(4-16)^+, \frac{2}{5}(0+0)\right\} = 0\\
V_2^P(HT) &= \max\left\{[4-S_2(HT)]^+, \frac{2}{5}[V_3^P(HTH)+V_3^P(HTT)]\right\} = \max\left\{(4-4)^+, \frac{2}{5}(0+2)\right\} = 0.8\\
V_2^P(TH) &= \max\left\{[4-S_2(TH)]^+, \frac{2}{5}[V_3^P(THH)+V_3^P(THT)]\right\} = \max\left\{(4-4)^+, \frac{2}{5}(0+2)\right\} = 0.8\\
V_2^P(TT) &= \max\left\{[4-S_2(TT)]^+, \frac{2}{5}[V_3^P(TTH)+V_3^P(TTT)]\right\} = \max\left\{(4-1)^+, \frac{2}{5}(2+3.5)\right\} = 3
\end{aligned}$$

A time one, the value of the put is calculated by

$$V_1^P(\cdot) = \max\left\{(4-S_1)^+, \widetilde{\mathbb{E}}_1\left[\frac{V_2^P}{1+r}\right]\right\} = \max\left\{(4-S_1)^+, \frac{\tilde{p}V_2^P(\cdot H) + \tilde{q}V_2^P(\cdot T)}{1+r}\right\}$$

Therefore

$$\begin{aligned}
V_1^P(H) &= \max\left\{[4-S_1(H)]^+, \frac{2}{5}[V_2^P(HH)+V_2^P(HT)]\right\} = \max\left\{(4-8)^+, \frac{2}{5}(0+0.8)\right\} = 0.32\\
V_1^P(T) &= \max\left\{[4-S_1(T)]^+, \frac{2}{5}[V_2^P(TH)+V_2^P(TT)]\right\} = \max\left\{(4-2)^+, \frac{2}{5}(0.8+3)\right\} = 2.
\end{aligned}$$

Finally, the value of the put at time zero is

$$V_0^P = \max\left\{(4-S_0)^+, \frac{2}{5}[V_1^P(H)+V_1^P(T)]\right\} = \max\left\{(4-4)^+, \frac{2}{5}(0.32+2)\right\} = 0.928.$$

□

(ii) Determine the price at time zero, denoted V_0^C, of the American call that expires at time three and has intrinsic value $g_C(s) = (s-4)^+$.

Solution. By Theorem 4.5.1, at the time zero, the price of the American call is the same as the price of the European call with the same strike. Therefore, at time three,

$$\begin{aligned}
&V_3^C(HHH) = (32-4)^+ = 28,\\
&V_3^C(HHT) = V_3^C(HTH) = V_3^C(THH) = (8-4)^+ = 4,\\
&V_3^C(HTT) = V_3^C(THT) = V_3^C(TTH) = (2-4)^+ = 0,\\
&V_3^C(TTT) = (0.5-4)^+ = 0.
\end{aligned}$$

At time two,

$$V_2^C(HH) = \frac{2}{5}(28+4) = 12.8,\ V_2^C(HT) = V_2^C(TH) = \frac{2}{5}(4+0) = 1.6,\ V_2^C(TT) = \frac{2}{5}(0+0) = 0.$$

At time one,

$$V_1^C(H) = \frac{2}{5}(12.8+1.6) = 5.76,\ V_1^C(T) = \frac{2}{5}(1.6+0) = 0.64.$$

And finally, at time zero,

$$V_0^C = \frac{2}{5}(5.76+0.64) = 2.56.$$

□

(iii) Determine the price at time zero, denoted V_0^S, of the American straddle that expires at time three and has intrinsic value $g_S(s) = g_P(s) + g_C(s)$.

Solution. $g_S(s) = |4-s|$. At time three,

$$\begin{aligned}
&V_3^S(HHH) = |4-32| = 28,\\
&V_3^S(HHT) = V_3^S(HTH) = V_3^S(THH) = |4-8| = 4,\\
&V_3^S(HTT) = V_3^S(THT) = V_3^S(TTH) = |4-2| = 2,\\
&V_3^S(TTT) = |4-0.5| = 3.5.
\end{aligned}$$

At time two,

$$\begin{aligned}
V_2^S(HH) &= \max\left\{|4-16|, \frac{2}{5}(28+4)\right\} = 12.8,\\
V_2^S(HT) = V_2^S(TH) &= \max\left\{|4-4|, \frac{2}{5}(4+2)\right\} = 2.4,\\
V_2^S(TT) &= \max\left\{|4-1|, \frac{2}{5}(2+0.5)\right\} = 3.
\end{aligned}$$

At time one,

$$\begin{aligned}
V_1^S(H) &= \max\left\{|4-8|, \frac{2}{5}(12.8+2.4)\right\} = 6.08,\\
V_1^S(T) &= \max\left\{|4-2|, \frac{2}{5}(2.4+3)\right\} = 2.16.
\end{aligned}$$

Finally, at time zero,

$$V_0^S = \max\left\{|4-4|, \frac{2}{5}(6.08+2.16)\right\} = 3.296.$$

□

(iv) Explain why $V_0^S < V_0^P + V_0^C$.

Solution. In our previous calculations,

$$V_0^S = 3.296 < V_0^P + V_0^C = 0.928 + 2.56.$$

To explain this, we first note the simple inequality

$$\max(a_1, b_1) + \max(a_2, b_2) \geq \max(a_1 + a_2, b_1 + b_2),$$

where "$>$" holds if and only if $b_1 > a_1$ and $b_2 < a_2$ *or* $b_1 < a_1$ and $b_2 > a_2$.

At time N, $V_N^S = g_S(S_N) = g_P(S_N) + g_C(S_N) = V_N^P + V_N^C$. By induction, we can show

$$\begin{aligned} V_n^S &= \max\left\{g_S(S_n), \frac{\tilde{p}V_{n+1}^S + \tilde{q}V_{n+1}^S}{1+r}\right\} \\ &\leq \max\left\{g_P(S_n) + g_C(S_n), \frac{\tilde{p}V_{n+1}^P + \tilde{q}V_{n+1}^P}{1+r} + \frac{\tilde{p}V_{n+1}^C + \tilde{q}V_{n+1}^C}{1+r}\right\} \\ &\leq \max\left\{g_P(S_n), \frac{\tilde{p}V_{n+1}^P + \tilde{q}V_{n+1}^P}{1+r}\right\} + \max\left\{g_C(S_n), \frac{\tilde{p}V_{n+1}^C + \tilde{q}V_{n+1}^C}{1+r}\right\} \\ &= V_n^P + V_n^C, \end{aligned}$$

$n = 0, 1, \cdots, N-1$.

To see when "$<$" holds, note for American call options, we always have $g_C(S_n) < \frac{\tilde{p}V_{n+1}^C + \tilde{q}V_{n+1}^C}{1+r}$ for $n = 0, 1, \cdots, N-1$. In view of our observation of the simple inequality at the beginning of our solution, "$<$" holds whenever

$$g_P(S_n) > \frac{\tilde{p}V_{n+1}^P + \tilde{q}V_{n+1}^P}{1+r}.$$

This is typically the case at the optimal exercise time of the put.

Beside the mathematical argument, intuitively, a straddle is not an American put plus an American call, since when exercised, a straddle has a payoff equal to the sum of the payoff of a put and the payoff of a call. However, the exercise time of a put typically differs from that of a call. So intuitively, we must have $V_0^S < V_0^P + V_0^C$. □

► **Exercise 4.2.** In Example 4.2.1, we computed the time-zero value of the American put with strike price 5 to be 1.36. Consider an agent who borrows 1.36 at time zero and buys the put. Explain how this agent can generate sufficient funds to pay off his loan (which grows by 25% each period) by trading in the stock and money markets and optimally exercising the put.

Solution. For this problem, we need Figure 4.2.1, Figure 4.4.1 and Figure 4.4.2. Then

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = -\frac{1}{12},\ \Delta_1(T) = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)} = -1,$$

and

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} \approx -0.433.$$

The optimal exercise time is $\tau = \inf\{n : V_n = G_n\}$. So

$$\tau(HH) = \infty,\ \tau(HT) = 2,\ \tau(TH) = \tau(TT) = 1.$$

Therefore, the agent borrows 1.36 at time zero and buys the put. At the same time, to hedge the long position, he needs to borrow again and buy 0.433 shares of stock at time zero.

At time one, if the result of coin toss is tail and the stock price goes down to 2, the value of the portfolio is $X_1(T) = (1+r)(-1.36 - 0.433S_0) + 0.433S_1(T) = (1+\frac{1}{4})(-1.36 - 0.433 \times 4) + 0.433 \times 2 = -3$. The agent should exercise the put at time one and get 3 to pay off his debt.

At time one, if the result of coin toss is head and the stock price goes up to 8, the value of the portfolio is $X_1(H) = (1+r)(-1.36 - 0.433S_0) + 0.433S_1(H) = -0.4$. The agent should borrow to buy $\frac{1}{12}$ shares of stock. At time two, if the result of coin toss is head and the stock price goes up to 16, the value of the portfolio is $X_2(HH) = (1+r)(X_1(H) - \frac{1}{12}S_1(H)) + \frac{1}{12}S_2(HH) = 0$, and the agent should let the put expire. If at time two, the result of coin toss is tail and the stock price goes down to 4, the value of the portfolio is $X_2(HT) = (1+r)(X_1(H) - \frac{1}{12}S_1(H)) + \frac{1}{12}S_2(HT) = -1$. The agent should exercise the put to get 1. This will pay off his debt. □

▶ **Exercise 4.3.** In the three-period model of Figure 1.2.2 of Chapter 1, let the interest rate be $r = \frac{1}{4}$ so the risk-neutral probabilities are $\tilde{p} = \tilde{q} = \frac{1}{2}$. Find the time-zero price and optimal exercise policy (optimal stopping time) for the path-dependent American derivative security whose intrinsic value at each time n, $n = 0, 1, 2, 3$, is $\left(4 - \frac{1}{n+1}\sum_{j=0}^{n} S_j\right)^+$. This intrinsic value is a put on the average stock price between time zero and time n.

Solution. We need Figure 1.2.2 for this problem, and calculate the intrinsic value process and price process of the put as follows.

For the intrinsic value process, $G_0 = 0$, $G_1(T) = 1$, $G_2(TH) = \frac{2}{3}$, $G_2(TT) = \frac{5}{3}$, $G_3(THT) = 1$, $G_3(TTH) = 1.75$, $G_3(TTT) = 2.125$. All the other outcomes of G is negative.

For the price process, $V_0 = 0.4$, $V_1(T) = 1$, $V_1(TH) = \frac{2}{3}$, $V_1(TT) = \frac{5}{3}$, $V_3(THT) = 1$, $V_3(TTH) = 1.75$, $V_3(TTT) = 2.125$. All the other outcomes of V is zero.

Therefore the time-zero price of the derivative security is 0.4 and the optimal exercise time satisfies

$$\tau(\omega) = \begin{cases} \infty & \text{if } \omega_1 = H, \\ 1 & \text{if } \omega_1 = T. \end{cases}$$

□

▶ **Exercise 4.4.** Consider the American put of Example 4.2.1, which has strike price 5. Suppose at time zero we sell this put to a purchaser who has inside information about the stock movements and uses the exercise rule ρ of (4.3.2). In particular, if the first toss is going to result in H, the owner of the put exercises at time zero, when the put has intrinsic value 1. If the first toss results in T and the second toss is going to result in H, the owner exercises at time one, when the put has intrinsic value 3. If the first two tosses result in TT, the owner exercises at time two, when the intrinsic value is 4. In summary, the owner of the put has the payoff random variable

$$Y(HH) = 1,\ Y(HT) = 1,\ Y(TH) = 3,\ Y(TT) = 4. \tag{4.8.1}$$

The risk-neutral expected value of this payoff, discounted from the time of payment back to zero, is

$$\widetilde{\mathbb{E}}\left[\left(\frac{4}{5}\right)^{\rho} Y\right] = \frac{1}{4}\left[1 + 1 + \frac{4}{5}\cdot 3 + \frac{16}{25}\cdot 4\right] = 1.74. \tag{4.8.2}$$

The time-zero price of the put computed in Example 4.2.1 is only 1.36. Do we need to charge the insider more than this amount if we are going to successfully hedge our short position after selling the put to her? Explain why or why not.

Solution. 1.36 is the cost of super-replicating the American derivative security. It enables us to construct a portfolio sufficient to pay off the derivative security, no matter when the derivative security is exercised. So to hedge our short position after selling the put, there is no need to charge the insider more than 1.36.

To understand the paradox that the risk-neutral expected value 1.74 of the insider's payoff is greater than the fair price 1.36, we note the rationale for risk-neutral pricing is the existence of a self-financing replicating portfolio:

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - C_n - \Delta_n S_n),\ n = 0, 1, \cdots, N-1,$$

where the hedging process $(\Delta_n)_n$ is assumed to be adapted, i.e. $\Delta_n \in \mathcal{F}_n$. This is in contradiction of the use of inside information. Therefore, it is fishy to use risk-neutral pricing formula when insider information exists. □

▶ **Exercise 4.5.** In equation (4.4.5), the maximum is computed over all stopping times in $\mathcal{S}_0$. List all the stopping times in $\mathcal{S}_0$ (there are 26), and from among them, list the stopping times that never exercise when the option is out of the money (there are 11). For each stopping time τ in the latter set, compute $\widetilde{\mathbb{E}}\left[1_{\{\tau\le 2\}}\left(\frac{4}{5}\right)^{\tau} G_\tau\right]$. Verify that the largest value for this quantity is given by the stopping time of (4.4.6), the one that makes this quantity equal to the 1.36 computed in (4.4.7).

Solution. The stopping times in $\mathcal{S}_0$ are
(1) $\tau \equiv 0$;
(2) $\tau \equiv 1$;
(3) $\tau(HT) = \tau(HH) = 1$, $\tau(TH)$, $\tau(TT) \in \{2, \infty\}$ (4 different ones);
(4) $\tau(HT)$, $\tau(HH) \in \{2, \infty\}$, $\tau(TH) = \tau(TT) = 1$ (4 different ones);
(5) $\tau(HT), \tau(HH), \tau(TH), \tau(TT) \in \{2, \infty\}$ (16 different ones).

When the option is out of money, the following stopping times do not exercise
(i) $\tau \equiv 0$;
(ii) $\tau(HT) \in \{2, \infty\}$, $\tau(HH) = \infty$, $\tau(TH), \tau(TT) \in \{2, \infty\}$ (8 different ones);
(iii) $\tau(HT) \in \{2, \infty\}$, $\tau(HH) = \infty$, $\tau(TH) = \tau(TT) = 1$ (2 different ones).

For (i), $\widetilde{\mathbb{E}}\left[1_{\{\tau\leq 2\}}\left(\frac{4}{5}\right)^{\tau} G_\tau\right] = G_0 = 1$. For (ii), $\widetilde{\mathbb{E}}\left[1_{\{\tau\leq 2\}}\left(\frac{4}{5}\right)^{\tau} G_\tau\right] \leq \widetilde{\mathbb{E}}\left[1_{\{\tau^*\leq 2\}}\left(\frac{4}{5}\right)^{\tau^*} G_{\tau^*}\right]$, where $\tau^*(HT) = 2$, $\tau^*(HH) = \infty$, $\tau^*(TH) = \tau^*(TT) = 2$. So

$$\widetilde{\mathbb{E}}\left[1_{\{\tau^*\leq 2\}}\left(\frac{4}{5}\right)^{\tau^*} G_{\tau^*}\right] = \frac{1}{4}\left[\left(\frac{4}{5}\right)^2 \cdot 1 + \left(\frac{4}{5}\right)^2 (1+4)\right] = 0.96.$$

For (iii), $\widetilde{\mathbb{E}}\left[1_{\{\tau\leq 2\}}\left(\frac{4}{5}\right)^{\tau} G_\tau\right]$ has the biggest value when τ satisfies $\tau(HT) = 2$, $\tau(HH) = \infty$, $\tau(TH) = \tau(TT) = 1$. This value is 1.36. □

▶ **Exercise 4.6 (Estimating American put prices).** For each n, where $n = 0, 1, \cdots, N$, let G_n be a random variable depending on the first n coin tosses. The time-zero value of a derivative security that can be exercised at nay time $n \leq N$ for payoff G_n but *must be exercised at time N if it has not been exercised before that time* is

$$V_0 = \max_{\tau \in \mathcal{S}_0, \tau \leq N} \widetilde{\mathbb{E}}\left[\frac{1}{(1+r)^\tau} G_\tau\right]. \tag{4.8.3}$$

In contrast to equation (4.4.1) in the Definition 4.4.1 for American derivative securities, here we consider only stopping times that take one of the values $0, 1, \cdots, N$ and not the value ∞.

(i) Consider $G_n = K - S_n$, the derivative security that permits its owner to sell one share of stock for payment K at any time up to and including N, but if the owner does not sell by time N, then she must do so at time N. Show that the optimal exercise policy is to sell the stock at time zero and that the value of this derivative security is $K - S_0$.

Proof. The value of the put at time N, if it is not exercised at previous times, is $K - S_N$. Hence

$$V_{N-1} = \max\left\{K - S_{N-1}, \widetilde{\mathbb{E}}_{N-1}\left[\frac{V_N}{1+r}\right]\right\} = \max\left\{K - S_{N-1}, \frac{K}{1+r} - S_{N-1}\right\} = K - S_{N-1}.$$

The second equality comes from the fact that discounted stock price process is a martingale under risk-neutral probability. By induction, we can show $V_n = K - S_n$ $(0 \leq n \leq N)$. So by Theorem 4.4.5, the optimal exercise policy is to sell the stock at time zero and the value of this derivative security is $K - S_0$. □

Remark 4.1. *We cheated a little bit by using American algorithm and Theorem 4.4.5, since they are developed for the case where τ is allowed to be ∞. But intuitively, results in this chapter should still hold for the case $\tau \leq N$, provided we replace "$\max\{G_n, 0\}$" with "G_n".*

(ii) Explain why a portfolio that holds the derivative security in (i) and a European call with strike K and expiration time N is at least as valuable as an American put struck at K with expiration time N. Denote the time-zero value of the European call by V_0^{EC} and the time-zero value of the American put by V_0^{AP}. Conclude that the upper bound

$$V_0^{AP} \leq K - S_0 + V_0^{EC} \tag{4.8.4}$$

on V_0^{AP} holds.

Proof. This is because at time N, if we have to exercise the put and $K - S_N < 0$, we can exercise the European call to set off the negative payoff. In effect, throughout the portfolio's lifetime, the portfolio has intrinsic values no less than that of an American put stuck at K with expiration time N. So, we must have $V_0^{AP} \leq K - S_0 + V_0^{EC}$. □

(iii) Use put-call parity (Exercise 2.11 of Chapter 2) to derive the lower bound on V_0^{AP}:

$$\frac{K}{(1+r)^N} - S_0 + V_0^{EC} \leq V_0^{AP}. \tag{4.8.5}$$

Proof. Let V_0^{EP} denote the time-zero value of a European put with strike K and expiration time N. Then

$$V_0^{AP} \geq V_0^{EP} = V_0^{EC} - \widetilde{\mathbb{E}}\left[\frac{S_N - K}{(1+r)^N}\right] = V_0^{EC} - S_0 + \frac{K}{(1+r)^N}.$$

□

▶ **Exercise 4.7.** For the class of derivative securities described in Exercise 4.6 whose time-zero price is given by (4.8.3), let $G_n = S_n - K$. This derivative security permits its owner to buy one share of stock in exchange for a payment of K at any time up to the expiration time N. If the purchase has not been made at time N, it must be made then. Determine the time-zero value and optimal exercise policy for this derivative security.[7]

Solution. $V_N = S_N - K$,

$$V_{N-1} = \max\left\{S_{N-1} - K, \widetilde{\mathbb{E}}_{N-1}\left[\frac{V_N}{1+r}\right]\right\} = \max\left\{S_{N-1} - K, S_{N-1} - \frac{K}{1+r}\right\} = S_{N-1} - \frac{K}{1+r}.$$

By induction, we can prove $V_n = S_n - \frac{K}{(1+r)^{N-n}}$ $(0 \leq n \leq N)$ and $V_n > G_n$ for $0 \leq n \leq N-1$. So the time-zero value is $S_0 - \frac{K}{(1+r)^N}$ and the optimal exercise time is N. □

5 Random Walk

▶ **Exercise 5.1.** For the symmetric random walk, consider the first passage time τ_m to the level m. The random variable $\tau_2 - \tau_1$ is the number of steps required for the random walk to rise from level 1 to level 2, and this random variable has the same distribution as τ_1, the number of steps required for the random walk to rise from level 0 to level 1. Furthermore, $\tau_2 - \tau_1$ and τ_1 are independent of one another; the latter depends only on the coin toss $1, 2, \cdots, \tau_1$, and the former depends only on the coin tosses $\tau_1 + 1$, $\tau_1 + 2$, $\cdots$, τ_2.

(i) Use these facts to explain why

$$\mathbb{E}\alpha^{\tau_2} = (\mathbb{E}\alpha^{\tau_1})^2 \text{ for all } \alpha \in (0,1).$$

Proof. $\mathbb{E}[\alpha^{\tau_2}] = \mathbb{E}[\alpha^{(\tau_2-\tau_1)+\tau_1}] = \mathbb{E}[\alpha^{(\tau_2-\tau_1)}]\mathbb{E}[\alpha^{\tau_1}] = \mathbb{E}[\alpha^{\tau_1}]^2$. □

Remark 5.1. *The i.i.d. property of $\tau_2 - \tau_1$ and τ_1 can be proven by strong Markov property of random walk.*

(ii) Without using (5.2.13), explain why for any positive integer m we must have

$$\mathbb{E}\alpha^{\tau_m} = (\mathbb{E}\alpha^{\tau_1})^m \text{ for all } \alpha \in (0,1). \tag{5.7.1}$$

Proof. If we define $M_n^{(m)} = M_{n+\tau_m} - M_{\tau_m}$ $(m = 1, 2, \cdots)$, then $(M_\cdot^{(m)})_m$ as random functions are i.i.d. with distributions the same as that of M. So $\tau_{m+1} - \tau_m = \inf\{n : M_n^{(m)} = 1\}$ are i.i.d. with distributions the same as that of τ_1. Therefore

$$\mathbb{E}[\alpha^{\tau_m}] = \mathbb{E}[\alpha^{(\tau_m-\tau_{m-1})+(\tau_{m-1}-\tau_{m-2})+\cdots+\tau_1}] = \mathbb{E}[\alpha^{\tau_1}]^m.$$

□

[7] Compare with American call option, Theorem 4.5.1.

Remark 5.2. *For those concerned with the meaning of "$(M^{(m)}_\cdot)_m$ as random functions...", note the function space $E := \{f : \mathbb{N} \to \mathbb{Z}\}$ is a Polish space with metric $\|f - g\| = \sum_{n=1}^{\infty} \frac{|f_n - g_n|}{2^n}$. Then each $M^{(m)}_\cdot : \Omega \to E$ is a random element from one probability space to another measurable space so that we can talk about its distribution.*

(iii) Would equation (5.7.1) still hold if the random walk is not symmetric? Explain why or why not.

Solution. Yes, it would still hold since the strong Markov property of random walk, hence the i.i.d. property of $(\tau_{m+1} - \tau_m)_{m=0}^{\infty}$, still holds and the argument of (ii) still works for asymmetric random walk. □

▶ **Exercise 5.2 (First passage time for random walk with upward drift).** Consider the asymmetric random walk with probability p for an up step and probability $q = 1 - p$ for a down step, where $\frac{1}{2} < p < 1$ so that $0 < q < \frac{1}{2}$. In the notation of (5.2.1), let τ_1 be the first time the random walk starting from level 0 reaches level 1. If the random walk never reaches this level, then $\tau_1 = \infty$.
(i) Define $f(\sigma) = pe^{\sigma} + qe^{-\sigma}$. Show that $f(\sigma) > 1$ for all $\sigma > 0$.

Proof. For all $\sigma > 0$, $f'(\sigma) = pe^{\sigma} - qe^{-\sigma} > p - q > 0$. So $f(\sigma) > f(0) = 1$ for all $\sigma > 0$. □

(ii) Show that, when $\sigma > 0$, the process

$$S_n = e^{\sigma M_n} \left(\frac{1}{f(\sigma)} \right)^n$$

is a martingale.

Proof. $\mathbb{E}_n \left[\frac{S_{n+1}}{S_n} \right] = \mathbb{E}_n \left[e^{\sigma X_{n+1}} \frac{1}{f(\sigma)} \right] = pe^{\sigma} \frac{1}{f(\sigma)} + qe^{-\sigma} \frac{1}{f(\sigma)} = 1.$ □

(iii) Show that, for $\sigma > 0$,

$$e^{-\sigma} = \mathbb{E} \left[1_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right].$$

Conclude that $\mathbb{P}(\tau_1 < \infty) = 1$.

Proof. By optional sampling theorem (Theorem 4.3.2), $\mathbb{E}[S_{n \wedge \tau_1}] = \mathbb{E}[S_0] = 1$. Note

$$S_{n \wedge \tau_1} = e^{\sigma M_{n \wedge \tau_1}} \left(\frac{1}{f(\sigma)} \right)^{n \wedge \tau_1} \leq e^{\sigma}$$

for all $\sigma > 0$, by bounded convergence theorem and the fact $0 < 1_{\{\tau_1 = \infty\}} S_{n \wedge \tau_1} \leq e^{\sigma} \left(\frac{1}{f(\sigma)} \right)^n \to 0$ as $n \to \infty$, we have

$$\mathbb{E}[1_{\{\tau_1 < \infty\}} S_{\tau_1}] = \mathbb{E}[\lim_{n \to \infty} S_{n \wedge \tau_1}] = \lim_{n \to \infty} \mathbb{E}[S_{n \wedge \tau_1}] = 1.$$

That is, $\mathbb{E} \left[1_{\{\tau_1 < \infty\}} e^{\sigma} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] = 1$. So $e^{-\sigma} = \mathbb{E} \left[1_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right]$. Let $\sigma \downarrow 0$, again by bounded convergence theorem, $1 = \mathbb{E} \left[1_{\{\tau_1 < \infty\}} \left(\frac{1}{f(0)} \right)^{\tau_1} \right] = P(\tau_1 < \infty)$. □

(iv) Compute $\mathbb{E}\alpha^{\tau_1}$ for $\alpha \in (0, 1)$.

Solution. Set $\alpha = \frac{1}{f(\sigma)} = \frac{1}{pe^{\sigma} + qe^{-\sigma}}$, then as σ varies from 0 to ∞, α can take all the values in $(0, 1)$. Write σ in terms of α, we have (note $4pq\alpha^2 < 4(\frac{p+q}{2})^2 \cdot 1^2 = 1$)

$$e^{\sigma} = \frac{1 \pm \sqrt{1 - 4pq\alpha^2}}{2p\alpha}.$$

We want to choose $\sigma > 0$, so we should take $\sigma = \ln\left(\frac{1+\sqrt{1-4pq\alpha^2}}{2p\alpha}\right)$. Therefore

$$\mathbb{E}[\alpha^{\tau_1}] = \frac{2p\alpha}{1+\sqrt{1-4pq\alpha^2}} = \frac{1-\sqrt{1-4pq\alpha^2}}{2q\alpha}.$$

□

(v) Compute $\mathbb{E}\tau_1$.

Solution. $\frac{\partial}{\partial\alpha}\mathbb{E}[\alpha^{\tau_1}] = \mathbb{E}[\frac{\partial}{\partial\alpha}\alpha^{\tau_1}] = \mathbb{E}[\tau_1\alpha^{\tau_1-1}]$, and

$$\begin{aligned}
&\left(\frac{1-\sqrt{1-4pq\alpha^2}}{2q\alpha}\right)' \\
=\ & \frac{1}{2q}\left[(1-\sqrt{1-4pq\alpha^2})\alpha^{-1}\right]' \\
=\ & \frac{1}{2q}\left[-\frac{1}{2}(1-4pq\alpha^2)^{-\frac{1}{2}}(-4pq2\alpha)\alpha^{-1} + (1-\sqrt{1-4pq\alpha^2})(-1)\alpha^2\right].
\end{aligned}$$

Therefore

$$\mathbb{E}[\tau_1] = \lim_{\alpha\uparrow 1}\frac{\partial}{\partial\alpha}\mathbb{E}[\alpha^{\tau_1}] = \frac{1}{2q}\left[-\frac{1}{2}(1-4pq)^{-\frac{1}{2}}(-8pq) - (1-\sqrt{1-4pq})\right] = \frac{1}{2p-1} = \frac{1}{p-q}.$$

□

▶ **Exercise 5.3 (First passage time for random walk with downward drift).** Modify Exercise 5.2 by assuming $0 < p < \frac{1}{2}$ so that $\frac{1}{2} < q < 1$.

(i) Find a positive number σ_0 such that the function $f(\sigma) = pe^{\sigma} + qe^{-\sigma}$ satisfies $f(\sigma_0) = 1$ and $f(\sigma) > 1$ for all $\sigma > \sigma_0$.

Solution. Solve the equation $pe^{\sigma} + qe^{-\sigma} = 1$ and a positive solution is

$$\ln\frac{1+\sqrt{1-4pq}}{2p} = \ln\frac{1-p}{p} = \ln q - \ln p.$$

Set $\sigma_0 = \ln q - \ln p$, then $f(\sigma_0) = 1$ and

$$f'(\sigma) = pe^{\sigma} - qe^{-\sigma} = qe^{-\sigma}\left[e^{2\left(\sigma - \frac{\ln q - \ln p}{2}\right)} - 1\right] > 0$$

for $\sigma > \sigma_0$. So $f(\sigma) > f(\sigma_0) = 1$ for all $\sigma > \sigma_0$. □

(ii) Determine $\mathbb{P}\{\tau_1 < \infty\}$. (This quantity is no longer equal to 1.)

Solution. As in Exercise 5.2, $S_n = e^{\sigma M_n}\left(\frac{1}{f(\sigma)}\right)^n$ is a martingale, and

$$1 = \mathbb{E}[S_0] = E[S_{n\wedge\tau_1}] = \mathbb{E}\left[e^{\sigma M_{n\wedge\tau_1}}\left(\frac{1}{f(\sigma)}\right)^{\tau_1\wedge n}\right].$$

Suppose $\sigma > \sigma_0$, then by bounded convergence theorem and $f(\sigma) > 1$, we have

$$1 = \mathbb{E}\left[\lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}}\left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1}\right] = \mathbb{E}\left[1_{\{\tau_1<\infty\}}e^{\sigma}\left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right].$$

Let $\sigma \downarrow \sigma_0$, we get $\mathbb{P}(\tau_1 < \infty) = e^{-\sigma_0} = \frac{p}{q} < 1$. □

(iii) Compute $\mathbb{E}\alpha^{\tau_1}$ for $\alpha \in (0,1)$.

Solution. Choose $\sigma > \sigma_0$, then $f(\sigma) > 1$ and we have from (ii) $\mathbb{E}\left[1_{\{\tau_1<\infty\}}\left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right] = e^{-\sigma}$. Set $\alpha = \frac{1}{f(\sigma)}$, then

$$\mathbb{E}\left[\alpha^{\tau_1}1_{\{\tau_1<\infty\}}\right] = e^{-\sigma(\alpha)}$$

where $\sigma(\alpha)$ satisfies $e^{\sigma(\alpha)} = \frac{1\pm\sqrt{1-4pq\alpha^2}}{2p\alpha}$. To make sure $\sigma(\alpha) > \sigma_0$, we choose $\sigma(\alpha) = \ln\left(\frac{1+\sqrt{1-4pq\alpha^2}}{2p\alpha}\right)$. As a consequence,

$$\mathbb{E}\left[\alpha^{\tau_1}1_{\{\tau_1<\infty\}}\right] = \frac{1-\sqrt{1-4pq\alpha^2}}{2q\alpha}.$$

□

(iv) Compute $\mathbb{E}\left[1_{\{\tau_1<\infty\}}\tau_1\right]$. (Since $\mathbb{P}(\tau_1 = \infty) > 0$, we have $\mathbb{E}\tau_1 = \infty$.)

Solution. Using result in (iii) and bounded convergence theorem, we have

$$\begin{aligned}
\mathbb{E}\left[\tau_1 1_{\{\tau_1<\infty\}}\right] &= \mathbb{E}\left[\lim_{\alpha\uparrow 1}\left(\tau_1\alpha^{\tau_1-1}\right)\right] = \lim_{\alpha\uparrow 1}\mathbb{E}\left[\left(\tau_1\alpha^{\tau_1-1}\right)\right] = \lim_{\alpha\uparrow 1}\frac{\partial}{\partial\alpha}\mathbb{E}[\alpha^{\tau_1}] \\
&= \frac{1}{2q}\left[\frac{4pq}{\sqrt{1-4pq}} - (1-\sqrt{1-4pq})\right] = \frac{1}{2q}\left[\frac{4pq}{2q-1} - 1 + 2q - 1\right] \\
&= \frac{p}{q}\frac{1}{q-p}.
\end{aligned}$$

□

► **Exercise 5.4 (Distribution of τ_2).** Consider the symmetric random walk, and let τ_2 be the first time the random walk, starting from level 0, reaches the level 2. According to Theorem 5.2.3,

$$\mathbb{E}\alpha^{\tau_2} = \left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right)^2 \text{ for all } \alpha \in (0,1).$$

Using the power series (5.2.21), we may write the right-hand side as

$$\begin{aligned}
\left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right)^2 &= \frac{2}{\alpha}\cdot\frac{1-\sqrt{1-\alpha^2}}{\alpha} - 1 \\
&= -1 + \sum_{j=1}^{\infty}\left(\frac{\alpha}{2}\right)^{2j-2}\frac{(2j-2)!}{j!(j-1)!} \\
&= \sum_{j=2}^{\infty}\left(\frac{\alpha}{2}\right)^{2j-2}\frac{(2j-2)!}{j!(j-1)!} \\
&= \sum_{k=1}^{\infty}\left(\frac{\alpha}{2}\right)^{2k}\frac{(2k)!}{(k+1)!k!}.
\end{aligned}$$

(i) Use the power series above to determine $\mathbb{P}\{\tau_2 = 2k\}$, $k = 1, 2, \cdots$.

Solution. $\mathbb{E}[\alpha^{\tau_2}] = \sum_{k=1}^{\infty}\mathbb{P}(\tau_2 = 2k)\alpha^{2k} = \sum_{k=1}^{\infty}\left(\frac{\alpha}{2}\right)^{2k}\mathbb{P}(\tau_2 = 2k)4^k$. So $\mathbb{P}(\tau_2 = 2k) = \frac{(2k)!}{4^k(k+1)!k!}$. □

(ii) Use the reflection principle to determine $\mathbb{P}\{\tau_2 = 2k\}$, $k = 1, 2, \cdots$.

Solution. $\mathbb{P}(\tau_2 = 2) = \frac{1}{4}$. For $k \geq 2$, $\mathbb{P}(\tau_2 = 2k) = \mathbb{P}(\tau_2 \leq 2k) - \mathbb{P}(\tau_2 \leq 2k-2)$.

$$\begin{aligned}
\mathbb{P}(\tau_2 \leq 2k) &= \mathbb{P}(M_{2k} = 2) + \mathbb{P}(M_{2k} \geq 4) + \mathbb{P}(\tau_2 \leq 2k, M_{2k} \leq 0) \\
&= \mathbb{P}(M_{2k} = 2) + 2\mathbb{P}(M_{2k} \geq 4) \\
&= \mathbb{P}(M_{2k} = 2) + \mathbb{P}(M_{2k} \geq 4) + \mathbb{P}(M_{2k} \leq -4) \\
&= 1 - \mathbb{P}(M_{2k} = -2) - \mathbb{P}(M_{2k} = 0).
\end{aligned}$$

Similarly, $\mathbb{P}(\tau_2 \leq 2k-2) = 1 - \mathbb{P}(M_{2k-2} = -2) - \mathbb{P}(M_{2k-2} = 0)$. So

$$\begin{aligned}
\mathbb{P}(\tau_2 = 2k) &= \mathbb{P}(M_{2k-2} = -2) + \mathbb{P}(M_{2k-2} = 0) - \mathbb{P}(M_{2k} = -2) - \mathbb{P}(M_{2k} = 0) \\
&= \left(\frac{1}{2}\right)^{2k-2}\left[\frac{(2k-2)!}{k!(k-2)!} + \frac{(2k-2)!}{(k-1)!(k-1)!}\right] - \left(\frac{1}{2}\right)^{2k}\left[\frac{(2k)!}{(k+1)!(k-1)!} + \frac{(2k)!}{k!k!}\right] \\
&= \frac{(2k)!}{4^k(k+1)!k!}\left[\frac{4}{2k(2k-1)}(k+1)k(k-1) + \frac{4}{2k(2k-1)}(k+1)k^2 - k - (k+1)\right] \\
&= \frac{(2k)!}{4^k(k+1)!k!}\left[\frac{2(k^2-1)}{2k-1} + \frac{2(k^2+k)}{2k-1} - \frac{4k^2-1}{2k-1}\right] \\
&= \frac{(2k)!}{4^k(k+1)!k!}.
\end{aligned}$$

□

► **Exercise 5.5 (Joint distribution of random walk and maximum-to-date).** Let M_n be a symmetric random walk, and define its *maximum-to-date* process

$$M_n^* = \max_{1 \leq k \leq n} M_k. \tag{5.7.2}$$

Let n and m be even positive integers, and let b be an even integer less than or equal to m. Assume $m \leq n$ and $2m - b \leq n$.

(i) Use an argument based on reflected paths to show that

$$\begin{aligned}
\mathbb{P}\{M_n^* \geq m, M_n = b\} &= \mathbb{P}\{M_n = 2m - b\} \\
&= \frac{n!}{\left(\frac{n-b}{2} + m\right)!\left(\frac{n+b}{2} - m\right)!}\left(\frac{1}{2}\right)^n.
\end{aligned}$$

Proof. For every path that crosses level m by time n and resides at b at time n, there corresponds a reflected path that resides at time $2m - b$. So

$$\mathbb{P}(M_n^* \geq m,\ M_n = b) = \mathbb{P}(M_n = 2m - b) = \left(\frac{1}{2}\right)^n \frac{n!}{\left(m + \frac{n-b}{2}\right)!\left(\frac{n+b}{2} - m\right)!}.$$

□

(ii) If the random walk is asymmetric with probability p for an up step and probability $q = 1 - p$ for a down step, where $0 < p < 1$, what is $\mathbb{P}\{M_n^* \geq m, M_n = b\}$?

Solution. The argument is similar to (i), but we need to determine the powers of p and q from the following equations

$$\begin{cases} x_p + x_q = n \\ x_p - x_q = 2m - b. \end{cases}$$

Solving it gives us

$$\begin{cases} x_p = m + \frac{n-b}{2} \\ x_q = \frac{n+b}{2} - m. \end{cases}$$

Therefore

$$\mathbb{P}(M_n^* \geq m,\ M_n = b) = \mathbb{P}(M_n = 2m - b) = \frac{n!}{(m + \frac{n-b}{2})!\,(\frac{n+b}{2} - m)!} p^{m+\frac{n-b}{2}} q^{\frac{n+b}{2}-m}.$$

□

► **Exercise 5.6.** The value of the perpetual American put in Section 5.4 is the limit as $n \to \infty$ of the value of an American put with the same strike price 4 that expires at time n. When the initial stock price is $S_0 = 4$, the value of the perpetual American put is 1 (see (5.4.6) with $j = 2$). Show that the value of an American put in the same model when the initial stock price is $S_0 = 4$ is 0.80 if the put expires at time 1, 0.928 if the put expires at time 3, and 0.96896 if the put expires at time 5.

Proof. Instead of numerical examples, we give a rigorous proof.

On the infinite coin-toss space, we define $\mathcal{M}_n = \{\text{stopping times that takes values } 0, 1, \cdots, n, \infty\}$ and $\mathcal{M}_\infty = \{\text{stopping times that takes values } 0, 1, 2, \cdots\}$. Then the time-zero value V^* of the perpetual American put as in Section 5.4 can be defined as

$$V^* = \sup_{\tau \in \mathcal{M}_\infty} \widetilde{\mathbb{E}}\left[1_{\{\tau<\infty\}} \frac{(K - S_\tau)^+}{(1+r)^\tau}\right].$$

For an American put with the same strike price K that expires at time n, its time-zero value $V^{(n)}$ is

$$V^{(n)} = \max_{\tau \in \mathcal{M}_n} \widetilde{\mathbb{E}}\left[1_{\{\tau<\infty\}} \frac{(K - S_\tau)^+}{(1+r)^\tau}\right].$$

Clearly $(V^{(n)})_{n\geq 0}$ is nondecreasing and $V^{(n)} \leq V^*$ for every n. So $\lim_n V^{(n)}$ exists and $\lim_n V^{(n)} \leq V^*$.

For any given $\tau \in \mathcal{M}_\infty$, we define $\tau^{(n)} = \begin{cases} \infty, & \text{if } \tau = \infty \\ \tau \wedge n, & \text{if } \tau < \infty \end{cases}$, then $\tau^{(n)}$ is also a stopping time, $\tau^{(n)} \in \mathcal{M}_n$ and $\lim_{n\to\infty} \tau^{(n)} = \tau$. By bounded convergence theorem,

$$\widetilde{\mathbb{E}}\left[1_{\{\tau<\infty\}} \frac{(K - S_\tau)^+}{(1+r)^\tau}\right] = \lim_{n\to\infty} \widetilde{\mathbb{E}}\left[1_{\{\tau^{(n)}<\infty\}} \frac{(K - S_{\tau^{(n)}})^+}{(1+r)^{\tau^{(n)}}}\right] \leq \lim_{n\to\infty} V^{(n)}.$$

Take sup at the left hand side of the inequality, we get $V^* \leq \lim_{n\to\infty} V^{(n)}$. Therefore $V^* = \lim_n V^{(n)}$. □

► **Exercise 5.7 (Hedging a short position in the perpetual American put).** Suppose you have sold the perpetual American put of Section 5.4 and are hedging the short position in this put. Suppose that at the current time the stock price is s and the value of your hedging portfolio is $v(s)$. Your hedge is to first consume the amount[8]

$$c(s) = v(s) - \frac{4}{5}\left[\frac{1}{2}v(2s) + \frac{1}{2}v\left(\frac{s}{2}\right)\right] \tag{5.7.3}$$

and then take a position

$$\delta(s) = \frac{v(2s) - v\left(\frac{s}{2}\right)}{2s - \frac{s}{2}} \tag{5.7.4}$$

in the stock. (See Theorem 4.2.2 of Chapter 4. The processes C_n and Δ_n in that theorem are obtained by replacing the dummy variable s by the stock price S_n in (5.7.3) and (5.7.4); i.e. $C_n = c(S_n)$ and $\Delta_n = \delta(S_n)$.) If you hedge this way, then regardless of whether the stock goes up or down on the next step, the value of your hedging portfolio should agree with the value of the perpetual American put.

(i) Compute $c(s)$ when $s = 2^j$ for the three cases $j \leq 0$, $j = 1$, and $j \geq 2$.

[8] The text missed a factor of $\frac{1}{2}$ in fron of $v\left(\frac{s}{2}\right)$.

Solution. By (5.4.6),

$$v(2^j) = \begin{cases} 4 - 2^j & \text{if } j \le 1 \\ \frac{4}{2^j} & \text{if } j \ge 1. \end{cases}$$

For $j \le 0$,

$$c(2^j) = v(2^j) - \frac{4}{5}\left[\frac{1}{2}v(2^{j+1}) + \frac{1}{2}v(2^{j-1})\right] = (4 - 2^j) - \frac{2}{5}(4 - 2^{j+1} + 4 - 2^{j-1}) = \frac{4}{5}.$$

For $j = 1$,

$$c(2) = v(2) - \frac{4}{5}\left[\frac{1}{2}v(4) + \frac{1}{2}v(1)\right] = 2 - \frac{2}{5}(1 + 4 - 1) = \frac{2}{5}.$$

For $j \ge 2$,

$$c(2^j) = v(2^j) - \frac{2}{5}\left[v(2^{j+1}) + v(2^{j-1})\right] = \frac{4}{2^j} - \frac{2}{5}\left(\frac{4}{2^{j+1}} + \frac{4}{2^{j-1}}\right) = 0.$$

□

(ii) Compute $\delta(s)$ when $s = 2^j$ for the three cases $j \le 0$, $j = 1$, and $j \ge 2$.

Solution. For $j \le 0$,

$$\delta(2^j) = \frac{v(2^{j+1}) - v(2^{j-1})}{2^{j+1} - 2^{j-1}} = \frac{(4 - 2^{j+1}) - (4 - 2^{j-1})}{2^{j+1} - 2^{j-1}} = -1.$$

For $j = 1$,

$$\delta(2) = \frac{v(4) - v(1)}{4 - 1} = \frac{1 - (4 - 1)}{4 - 1} = -\frac{2}{3}.$$

For $j \ge 2$,

$$\delta(2^j) = \frac{v(2^{j+1}) - v(2^{j-1})}{2^{j+1} - 2^{j-1}} = \frac{\frac{4}{2^{j+1}} - \frac{4}{2^{j-1}}}{2^{j+1} - 2^{j-1}} = -\frac{1}{4^{j-1}}.$$

□

(iii) Verify in each of the three cases $s = 2^j$ for $j \le 0$, $j = 1$, and $j \ge 2$ that the hedge works (i.e., regardless of whether the stock goes up or down, the value of your hedging portfolio at the next time is equal to the value of the perpetual American put at that time).

Proof. The computation is too tedious; skipped for this version. □

► **Exercise 5.8 (Perpetual American call).** Like the perpetual American put of Section 5.4, the perpetual American call has no expiration. Consider a binomial model with up factor u, down factor d, and interest rate r that satisfies the no-arbitrage condition $0 < d < 1 + r < u$. The risk-neutral probabilities are

$$\tilde{p} = \frac{1 + r - d}{u - d}, \; \tilde{q} = \frac{u - 1 - r}{u - d}.$$

The intrinsic value of the perpetual American call is $g(s) = s - K$, where $K > 0$ is the strike price. The purpose of this exercise is to show that the value of the call is always the price of the underlying stock, and there is no optimal exercise time.

(i) Let $v(s) = s$. Show that $v(S_n)$ is always at least as large as the intrinsic value $g(S_n)$ of the call and $\left(\frac{1}{1+r}\right)^n v(S_n)$ is a supermartingale under the risk-neutral probabilities. In fact, $\left(\frac{1}{1+r}\right)^n v(S_n)$ is a martingale.[9] These are the analogues of properties (i) and (ii) for the perpetual American put of Section 5.4.

[9] The textbook said "supermartingale" by mistake.

Proof. $v(S_n) = S_n \geq S_n - K = g(S_n)$. Under risk-neutral probabilities, $\frac{1}{(1+r)^n} v(S_n) = \frac{S_n}{(1+r)^n}$ is a martingale by Theorem 2.4.4. □

(ii) to show that $v(s) = s$ is not too large to be the value of the perpetual American call, we must find a good policy for the purchaser of the call. Show that if the purchaser of the call exercises at time n, regardless of the stock price at that time, then the discounted risk-neutral expectation of her payoff is $S_0 - \frac{K}{(1+r)^n}$. Because this is true for every n, and

$$\lim_{n\to\infty} \left[S_0 - \frac{K}{(1+r)^n} \right] = S_0,$$

the value of the call at time zero must be at least S_0. (The same is true at all other times; the value of the call is at least as great as the current stock price.)

Proof. If the purchaser chooses to exercises the call at time n, then the discounted risk-neutral expectation of her payoff is $\widetilde{\mathbb{E}}\left[\frac{S_n - K}{(1+r)^n}\right] = S_0 - \frac{K}{(1+r)^n}$. Since $\lim_{n\to\infty}\left[S_0 - \frac{K}{(1+r)^n}\right] = S_0$, the value of the call at time zero is at least $\sup_n \left[S_0 - \frac{K}{(1+r)^n}\right] = S_0$. □

(iii) In place of (i) and (ii) above, we could verify that $v(s) = s$ is the value of the perpetual American call by checking that this function satisfies the equation (5.4.16) and boundary conditions (5.4.18). Do this verification.

Proof. $\max\left\{g(s), \frac{\widetilde{p}v(us)+\widetilde{q}v(ds)}{1+r}\right\} = \max\{s-K, \frac{\widetilde{p}u+\widetilde{q}v}{1+r}s\} = \max\{s-K, s\} = s = v(s)$, so equation (5.4.16) is satisfied. Clearly $v(s) = s$ also satisfies the boundary condition (5.4.18). □

(iv) Show that there is no optimal time to exercise the perpetual American call.

Proof. Suppose τ is an optimal exercise time, then according to result in (ii),

$$\widetilde{\mathbb{E}}\left[\frac{S_\tau - K}{(1+r)^\tau} 1_{\{\tau<\infty\}}\right] \geq \sup_n \left[S_0 - \frac{K}{(1+r)^n}\right] = S_0,$$

which implies $\mathbb{P}(\tau < \infty) \neq 0$ and hence $\widetilde{\mathbb{E}}\left[\frac{K}{(1+r)^\tau} 1_{\{\tau<\infty\}}\right] > 0$. So $\widetilde{\mathbb{E}}\left[\frac{S_\tau - K}{(1+r)^\tau} 1_{\{\tau<\infty\}}\right] < \widetilde{\mathbb{E}}\left[\frac{S_\tau}{(1+r)^\tau} 1_{\{\tau<\infty\}}\right]$. Since $\left(\frac{S_n}{(1+r)^n}\right)_{n\geq 0}$ is a martingale under the risk-neutral measure, by Fatou's lemma,

$$\widetilde{\mathbb{E}}\left[\frac{S_\tau}{(1+r)^\tau} 1_{\{\tau<\infty\}}\right] \leq \liminf_{n\to\infty} \widetilde{\mathbb{E}}\left[\frac{S_{\tau\wedge n}}{(1+r)^{\tau\wedge n}} 1_{\{\tau<\infty\}}\right] \leq \liminf_{n\to\infty} \widetilde{\mathbb{E}}\left[\frac{S_{\tau\wedge n}}{(1+r)^{\tau\wedge n}}\right] = \liminf_{n\to\infty} \widetilde{\mathbb{E}}[S_0] = S_0.$$

Combined, we have

$$S_0 \leq \widetilde{\mathbb{E}}\left[\frac{S_\tau - K}{(1+r)^\tau} 1_{\{\tau<\infty\}}\right] < \widetilde{\mathbb{E}}\left[\frac{S_\tau}{(1+r)^\tau} 1_{\{\tau<\infty\}}\right] \leq S_0.$$

Contradiction. So there is no optimal time to exercise the perpetual American call. Simultaneously, we have shown $\widetilde{\mathbb{E}}\left[\frac{S_\tau - K}{(1+r)^\tau} 1_{\{\tau<\infty\}}\right] < S_0$ for any stopping time τ. Combined with (ii), we conclude S_0 is the least upper bound for all the prices acceptable to the buyer. □

▶ **Exercise 5.9.** (Provided by Irene Villegas.) Here is a method for solving equation (5.4.13) for the value of the perpetual American put in Section 5.4.

(i) We first determine $v(s)$ for large values of s. When s is large, it is not optimal to exercise the put, so the maximum in (5.4.13) will be given by the second term

$$\frac{4}{5}\left[\frac{1}{2}v(2s) + \frac{1}{2}v\left(\frac{s}{2}\right)\right] = \frac{2}{5}v(2s) + \frac{2}{5}v\left(\frac{s}{2}\right).$$

We thus seek solutions to the equation

$$v(s) = \frac{2}{5}v(2s) + \frac{2}{5}v\left(\frac{s}{2}\right). \tag{5.7.5}$$

All such solutions are of the form s^p for some constant p or linear combination of functions of this form. Substitute s^p into (5.7.5), obtain a quadratic equation for 2^p, and solve to obtain $2^p = 2$ or $2^p = \frac{1}{2}$. This leads to the values $p = 1$ and $p = -1$, i.e., $v_1(s) = s$ and $v_2(s) = \frac{1}{2}$ are soltuions to (5.7.5).

Proof. Suppose $v(s) = s^p$, then we have $s^p = \frac{2}{5}2^p s^p + \frac{2}{5}\frac{s^p}{2^p}$. So $1 = \frac{2^{p+1}}{5} + \frac{2^{1-p}}{5}$. Solve it for p, we get $p = 1$ or $p = -1$. □

(ii) The general solution to (5.7.5) is a linear combination of $v_1(s)$ and $v_2(s)$, i.e.,

$$v(s) = As + \frac{B}{s}. \tag{5.7.6}$$

For large values of s, the value of the perpetual American put must be given by (5.7.6). It remains to evaluate A and B. Using the second boundary condition in (5.4.15), show that A must be zero.

Proof. Since $\lim_{s\to\infty} v(s) = \lim_{s\to\infty}(As + \frac{B}{s}) = 0$, we must have $A = 0$. □

(iii) We have thus established that for large values of s, $v(s) = \frac{B}{s}$ for some constant B still to be determined. For small values of s, the value of the put is its intrinsic value $4 - s$. We must choose B so these two functions coincide at some point, i.e., we must find a value for B so that, for some $s > 0$,

$$f_B(s) = \frac{B}{s} - (4 - s)$$

equals zero. Show that, when $B > 4$, this function does not take the value 0 for any $s > 0$, but, when $B \le 4$, the equation $f_B(s) = 0$ has a solution.

Proof. $f_B(s) = 0$ if and only if $B + s^2 - 4s = 0$. The discriminant $\Delta = (-4)^2 - 4B = 4(4 - B)$. So for $B \le 4$, the equation has roots and for $B > 4$, this equation does not have roots. □

(iv) Let B be less than or equal to 4, and let s_B be a solution of the equation $f_B(s) = 0$. Suppose s_B is a stock price that can be attained in the model (i.e., $s_B = 2^j$ for some integer j). Suppose further that the owner of the perpetual American put exercises the first time the stock price is s_B or smaller. Then the discounted risk-neutral expected payoff of the put is $v_B(S_0)$, where $v_B(s)$ is given by the formula

$$v_B(s) = \begin{cases} 4 - s, & \text{if } s \le s_B, \\ \frac{B}{s}, & \text{if } s \ge s_B. \end{cases} \tag{5.7.7}$$

Which values of B and s_B give the owner the largest option value?

Proof. Suppose $B \le 4$, then the equation $s^2 - 4s + B = 0$ has solution $2 \pm \sqrt{4 - B}$. By drawing graphs of $4 - s$ and $\frac{B}{s}$, we can see the largest $v_B(s)$ is obtained by making $4 - s$ and $\frac{B}{s}$ tangent to each other, which implies

$$\begin{cases} \Delta = 4(4 - B) = 0 \\ 4 - s_B = \frac{B}{s_B}. \end{cases}$$

Solving it gives us $B = 4$ and $s_B = 2$. □

(v) For $s < s_B$, the derivative of $v_B(s)$ is $v'_B(s) = -1$. For $s > s_B$, this derivative is $v'_B(s) = -\frac{B}{s^2}$. Show that the best value of B for the option owner makes the derivative of $v_B(s)$ continuous at $s = S_B$ (i.e., the two formulas for $v'_B(s)$ give the same answer at $s = s_B$).

Proof. To have continuous derivative, we must have $-1 = -\frac{B}{s_B^2}$. Plug $B = s_B^2$ back into $4 - s_B = \frac{B}{s_B}$, we get $s_B = 2$. This gives $B = 4$. □

6 Interest-Rate-Dependent Assets

★ **Comments**:

1) In previous chapters, we started with a real probability measure and derived the risk-neutral measure based on the no-arbitrage argument. The concrete form of the risk-neutral measure is known in that setting. In the current chapter, models are built under the risk-neutral measure, whose existence is assumed *a priori* and whose concrete form (in terms of the real probability measure) is unknown. The justification for this is the following result by Dalang et al. [2]:

Theorem 2 (Dalang-Morton-Willinger, *Fundamental Theorem of Asset Pricing*)**.** *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0,\cdots,T}, \mathbb{P})$ be a (general) filtered probability space and let $S = \{S_t\}_{t\in\{0,\cdots,T\}}$ be an adapted, $\mathbb{R}^d$-valued stochastic process describing the discounted prices of $d \in \mathbb{N}$ financial assets. Then the following properties are equivalent:*
(a) The financial market model is free of arbitrage.
(b) There exists a probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that
- *$\mathbb{P}^*$ is equivalent to $\mathbb{P}$ and the Radon-Nikodým density $\varrho := d\mathbb{P}^*/d\mathbb{P}$ is bounded, i.e. in $L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$,*
- *Integrability: $S_t \in L^1(\Omega, \mathcal{F}_t, \mathbb{P}^*)$ for all $t \in \{0, \cdots, T\}$,*
- *Martingale property w.r.t. $\mathbb{P}$:*

$$\mathbb{E}_{\mathbb{P}^*}[S_t|\mathcal{F}_{t-1}] \overset{a.s.}{=} S_{t-1} \text{ for all } t \in \{0, \cdots, T\}.$$

According to the *Fundamental Theorem of Asset Pricing* (FTAP), the no-arbitrage property is associated with the existence of a probability measure called *equivalent martingale measure* (EMM) and a positive process called *numéraire*, such that the price processes of tradable assets discounted by the numéraire are martingales under the equivalent martingale measure. In the current chapter, the numéraire is the value process of the money market account

$$M_n = (1 + R_0) \cdots (1 + R_{n-1}),\ n = 1, 2, \cdots, N; M_0 = 1.$$

And the risk-neutral measure is the EMM associated with this numéraire. This justifies Definition 6.2.4 (zero-coupon bond prices) since

$$D_n B_{n,m} = \widetilde{\mathbb{E}}_n [D_m B_{m,m}] \Rightarrow B_{n,m} = \widetilde{\mathbb{E}}_n \left[\frac{D_m \cdot 1}{D_n}\right].$$

2) The wealth equation (6.2.6) essentially assumes all coupons and other payouts of cash are reinvested in the portfolio.

3) To see the intuition of the forward interest rate at time n for investing at time m (Definition 6.3.4), note at time n, we can invest \$1 to buy $\frac{1}{B_{n,m}}$ shares of zero-coupon bond maturing at time m; at time m, we reinvest the payoff $\frac{1}{B_{n,m}}$ at rate $F_{n,m}$ to get paid at time $(m + 1)$. To eliminate arbitrage, this investing strategy should have the same return as investing \$1 at time n in the zero-coupon bond maturing at time $(m + 1)$:

$$\frac{1}{B_{n,m}}(1 + F_{n,m}) = \frac{1}{B_{n,m+1}}.$$

This gives formula (6.3.3)

$$F_{n,m} = \frac{B_{n,m}}{B_{n,m+1}} - 1 = \frac{B_{n,m} - B_{n,m+1}}{B_{n,m+1}}.$$

4) Whenever there is a hedge, static or not, the discounted value of the hedging portfolio is a martingale under the risk-neutral measure, and the risk-neutral pricing formula applies. This is regardless of whether or not the market is complete.

i) *Forward contract.* a) Hedging of a short position: at time n, short $\frac{S_n}{B_{n,m}}$ zero-coupon bond maturing at time m and long 1 share of asset at price of S_n, hold the portfolio until maturity time m. b) Risk-neutral pricing:

$$\widetilde{\mathbb{E}}_n [D_m(K - S_m)] = K D_n B_{n,m} - D_n S_n = D_n V_n,$$

where V_n is the contract value at time n. By setting $V_n = 0$, we obtain m-forward price of the asset at time n (Theorem 6.3.2).

ii) *Forward rate agreement* (FRA).[10] a) Hedging of a short position: the portfolio whose value is R_m at time $(m+1)$ can be replicated by a cash inflow of \$1 at time m and a cash outflow of \$1 at time $(m+1)$, since the inflow of \$1 at time m can be invested in the money market account to return $(1+R_m)$ at time $(m+1)$. The cash inflow of \$1 at time m can be obtained by longing at time n a zero-coupon bond maturing at time m, while the cash outflow of \$1 at time $(m+1)$ can be obtained by shorting at time n a zero-coupon bond maturing at time $(m+1)$. So the value of the portfolio at time n is

$$V_n = B_{n,m} - B_{n,m+1}.$$

b) Risk-neutral pricing (Exercise 6.3):

$$V_n = \widetilde{\mathbb{E}}_n\left[\frac{D_{m+1}R_m}{D_n}\right] = \widetilde{\mathbb{E}}_n\left[\frac{D_m - D_{m+1}}{D_n}\right] = B_{n,m} - B_{n,m+1}.$$

This is Theorem 6.3.5.

iii) *Interest rate swap.* A long swap position ("receive fixed") is a long position in a coupon bond plus a short position in a series of FRA's. Therefore, an interest rate swap can be hedged and risk-neutral pricing formula applies. This is Theorem 6.3.7. More specifically,

$$Swap_m = K\sum_{n=1}^{m} + B_{0,m} - 1$$

suggests a long position in a bond with coupon K and maturity m and a short position of \$1 in the money market account. A short position of \$1 in the money market account at the beginning of each period will result in $-(1+R_n)$ at the end of the period.

iv) *Interest rate caps and floors.* Suppose the replication uses the money market account and zero-coupon bonds with various maturities. The wealth equation satisfies

$$X_{n+1} = (1+R_n)(X_n - \Delta_n B_{n,m}) + \Delta_n B_{n+1,m}.$$

To successfully replicate V_{n+1}, it's necessary and sufficient that $B_{n+1,m}(H) \neq B_{n+1,m}(T)$. See Exercise 6.4 for illustration.

5) According to the *Fundamental Theorem of Asset Pricing* (FTAP), the no-arbitrage property is associated with the existence of a probability measure called *equivalent martingale measure* (EMM) and a positive process called *numéraire*, such that the price processes of tradable assets discounted by the numéraire are martingales under the given probability measure (hence the name *martingale measure*).

The *m-forward measure* is the EMM associated with the zero-coupon bond $(B_{n,m})_{n=0}^m$ as the numéraire. For a European contingent claim with maturity m, we have the risk-neutral pricing formula (note $B_{m,m} \equiv 1$)

$$V_n = B_{n,m}\widetilde{\mathbb{E}}_n^m[V_m] = \frac{1}{D_n}\widetilde{\mathbb{E}}_n[D_m V_m].$$

If we denote the Radon-Nikodým derivative process by $(Z_{n,m})_{n=0}^m$, that is,

$$Z_{m,m} := \frac{d\widetilde{\mathbb{P}}^m}{d\widetilde{\mathbb{P}}},\ Z_{n,m} := \widetilde{\mathbb{E}}_n[Z_{m,m}],$$

by setting $n = 0$ in the risk-neutral pricing formula, we have $Z_{m,m} = \frac{D_m}{B_{0,m}}$ and hence

$$Z_{n,m} = \frac{\widetilde{\mathbb{E}}_n[D_m]}{B_{0,m}} = \frac{D_n B_{n,m}}{B_{0,m}},\ n = 0, \cdots, m.$$

[10] A FRA is a contract involving three time instants: the current time t, the expiry time $T > t$, and the maturity time $S > T$. The contract gives its holder an interest-rate payment for the period between T and S. See Brigo and Mercurio [1, page 11] for a general definition.

This gives an alternative presentation of the m-forward measure $\widetilde{\mathbb{P}}^m$. A notable feature of working under this measure is that any asset price process discounted by the zero-coupon bond $(B_{n,m})_{n=0}^m$ is exactly its m-forward price process (Theorem 6.3.2).

► **Exercise 6.1.** Prove Theorem 2.3.2 when conditional expectation is defined by Definition 6.2.2.

Proof. The proof is tedious and is not a lot simpler than the one using the most general definition of conditional expectation (see, for example, Shiryaev [5]); skipped for this version. □

► **Exercise 6.2.** Verify that the discounted value of the static hedging portfolio constructed in the proof of Theorem 6.3.2 is a martingale under $\widetilde{\mathbb{P}}$.

Proof. The static hedging portfolio consists of shorting one forward contract, shorting $\frac{S_n}{B_{n,m}}$ zero-coupon bonds maturing at time m, and longing one share of the asset. So for any $k = n, \cdots, m$, the value of the portfolio at time k is

$$X_k = S_k - \widetilde{\mathbb{E}}_k[D_m(S_m - K)]D_k^{-1} - \frac{S_n}{B_{n,m}} B_{k,m}.$$

Then

$$\begin{aligned}
\widetilde{\mathbb{E}}_{k-1}[D_k X_k] &= \widetilde{\mathbb{E}}_{k-1}\left[D_k S_k - \widetilde{\mathbb{E}}_k[D_m(S_m - K)] - \frac{S_n}{B_{n,m}} B_{k,m} D_k\right] \\
&= D_{k-1}S_{k-1} - \widetilde{\mathbb{E}}_{k-1}[D_m(S_m - K)] - \frac{S_n}{B_{n,m}} \widetilde{\mathbb{E}}_{k-1}\left[\widetilde{\mathbb{E}}_k[D_m \cdot 1]\right] \\
&= D_{k-1}\left\{S_{k-1} - \widetilde{\mathbb{E}}_{k-1}[D_m(S_m - K)]D_{k-1}^{-1} - \frac{S_n}{B_{n,m}} B_{k-1,m}\right\} \\
&= D_{k-1}X_{k-1}.
\end{aligned}$$

This shows the discounted value of the static hedging portfolio constructed in the proof of Theorem 6.3.2 is a martingale under $\widetilde{\mathbb{P}}$. □

► **Exercise 6.3.** Let $0 \leq n \leq m \leq N-1$ be given. According to the risk-neutral pricing formula, the contract that pays R_m at time $m+1$ has time-n price $\frac{1}{D_n}\widetilde{\mathbb{E}}_n[D_{m+1}R_m]$. Use the properties of conditional expectations to show that this gives the same result as Theorem 6.3.5, i.e.,

$$\frac{1}{D_n}\widetilde{\mathbb{E}}_n[D_{m+1}R_m] = B_{n,m} - B_{n,m+1}.$$

Proof.

$$\frac{1}{D_n}\widetilde{\mathbb{E}}_n[D_{m+1}R_m] = \frac{1}{D_n}\widetilde{\mathbb{E}}_n[D_m(1+R_m)^{-1}R_m] = \frac{1}{D_n}\widetilde{\mathbb{E}}_n\left[D_m - D_{m+1}\right] = B_{n,m} - B_{n,m+1}.$$

□

► **Exercise 6.4 (Short position hedge for caplets).** Using the data in Example 6.3.9, this exercise constructs a hedge for a short position in the caplet paying $\left(R_2 - \frac{1}{3}\right)^+$ at time three. We observe from the second table in Example 6.3.9 that the payoff at time three of this caplet is

$$V_3(HH) = \frac{2}{3},\ V_3(HT) = V_3(TH) = V_3(TT) = 0.$$

Since this payoff depends on only the first two coin tosses, the price of the caplet at time two can be determined by discounting:

$$V_2(HH) = \frac{1}{1+R_2(HH)} V_3(HH) = \frac{1}{3},\ V_2(HT) = V_2(TH) = V_2(TT) = 0.$$

Indeed, if one is hedging a short position in the caplet and has a portfolio valued at $\frac{1}{3}$ at time two in the event $\omega_1 = H$, $\omega_2 = H$, then one can simply invest this $\frac{1}{3}$ in the money market in order to have the $\frac{2}{3}$ required to pay off the caplet at time three.

In Example 6.3.9, the time-zero price of the caplet is determined to be $\frac{2}{21}$ (see (6.3.10)).

(i) Determine $V_1(H)$ and $V_1(T)$, the price at time one of the caplet in the events $\omega_1 = H$ and $\omega_1 = T$, respectively.

Solution. $D_1 V_1 = \widetilde{\mathbb{E}}_1[D_2 V_2] = D_2 \widetilde{\mathbb{E}}_1[V_2]$. So $V_1 = \frac{D_2}{D_1}\widetilde{\mathbb{E}}_1[V_2] = \frac{1}{1+R_1}\widetilde{\mathbb{E}}_1[V_2]$. In particular,

$$V_1(H) = \frac{V_2(HH)P(\omega_2 = H|\omega_1 = H) + V_2(HT)P(\omega_2 = T|\omega_1 = H)}{1 + R_1(H)} = \frac{6}{7}\left(\frac{1}{3}\cdot\frac{2}{3} + 0\cdot\frac{1}{3}\right) = \frac{4}{21},$$

$$V_1(T) = \frac{V_2(TH)P(\omega_2 = H|\omega_1 = T) + V_2(TT)P(\omega_2 = T|\omega_1 = T)}{1 + R_1(T)} = 0.$$

□

(ii) Show how to begin with $\frac{2}{21}$ at time zero and invest in the money market and the maturity two bond in order to have a portfolio value X_1 at time one that agrees with V_1, regardless of the outcome of the first coin toss. Why do we invest in the maturity two bond rather than the maturity three bond to do this?

Proof. Let $X_0 = \frac{2}{21}$. Suppose we buy Δ_0 shares of the maturity two bond, then at time one, the value of our portfolio is $X_1 = (1 + R_0)(X_0 - \Delta_0 B_{0,2}) + \Delta_0 B_{1,2}$. To replicate the value V_1, we must have

$$\begin{cases} V_1(H) = (1 + R_0)(X_0 - \Delta_0 B_{0,2}) + \Delta_0 B_{1,2}(H) \\ V_1(T) = (1 + R_0)(X_0 - \Delta_0 B_{0,2}) + \Delta_0 B_{1,2}(T). \end{cases}$$

So

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{B_{1,2}(H) - B_{1,2}(T)} = \frac{\frac{4}{21} - 0}{\frac{6}{7} - \frac{5}{7}} = \frac{4}{3}.$$

The hedging strategy is therefore to borrow $\frac{4}{3}B_{0,2} - \frac{2}{21} = \frac{20}{21}$ and buy $\frac{4}{3}$ share of the maturity two bond. The reason why we do not invest in the maturity three bond is that $B_{1,3}(H) = B_{1,3}(T)(= \frac{4}{7})$ and the portfolio will therefore have the same value at time one regardless the outcome of first coin toss. This makes impossible the replication of V_1, since $V_1(H) \neq V_1(T)$. □

(iii) Show how to take the portfolio value X_1 at time one and invest in the money market and the maturity three bond in order to have a portfolio value X_2 at time two that agrees with V_2, regardless of the outcome of the first two coin tosses. Why do we invest in the maturity three bond rather than the maturity two bond to do this?

Proof. Suppose we buy Δ_1 share of the maturity three bond at time one, then to replicate V_2 at time two, we must have $V_2 = (1 + R_1)(X_1 - \Delta_1 B_{1,3}) + \Delta_1 B_{2,3}$. So

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{B_{2,3}(HH) - B_{2,3}(HT)} = -\frac{2}{3},\ \Delta_1(T) = \frac{V_2(TH) - V_2(TT)}{B_{2,3}(TH) - B_{2,3}(TT)} = 0.$$

The hedging strategy is as follows. If the outcome of first coin toss is T, then we keep everything in the money market account. If the outcome of first coin toss is H, we short $\frac{2}{3}$ shares of the maturity three bond and invest the income into the money market account. We do not invest in the maturity two bond, because at time two, the value of the bond is its face value and our portfolio will therefore have the same value regardless outcomes of coin tosses. This makes impossible the replication of V_2. □

► **Exercise 6.5.** Let m be given with $0 \leq m \leq N - 1$, and consider the forward interest rate

$$F_{n,m} = \frac{B_{n,m} - B_{n,m+1}}{B_{n,m+1}},\ n = 0, 1, \cdots, m.$$

(i) Use (6.4.8) and (6.2.5) to show that $F_{n,m}$, $n = 0, 1, \cdots, m$, is a martingale under the $(m + 1)$-forward measure $\widetilde{\mathbb{P}}^{m+1}$.

Proof. Before we start the computation, note $F_{n,m} = \frac{B_{n,m}-B_{n,m+1}}{B_{n,m+1}}$ is a tradable asset discounted by $(B_{n,m+1})_n$, so it must be a martingale under the equivalent martingale measure $\widetilde{\mathbb{P}}^{m+1}$ associated with the numéraire $(B_{n,m+1})_n$. To verify this, suppose $1 \le n \le m$, then

$$\begin{aligned}
\widetilde{\mathbb{E}}^{m+1}_{n-1}[F_{n,m}] &= \widetilde{\mathbb{E}}_{n-1}[B^{-1}_{n,m+1}(B_{n,m} - B_{n,m+1})Z_{n,m+1}Z^{-1}_{n-1,m+1}] \\
&= \widetilde{\mathbb{E}}_{n-1}\left[\left(\frac{B_{n,m}}{B_{n,m+1}} - 1\right)\frac{B_{n,m+1}D_n}{B_{n-1,m+1}D_{n-1}}\right] \\
&= \frac{D_n}{B_{n-1,m+1}D_{n-1}}\widetilde{\mathbb{E}}_{n-1}[B_{n,m} - B_{n,m+1}] \\
&= \frac{D_n}{B_{n-1,m+1}D_{n-1}}\widetilde{\mathbb{E}}_{n-1}[D_n^{-1}\widetilde{\mathbb{E}}_n[D_m] - D_n^{-1}\widetilde{\mathbb{E}}_n[D_{m+1}]] \\
&= \frac{\widetilde{\mathbb{E}}_{n-1}[D_m - D_{m+1}]}{B_{n-1,m+1}D_{n-1}} \\
&= \frac{B_{n-1,m} - B_{n-1,m+1}}{B_{n-1,m+1}} \\
&= F_{n-1,m}.
\end{aligned}$$

□

(ii) Compute $F_{0,2}$, $F_{1,2}(H)$, and $F_{1,2}(T)$ in Example 6.4.4 and verify the martingale property

$$\widetilde{\mathbb{E}}^3[F_{1,2}] = F_{0,2}.$$

Solution.

$$\begin{aligned}
F_{0,2} &= \frac{B_{0,2}}{B_{0,3}} - 1 = \frac{0.9071}{0.8639} - 1 = 0.05 \\
F_{1,2}(H) &= \frac{B_{1,2}(H)}{B_{1,3}(H)} - 1 = \frac{0.9479}{0.8985} - 1 = 0.055 \\
F_{1,2}(T) &= \frac{B_{1,2}(T)}{B_{1,3}(T)} - 1 = \frac{0.9569}{0.9158} - 1 = 0.045
\end{aligned}$$

Therefore

$$\widetilde{\mathbb{E}}^3[F_{1,2}] = \widetilde{\mathbb{P}}^3(H)F_{1,2}(H) + \widetilde{\mathbb{P}}^3(T)F_{1,2}(T) = 0.4952 \cdot 0.055 + 0.5048 \cdot 0.045 = 0.05 = F_{0,2}.$$

□

► **Exercise 6.6.** Let S_m be the price at time m of an asset in a binomial interest rate model. For $n = 0, 1, \cdots, m$, the forward price is $\text{For}_{n,m} = \frac{S_n}{B_{n,m}}$ and the futures price is $\text{Fut}_{n,m} = \widetilde{\mathbb{E}}_n[S_m]$.[11]

(i) Suppose that at each time n an agent takes a long forward position and sells this contract at time $n+1$. Show that this generates cash flow $S_{n+1} - \frac{S_n B_{n+1,m}}{B_{n,m}}$ at time $n+1$.

Proof. At time $n+1$, the forward contract has a value of

$$\frac{1}{D_{n+1}}\widetilde{\mathbb{E}}_{n+1}\left[D_m(S_m - \text{Fut}_{n,m})\right] = S_{n+1} - \text{Fut}_{n,m}B_{n+1,m} = S_{n+1} - \frac{S_n B_{n+1,m}}{B_{n,m}}.$$

So if the agent takes a long forward position at time n for no cost and sells this contract at time $n+1$, he will receive a cash flow of $S_{n+1} - \frac{S_n B_{n+1,m}}{B_{n,m}}$. □

[11] The textbook said "S_n" by mistake.

(ii) Show that if the interest rate is a constant r and at each time n an agent takes a long position of $(1+r)^{m-n-1}$ forward contracts, selling these contracts at time $n+1$, then the resulting cash flow is the same as the difference in the futures price $\text{Fut}_{n+1,m} - \text{Fut}_{n,m}$.

Proof. By (i), the cash flow generated at time $n+1$ is

$$\begin{aligned}
& (1+r)^{m-n-1}\left(S_{n+1} - \frac{S_n B_{n+1,m}}{B_{n,m}}\right) \\
= \; & (1+r)^{m-n-1}\left(S_{n+1} - \frac{\frac{S_n}{(1+r)^{m-n-1}}}{\frac{1}{(1+r)^{m-n}}}\right) \\
= \; & (1+r)^m \frac{S_{n+1}}{(1+r)^{n+1}} - (1+r)^m \frac{S_n}{(1+r)^n} \\
= \; & (1+r)^m \widetilde{\mathbb{E}}_{n+1}\left[\frac{S_m}{(1+r)^m}\right] - (1+r)^m \widetilde{\mathbb{E}}_n\left[\frac{S_m}{(1+r)^m}\right] \\
= \; & \text{Fut}_{n+1,m} - \text{Fut}_{n,m}.
\end{aligned}$$

□

► **Exercise 6.7.** Consider a binomial interest rate model in which the interest rate at time n depends on only the number of heads in the first n coin tosses. In other words, for each n there is a function $r_n(k)$ such that

$$R_n(\omega_1, \cdots, \omega_n) = r_n(\#H(\omega_1, \cdots, \omega_n)).$$

Assume the risk-neutral probabilities are $\tilde{p} = \tilde{q} = \frac{1}{2}$. The Ho-Lee model (Example 6.4.4) and the Black-Derman-Toy model (Example 6.5.5) satisfy these conditions.

Consider a derivative security that pays 1 at time n if and only if there are k heads in the first n tosses; i.e., the payoff is $V_n(k) = 1_{\{\#H(\omega_1,\cdots,\omega_n)=k\}}$. Define $\psi_0(0) = 1$ and , for $n = 1, 2, \cdots$, define

$$\psi_n(k) = \widetilde{\mathbb{E}}\left[D_n V_n(k)\right], \; k = 0, 1, \cdots, n,$$

to be the price of this security at time zero. Show that the functions $\psi_n(k)$ can be computed by the recursion

$$\begin{aligned}
\psi_{n+1}(0) &= \frac{\psi_n(0)}{2(1+r_n(0))} \\
\psi_{n+1}(k) &= \frac{\psi_n(k-1)}{2(1+r_n(k-1))} + \frac{\psi_n(k)}{2(1+r_n(k))}, \; k = 1, \cdots, n, \\
\psi_{n+1}(n+1) &= \frac{\psi_n(n)}{2(1+r_n(n))}.
\end{aligned}$$

Proof.

$$\begin{aligned}
\psi_{n+1}(0) &= \widetilde{\mathbb{E}}\left[D_{n+1} V_{n+1}(0)\right] \\
&= \widetilde{\mathbb{E}}\left[\frac{D_n}{1+r_n(0)} 1_{\{\#H(\omega_1 \cdots \omega_{n+1})=0\}}\right] \\
&= \widetilde{\mathbb{E}}\left[\frac{D_n}{1+r_n(0)} 1_{\{\#H(\omega_1 \cdots \omega_n)=0\}} 1_{\{\omega_{n+1}=T\}}\right] \\
&= \frac{1}{2(1+r_n(0))} \widetilde{\mathbb{E}}\left[D_n 1_{\{\#H(\omega_1 \cdots \omega_n)=0\}}\right] \\
&= \frac{\psi_n(0)}{2(1+r_n(0))}.
\end{aligned}$$

For $k = 1, 2, \cdots, n$,

$$\begin{aligned}
\psi_{n+1}(k) &= \widetilde{\mathbb{E}}\left[\frac{D_n}{1+r_n(\#H(\omega_1\cdots\omega_n))}1_{\{\#H(\omega_1\cdots\omega_{n+1})=k\}}\right] \\
&= \widetilde{\mathbb{E}}\left[\frac{D_n}{1+r_n(k)}1_{\{\#H(\omega_1\cdots\omega_n)=k\}}1_{\{\omega_{n+1}=T\}}\right] + \widetilde{\mathbb{E}}\left[\frac{D_n}{1+r_n(k-1)}1_{\{\#H(\omega_1\cdots\omega_n)=k-1\}}1_{\{\omega_{n+1}=H\}}\right] \\
&= \frac{1}{2}\frac{\widetilde{\mathbb{E}}[D_nV_n(k)]}{1+r_n(k)} + \frac{1}{2}\frac{\widetilde{\mathbb{E}}[D_nV_n(k-1)]}{1+r_n(k-1)} \\
&= \frac{\psi_n(k)}{2(1+r_n(k))} + \frac{\psi_n(k-1)}{2(1+r_n(k-1))}.
\end{aligned}$$

Finally,

$$\psi_{n+1}(n+1) = \widetilde{\mathbb{E}}[D_{n+1}V_{n+1}(n+1)] = \widetilde{\mathbb{E}}\left[\frac{D_n}{1+r_n(n)}1_{\{\#H(\omega_1\cdots\omega_n)=n\}}1_{\{\omega_{n+1}=H\}}\right] = \frac{\psi_n(n)}{2(1+r_n(n))}.$$

□

Remark 6.1. *In the above proof, we have used the independence of ω_{n+1} and $(\omega_1, \cdots, \omega_n)$ under $\widetilde{\mathbb{P}}$. This is guaranteed by the assumption that $\widetilde{\mathbb{P}}(\omega_{n+1} = H|\omega_1, \cdots, \omega_n) = \tilde{p}$ and $\widetilde{\mathbb{P}}(\omega_{n+1} = T|\omega_1, \cdots, \omega_n) = \tilde{q}$, which are deterministic. In case the binomial model has stochastic up- and down-factor u_n and d_n, we have $\widetilde{\mathbb{P}}(\omega_{n+1} = H|\omega_1, \cdots, \omega_n) = \tilde{p}_n$ and $\widetilde{\mathbb{P}}(\omega_{n+1} = T|\omega_1, \cdots, \omega_n) = \tilde{q}_n$, where $\tilde{p}_n = \frac{1+r_n-d_n}{u_n-d_n}$ and $\tilde{q}_n = \frac{u_n-1-r_n}{u_n-d_n}$. Then for any $X \in \mathcal{F}_n = \sigma(\omega_1, \cdots, \omega_n)$, we have $\widetilde{\mathbb{E}}[Xf(\omega_{n+1})] = \widetilde{\mathbb{E}}[X\widetilde{\mathbb{E}}[f(\omega_{n+1})|\mathcal{F}_n]] = \widetilde{\mathbb{E}}[X(\tilde{p}_nf(H) + \tilde{q}_nf(T))]$.*

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