Analysis on Manifolds
Solution of Exercise Problems

Yan Zeng

Version 0.1.1, last revised on 2014-03-25.

Abstract

This is a solution manual of selected exercise problems from *Analysis on manifolds*, by James R. Munkres [1]. If you find any typos/errors, please email me at zypublic@hotmail.com.

Contents

1 Review of Linear Algebra

A good textbook on linear algebra from the viewpoint of finite-dimensional spaces is Lax [2]. In the below, we make connections between the results presented in the current section and that reference.

Theorem 1.1 (page 2) corresponds to Lax [2, page 5], Chapter 1, Lemma 1.

Theorem 1.2 (page 3) corresponds to Lax [2, page 6], Chapter 1, Theorem 4.

Theorem 1.5 (page 7) corresponds to Lax [2, page 37], Chapter 4, Theorem 2 and the paragraph below Theorem 2.

2. (Theorem 1.3, page 5) If A is an n by m matrix and B is an m by p matrix, show that

$$|A \cdot B| \leq m|A||B|.$$

Proof. For any $i = 1, \cdots n, j = 1, \cdots, p$, we have

$$\left|\sum_{k=1}^{m} a_{ik} b_{kj}\right| \leq \sum_{k=1}^{m} |a_{ik} b_{kj}| \leq |A| \sum_{k=1}^{m} |b_{kj}| \leq m|A||B|.$$

Therefore,

$$|A \cdot B| = \max \left\{ \left|\sum_{k=1}^{m} a_{ik} b_{kj}\right|; i = 1, \cdots n, j = 1, \cdots, p \right\} \leq m|A||B|.$$

□

3. Show that the sup norm on $\mathbb{R}^2$ is not derived from an inner product on $\mathbb{R}^2$. [*Hint*: Suppose $\langle x, y\rangle$ is an inner product on $\mathbb{R}^2$ (*not* the dot product) having the property that $|x| = \langle x, x\rangle^{1/2}$. Compute $\langle x \pm y, x \pm y\rangle$ and apply to the case $x = e_1$ and $y = e_2$.]

Proof. Suppose $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^2$ having the property that $|x| = \langle x, x\rangle^{\frac{1}{2}}$, where $|x|$ is the sup norm. By the equality $\langle x, y\rangle = \frac{1}{4}(|x+y|^2 - |x-y|^2)$, we have

$$\begin{aligned}
\langle e_1, e_1 + e_2\rangle &= \frac{1}{4}(|2e_1 + e_2|^2 - |e_2|^2) = \frac{1}{4}(4-1) = \frac{3}{4}, \\
\langle e_1, e_2\rangle &= \frac{1}{4}(|e_1 + e_2|^2 - |e_1 - e_2|^2) = \frac{1}{4}(1-1) = 0, \\
\langle e_1, e_1\rangle &= |e_1|^2 = 1.
\end{aligned}$$

So $\langle e_1, e_1 + e_2\rangle \neq \langle e_1, e_2\rangle + \langle e_1, e_1\rangle$, which implies $\langle \cdot, \cdot \rangle$ cannot be an inner product. Therefore, our assumption is not true and the sup norm on $\mathbb{R}^2$ is not derived from an inner product on $\mathbb{R}^2$. □

2 Matrix Inversion and Determinants

1. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$$

(a) Find two different left inverse for A.

(b) Show that A has no right inverse.

(a)

Proof. $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$. Then $BA = \begin{pmatrix} b_{11}+b_{12} & 2b_{11}-b_{12}+b_{13} \\ b_{21}+b_{22} & 2b_{21}-b_{12}+b_{23} \end{pmatrix}$. So $BA = I_2$ if and only if

$$\begin{cases} b_{11}+b_{12}=1 \\ b_{21}+b_{22}=0 \\ 2b_{11}-b_{12}+b_{13}=0 \\ 2b_{21}-b_{22}+b_{23}=1. \end{cases}$$

Plug $-b_{12} = b_{11}-1$ and $-b_{22} = b_{21}$ into the las two equations, we have

$$\begin{cases} 3b_{11}+b_{13}=1 \\ 3b_{21}+b_{23}=1. \end{cases}$$

So we can have the following two different left inverses for A: $B_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 1 & 0 & -2 \\ 1 & -1 & -2 \end{pmatrix}$. □

(b)

Proof. By Theorem 2.2, A has no right inverse. □

2.

Proof. (a) By Theorem 1.5, $n \geq m$ and among the n row vectors of A, there are exactly m of them are linearly independent. By applying elementary row operations to A, we can reduce A to the echelon form $\begin{bmatrix} I_m \\ 0 \end{bmatrix}$. So we can find a matrix D that is a product of elementary matrices such that $DA = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$.

(b) If rank$A = m$, by part (a) there exists a matrix D that is a product of elementary matrices such that

$$DA = \begin{bmatrix} I_m \\ 0 \end{bmatrix}.$$

Let $B = [I_m, 0]D$, then $BA = I_m$, i.e. B is a left inverse of A. Conversely, if B is a left inverse of A, it is easy to see that A as a linear mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$ is injective. This implies the column vectors of A are linearly independent, i.e. rank$A = m$.

(c) A has a right inverse if and only if A^{tr} has a left inverse. By part (b), this implies rank$A = $ rank$A^{tr} = n$. □

4.

Proof. Suppose $(D_k)_{k=1}^K$ is a sequence of elementary matrices such that $D_K \cdots D_2 D_1 A = I_n$. Note $D_K \cdots D_2 D_1 A = D_K \cdots D_2 D_1 I_n A$, we can conclude $A^{-1} = D_K \cdots D_2 D_1 I_n$. □

5.

Proof. $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \frac{1}{d-bc}$ by Theorem 2.14. □

3 Review of Topology in $\mathbb{R}^n$

2.

Proof. $X = \mathbb{R}$, $Y = (0, 1]$, and $A = Y$. □

3.

Proof. For any closed subset C of Y, $f^{-1}(C) = [f^{-1}(C) \cap A] \cup [f^{-1}(C) \cap B]$. Since $f^{-1}(C) \cap A$ is a closed subset of A, there must be a closed subset D_1 of X such that $f^{-1}(C) \cap A = D_1 \cap A$. Similarly, there is a closed subset D_2 of X such that $f^{-1}(C) \cap B = D_2 \cap B$. So $f^{-1}(C) = [D_1 \cap A] \cup [D_2 \cap B]$. A and B are closed in X, so $D_1 \cap A$, $D_2 \cap B$ and $[D_1 \cap A] \cup [D_2 \cap B]$ are all closed in X. This shows f is continuous. □

7.

Proof. (a) Take $f(x) \equiv y_0$ and let g be such that $g(y_0) \neq z_0$ but $g(y) \to z_0$ as $y \to y_0$. □

4 Compact Subspaces and Connected Subspace of $\mathbb{R}^n$

1.

Proof. (a) Let $x_n = (2n\pi + \frac{\pi}{2})^{-1}$ and $y_n = (2n\pi - \frac{\pi}{2})^{-1}$. Then as $n \to \infty$, $|x_n - y_n| \to 0$ but $|\sin\frac{1}{x_n} - \sin\frac{1}{y_n}| = 2$. □

3.

Proof. The boundedness of X is clear. Since for any $i \neq j$, $\|e_i - e_j\| = 1$, the sequence $(e_i)_{i=1}^{\infty}$ has no accumulation point. So X cannot be compact. Also, the fact $\|e_i - e_j\| = 1$ for $i \neq j$ shows each e_i is an isolated point of X. Therefore X is closed. Combined, we conclude X is closed, bounded, and non-compact. □

5 The Derivative

1.

Proof. By definition, $\lim_{t\to 0}\frac{f(a+tu)-f(a)}{t}$ exists. Consequently, $\lim_{t\to 0}\frac{f(a+tu)-f(a)}{t} = \lim_{t\to 0}\frac{f(a+tcu)-f(a)}{ct}$ exists and is equal to $cf'(a;u)$. □

2.

Proof. (a) $f(u) = f(u_1, u_2) = \frac{u_1 u_2}{u_1^2 + u_2^2}$. So

$$\frac{f(tu) - f(0)}{t} = \frac{1}{t}\frac{t^2 u_1 u_2}{t^2(u_1^2 + u_2^2)} = \frac{1}{t}\frac{u_1 u_2}{u_1^2 + u_2^2}.$$

In order for $\lim_{t\to 0}\frac{f(tu)-f(0)}{t}$ to exist, it is necessary and sufficient that $u_1 u_2 = 0$ and $u_1^2 + u_2^2 \neq 0$. So for vectors $(1,0)$ and $(0,1)$, $f'(0;u)$ exists, and we have $f'(0;(1,0)) = f'(0;(0,1)) = 0$.

(b) Yes, $D_1 f(0) = D_2 f(0) = 0$.

(c) No, because f is not continuous at 0: $\lim_{(x,y)\to 0, y=kx} f(x,y) = \frac{kx^2}{x^2+k^2x^2} = \frac{k}{1+k^2}$. For $k \neq 0$, the limit is not equal to $f(0)$.

(d) See (c). □

6 Continuously Differentiable Functions

1.

Proof. We note

$$\frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{1}{2}\frac{x^2+y^2}{\sqrt{x^2+y^2}} = \frac{1}{2}\sqrt{x^2+y^2}.$$

So $\lim_{(x,y)\to 0}\frac{|xy|}{\sqrt{x^2+y^2}} = 0$. This shows $f(x,y) = |xy|$ is differentiable at 0 and the derivative is 0. However, for any fixed y, $f(x,y)$ is not a differentiable function of x at 0. So its partial derivative w.r.t. x does not exist in a neighborhood of 0, which implies f is not of class C^1 in a neighborhood of 0. □

7 The Chain Rule

8 The Inverse Function Theorem

9 The Implicit Function Theorem

10 The Integral over a Rectangle

6.

Proof. (a) Straightforward from the Riemann condition (Theorem 10.3).

(b) Among all the sub-rectangles determined by P, those whose sides contain the newly added point have a combined volume no greater than $(\text{mesh}P)(\text{width}(Q))^{n-1}$. So

$$0 \le L(f, P^{''}) - L(f, P) \le 2M(\text{mesh}P)(\text{width}Q)^{n-1}.$$

The result for upper sums can be derived similarly.

(c) Given $\varepsilon > 0$, choose a partition P' such that $U(f, P') - L(f, P') < \frac{\varepsilon}{2}$. Let N be the number of partition points in P' and let

$$\delta = \frac{\varepsilon}{8MN(\text{width}Q)^{n-1}}.$$

Suppose P has mesh less than δ, the common refinement P'' of P and P' is obtained by adjoining at most N points to P. So by part (b)

$$0 \le L(f, P'') - L(f, P) \le N \cdot 2M(\text{mesh}P)(\text{width}Q)^{n-1} \le 2MN(\text{width}Q)^{n-1}\frac{\varepsilon}{8MN(\text{width}Q)^{n-1}} = \frac{\varepsilon}{4}.$$

Similarly, we can show $0 \le U(f, P) - U(f, P'') \le \frac{\varepsilon}{4}$. So

$$\begin{aligned} U(f, P) - L(f, P) &= [U(f, P) - U(f, P'')] + [L(f, P'') - L(f, P)] + [U(f, P'') - L(f, P'')] \\ &\le \frac{\varepsilon}{4} + \varepsilon 4 + [U(f, P') - L(f, P')] \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

This shows for any given $\varepsilon > 0$, there is a $\delta > 0$ such that $U(f, P) - L(f, P) < \varepsilon$ for every partition P of mesh less than δ. □

7.

Proof. (Sufficiency) Note $|\sum_R f(x_R)v(R) - A| < \varepsilon$ can be written as

$$A - \varepsilon < \sum_R f(x_R)v(R) < A + \varepsilon.$$

This shows $U(f, P) \le A + \varepsilon$ and $L(f, P) \ge A - \varepsilon$. So $U(f, P) - L(f, P) \le 2\varepsilon$. By Problem 6, we conclude f is integrable over Q, with $\int_Q f \in [A - \varepsilon, A + \varepsilon]$. Since ε is arbitrary, we conclude $\int_Q f = A$.

(Necessity) By Problem 6, for any given $\varepsilon > 0$, there is a $\delta > 0$ such that $U(f, P) - L(f, P) < \varepsilon$ for every partition P of mesh less than δ. For any such partition P, if for each sub-rectangle R determined by P, x_R is a point of R, we must have

$$L(f, P) - A \le \sum_R f(x_R)v(R) - A \le U(f, P) - A.$$

Since $L(f, P) \le A \le U(f, P)$, we conclude

$$|\sum_R f(x_R)v(R) - A| \le U(f, P) - L(f, P) < \varepsilon.$$

□

11 Existence of the Integral

12 Evaluation of the Integral

13 The Integral over a Bounded Set

14 Rectifiable Sets

15 Improper Integrals

16 Partition of Unity

17 The Change of Variables Theorem

18 Diffeomorphisms in $\mathbb{R}^n$

19 Proof of the Change of Variables Theorem

20 Applications of Change of Variables

21 The Volume of a Parallelepiped

1. (a)

Proof. Let $v = (a, b, c)$, then $X^{tr}X = (I_3, v^{tr})\begin{pmatrix} I_3 \\ v \end{pmatrix} = I_3 + \begin{pmatrix} a \\ b \\ c \end{pmatrix}(a, b, c) = \begin{pmatrix} 1+a^2 & ab & ac \\ ab & 1+b^2 & bc \\ ca & cb & 1+c^2 \end{pmatrix}$. □

(b)

Proof. We use both methods:

$$V(X) = [\det(X^{tr} \cdot X)]^{1/2} = [(1+a^2)(1+b^2+c^2) - ab \cdot ab + ca \cdot (-ac)]^{1/2} = (1+a^2+b^2+c^2)^{1/2}$$

and

$$V(X) = \left[\det{}^2 I_3 + \det{}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix} + \det{}^2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix} + \det{}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix}\right]^{1/2} = (1+c^2+a^2+b^2)^{1/2}.$$

□

2.

Proof. Let $X = (x_1, \cdots, x_i, \cdots, x_k)$ and $Y = (x_1, \cdots, \lambda x_i, \cdots, x_k)$. Then $V(Y) = [\sum_{[I]} \det^2 Y_I]^{1/2} = [\sum_{[I]} \lambda^2 \det^2 X_I]^{1/2} = |\lambda|[\sum_{[I]} \det^2 X_I]^{\frac{1}{2}} = |\lambda| V(X)$. □

3.

Proof. Suppose $\mathcal{P}$ is determined by $x_1, \cdots, x_k$. Then $V(h(\mathcal{P})) = V(\lambda x_1, \cdots, \lambda x_k) = |\lambda| V(x_1, \lambda x_2, \cdots, \lambda x_k) = \cdots = |\lambda|^k V(x_1, x_2, \cdots, x_k) = |\lambda|^k V(\mathcal{P})$. □

4. (a)

Proof. Straightforward. □

(b)

Proof.

$$\begin{aligned}
\|a\|^2\|b\|^2 - \langle a,b\rangle^2 &= (\sum_{i=1}^3 a_i^2)(\sum_{j=1}^3 b_j^2) - (\sum_{k=1}^3 a_k b_k)^2 \\
&= \sum_{i,j=1}^3 a_i^2 b_j^2 - \sum_{k=1}^3 a_k^2 b_k^2 - 2(a_1b_1a_2b_2 + a_1b_1a_3b_3 + a_2b_2a_3b_3) \\
&= \sum_{i,j=1, i\neq j}^3 a_i^2 b_j^2 - 2(a_1b_1a_2b_2 + a_1b_1a_3b_3 + a_2b_2a_3b_3) \\
&= (a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2 \\
&= \det{}^2 \begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} + \det{}^2 \begin{pmatrix} a_1 & b_1 \\ a_3 & b_3 \end{pmatrix} + \det{}^2 \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}.
\end{aligned}$$

□

5. (a)

Proof. Suppose V_1 and V_2 both satisfy conditions (i)-(iv). Then by the Gram-Schmidt process, the uniqueness is reduced to $V_1(x_1, \cdots, x_k) = V_2(x_1, \cdots, x_k)$, where x_1, $\cdots$, x_k are orthonormal. □

(b)

Proof. Following the hint, we can assume without loss of generality that $W = \mathbb{R}^n$ and the inner product is the dot product on $\mathbb{R}^n$. Let $V(x_1, \cdots, x_k)$ be the volume function, then (i) and (ii) are implied by Theorem 21.4, (iii) is Problem 2, and (iv) is implied by Theorem 21.3: $V(x_1, \cdots, x_k) = [\det(X^{tr}X)]^{1/2}$. □

22 The Volume of a Parametrized-Manifold

1.

Proof. By definition, $v(Z_\beta) = \int_A V(D\beta)$. Let x denote the general point of A; let $y = \alpha(x)$ and $z = h\circ\alpha(x) = \beta(y)$. By chain rule, $D\beta(x) = Dh(y) \cdot D\alpha(x)$. So $[V(D\beta(x))]^2 = \det(D\alpha(x)^{tr} Dh(y)^{tr} Dh(y) D\alpha(x)) = [V(D\alpha(x))]^2$ by Theorem 20.6. So $v(Z_\beta) = \int_A V(D\beta) = \int_A V(D\alpha) = v(Y_\alpha)$. □

2.

Proof. Let x denote the general point of A. Then

$$D\alpha(x) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ D_1 f(x) & D_2 f(x) & \cdots & D_k f(x) \end{pmatrix}$$

and by Theorem 21.4, $V(D\alpha(x)) = \left[1 + \sum_{i=1}^k (D_i f(x))^2\right]^{1/2}$. So $v(Y_\alpha) = \int_A \left[1 + \sum_{i=1}^k (D_i f(x))^2\right]^{1/2}$. □

3. (a)

Proof. $v(Y_\alpha) = \int_A V(D\alpha)$ and $\int_{Y_\alpha} \pi_i dV = \int_A \pi_i \circ \alpha V(D\alpha)$. Since $D\alpha(t) = \begin{pmatrix} -a\sin t \\ a\cos t \end{pmatrix}$, $V(D\alpha) = |a|$. So $v(Y_\alpha) = |a|\pi$, $\int_{Y_\alpha} \pi_1 dV = \int_0^\pi a\cos t|a| = 0$, and $\int_{Y_\alpha} \pi_2 dV = \int_0^\pi a\sin t|a| = 2a|a|$. Hence the centroid is $(0, 2a/\pi)$. □

(b)

Proof. By Example 4, $v(Y_\alpha) = 2\pi a^2$ and

$$\int_{Y_\alpha} \pi_1 dV = \int_A x\frac{a}{\sqrt{a^2-x^2-y^2}} = \int_0^{2\pi}\int_0^a \frac{r\cos\theta \cdot ar}{\sqrt{a^2-r^2}} = 0,$$

$$\int_{Y_\alpha} \pi_2 dV = \int_A y\frac{a}{\sqrt{a^2-x^2-y^2}} = \int_0^{2\pi}\int_0^a \frac{r\sin\theta \cdot ar}{\sqrt{a^2-r^2}} = 0,$$

$$\int_{Y_\alpha} \pi_3 dV = \int_A \sqrt{a^2-x^2-y^2}\frac{a}{\sqrt{a^2-x^2-y^2}} = a^3\pi.$$

So the centroid is $(0, 0, \frac{a}{2})$. □

4. (a)

Proof. $v(\Delta_1(R)) = \int_A V(D\alpha)$, where A is the (open) triangle in $\mathbb{R}^2$ with vertices (a,b), $(a+h,b)$ and $(a+h,b+h)$. $V(D\alpha)$ is a continuous function on the compact set $\bar{A}$, so it achieves its maximum M and minimum m on $\bar{A}$. Let $x_1, x_2 \in \bar{A}$ be such that $V(D\alpha(x_1)) = M$ and $V(D\alpha(x_2)) = m$, respectively. Then

$$v(A)\cdot m \le v(\Delta_1(R)) \le v(A)\cdot M.$$

By considering the segment connecting x_1 and x_2, we can find a point $\xi \in \bar{A}$ such that $V(D\alpha(\xi))v(A) = \int_A V(D\alpha)$. This shows there is a point ξ of R such that

$$v(\Delta_1(R)) = \int_A V(D\alpha) = V(D\alpha(\xi))v(A) = \frac{1}{2}V(D\alpha(\xi))\cdot v(R).$$

A similar result for $v(\Delta_2(R))$ can be proved similarly. □

(b)

Proof. $V(D\alpha)$ as a continuous function is uniformly continuous on the compact set Q. □

(c)

Proof.

$$\begin{aligned} \left|A(P) - \int_Q V(D\alpha)\right| &\le \sum_R \left|v(\Delta_1(R)) + v(\Delta_2(R)) - \int_R V(D\alpha)\right| \\ &= \sum_R \left|\frac{1}{2}\left[V(D\alpha(\xi_1(R))) + V(D\alpha(\xi_2(R)))\right]v(R) - \int_R V(D\alpha)\right| \\ &\le \sum_R \int_R \left|\frac{V(D\alpha(\xi_1(R))) + V(D\alpha(\xi_2(R)))}{2} - V(D\alpha)\right|. \end{aligned}$$

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x_1, x_2 \in Q$ with $|x_1 - x_2| < \delta$, we must have $|V(D\alpha(x_1)) - V(D\alpha(x_2))| < \frac{\varepsilon}{v(Q)}$. So for every partition P of Q of mesh less than δ,

$$\left|A(P) - \int_Q V(D\alpha)\right| < \sum_R \int_R \frac{\varepsilon}{v(Q)} = \varepsilon.$$

□

23 Manifolds in $\mathbb{R}^n$

1.

Proof. In this case, we set $U = \mathbb{R}$ and $V = M = \{(x, x^2) : x \in \mathbb{R}\}$. Then α maps U onto V in a one-to-one fashion. Moreover, we have
(1) α is of class C^∞.
(2) $\alpha^{-1}((x, x^2)) = x$ is continuous, for $(x_n, x_n^2) \to (x, x^2)$ as $n \to \infty$ implies $x_n \to x$ as $n \to \infty$.
(3) $D\alpha(x) = \begin{bmatrix} 1 \\ 2x \end{bmatrix}$ has rank 1 for each $x \in U$.

So M is a 1-manifold in $\mathbb{R}^2$ covered by the single coordinate patch α. □

2.

Proof. We let $U = \mathbb{H}^1$ and $V = N = \{(x, x^2) : x \in \mathbb{H}^1\}$. Then β maps U onto V in a one-to-one fashion. Moreover, we have
(1) β is of class C^∞.
(2) $\beta^{-1}((x, x^2)) = x$ is continuous.
(3) $D\beta(x) = \begin{bmatrix} 1 \\ 2x \end{bmatrix}$ has rank 1 for each $x \in \mathbb{H}^1$.

So N is a 1-manifold in $\mathbb{R}^2$ covered by the single coordinate patch β. □

3. (a)

Proof. For any point $p \in S^1$ with $p \neq (1, 0)$, we let $U = (0, 2\pi)$, $V = S^1 - (1, 0)$, and $\alpha : U \to V$ be defined by $\alpha(\theta) = (\cos\theta, \sin\theta)$. Then α maps U onto V continuously in a one-to-one fashion. Moreover,
(1) α is of class C^∞.
(2) α^{-1} is continuous, for $(\cos\theta_n, \sin\theta_n) \to (\cos\theta, \sin\theta)$ as $n \to \infty$ implies $\theta_n \to \theta$ as $n \to \infty$.
(3) $D\alpha(\theta) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ has rank 1.

So α is a coordinate patch. For $p = (1, 0)$, we consider $U = (-\pi, \pi)$, $V = S^1 - (-1, 0)$, and $\beta : U \to V$ be defined by $\beta(\theta) = (\cos\theta, \sin\theta)$. We can prove in a similar way that β is a coordinate patch. Combined, we can conclude the unit circle S^1 is a 1-manifold in $\mathbb{R}^2$. □

(b)

Proof. We claim α^{-1} is not continuous. Indeed, for $t_n = 1 - \frac{1}{n}$, $\alpha(t_n) \to (1, 0)$ on S^1 as $n \to \infty$, but $\alpha^{-1}(\alpha(t_n)) = t_n \to 1 \neq \alpha^{-1}((1, 0)) = 0$ as $n \to \infty$. □

4.

Proof. Let $U = A$ and $V = \{(x, f(x)) : x \in A\}$. Define $\alpha : U \to V$ by $\alpha(x) = (x, f(x))$. Then α maps U onto V in a one-to-one fashion. Moreover,
(1) α is of class C^r.
(2) α^{-1} is continuous, for $(x_n, f(x_n)) \to (x, f(x))$ as $n \to \infty$ implies $x_n \to x$ as $n \to \infty$.
(3) $D\alpha(x) = \begin{bmatrix} I_k \\ Df(x) \end{bmatrix}$ has rank k.

So V is a k-manifold in $\mathbb{R}^{k+1}$ with a single coordinate patch α. □

5.

Proof. For any $x \in M$ and $y \in N$, there is a coordinate patch α for x and a coordinate patch β for y, respectively. Denote by U the domain of α, which is open in $\mathbb{R}^k$ and by W the domain of β, which is open in either $\mathbb{R}^l$ or $\mathbb{H}^l$. Then $U \times W$ is open in either $\mathbb{R}^{k+l}$ or $\mathbb{H}^{k+l}$, depending on W is open in $\mathbb{R}^l$ or $\mathbb{H}^l$. This is the essential reason why we need at least one manifold to have no boundary: if both M and N have boundaries, $U \times W$ may not be open in $\mathbb{R}^{k+l}$ or $\mathbb{H}^{k+l}$.

The rest of the proof is routine. We define a map $f : U \times W \to \alpha(U) \times \beta(W)$ by $f(x, y) = (\alpha(x), \beta(y))$. Since $\alpha(U)$ is open in M and $\beta(W)$ is open in N by the definition of coordinate patch, $f(U \times W) = \alpha(U) \times \beta(W)$ is open in $M \times N$ under the product topology. f is one-to-one and continuous, since α and β enjoy such properties. Moreover,
(1) f is of class C^r, since α and β are of class C^r.
(2) $f^{-1} = (\alpha^{-1}, \beta^{-1})$ is continuous since α^{-1} and β^{-1} are continuous.
(3) $Df(x, y) = \begin{bmatrix} D\alpha(x) & 0 \\ 0 & D\beta(y) \end{bmatrix}$ clearly has rank $k + l$ for each $(x, y) \in U \times W$.

Therefore, we conclude $M \times N$ is a $k + l$ manifold in $\mathbb{R}^{m+n}$. □

6. (a)

Proof. We define $\alpha_1 : [0, 1) \to [0, 1)$ by $\alpha_1(x) = x$ and $\alpha_2 : [0, 1) \to (0, 1]$ by $\alpha_2(x) = -x + 1$. Then it's easy to check α_1 and α_2 are both coordinate patches. □

(b)

Proof. Intuitively $I \times I$ cannot be a 2-manifold since it has "corners". For a formal proof, assume $I \times I$ is a 2-manifold of class C^r with $r \geq 1$. By Theorem 24.3, $\partial(I \times I)$, the boundary of $I \times I$, is a 1-manifold without boundary of class C^r. Assume α is a coordinate patch of $\partial(I \times I)$ whose image includes one of those corner points. Then $D\alpha$ cannot exist at that corner point, contradiction. In conclusion, $I \times I$ cannot be a 2-manifold of class C^r with $r \geq 1$. □

24 The Boundary of a Manifold

1.

Proof. The equation for the solid torus N in cartesian coordinates is $(b - \sqrt{x^2 + y^2})^2 + z^2 \leq a^2$, and the equation for the torus T in cartesian coordinates is $(b - \sqrt{x^2 + y^2})^2 + z^2 = a^2$. Define $\mathcal{O} = \mathbb{R}$ and $f : \mathcal{O} \to \mathbb{R}$ by $f(x, y, z) = a^2 - z^2 - (b - \sqrt{x^2 + y^2})^2$. Then $Df(x, y, z) = \begin{bmatrix} 2x - \frac{2xb}{\sqrt{x^2+y^2}} \\ 2y - \frac{2yb}{\sqrt{x^2+y^2}} \\ -2z \end{bmatrix}$ has rank 1 at each point of T. By Theorem 24.4, N is a 3-manifold and $T = \partial N$ is a 2-manifold without boundary. □

2.

Proof. We first prove a regularization result.

Lemma 24.1. *Let $f : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be of class C^r. Assume Df has rank n at a point p, then there is an open set $W \subset \mathbb{R}^{n+k}$ and a C^r-function $G : W \to \mathbb{R}^{n+k}$ with C^r-inverse such that $G(W)$ is an open neighborhood of p and $f \circ G : W \to \mathbb{R}^n$ is the projection mapping to the first n coordinates.*

Proof. We write any point $x \in \mathbb{R}^{n+k}$ as (x_1, x_2) with $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^k$. We first assume $D_{x_1} f(p)$ has rank n. Define $F(x) = (f(x), x_2)$, then $DF = \begin{bmatrix} D_{x_1} f & D_{x_2} f \\ 0 & I_k \end{bmatrix}$. So $\det DF(p) = \det D_{x_1} f(p) \neq 0$. By the inverse function theorem, there is an open set U of $\mathbb{R}^{n+k}$ containing p such that F carries U in a one-to-one fashion onto an open set W of $\mathbb{R}^{n+k}$ and its inverse function G is of class C^r. Denote by $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ the projection $\pi(x) = x_1$, then $f \circ G(x) = \pi \circ F \circ G(x) = \pi(x)$ on W.

In general, since $Df(p)$ has rank n, there will be $j_1 < \cdots < j_n$ such that the matrix $\frac{\partial(f_1, \cdots, f_n)}{\partial(x^{j_1}, \cdots, x^{j_n})}$ has rank n at p. Here x^j denotes the j-th coordinate of x. Define $H : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ as the permutation that swaps the pairs (x^1, x^{j_1}), (x^2, x^{j_2}), $\cdots$, (x^n, x^{j_n}), i.e. $H(x) = (x^{j_1}, x^{j_2}, \cdots, x^{j_n}, \cdots) - (p^{j_1}, p^{j_2}, \cdots, p^{j_n}, \cdots) + p$. Then $H(p) = p$ and $D(f \circ H)(p) = Df(H(p))DH(p) = Df(p) \cdot DH(p)$. So $D_{x_1}(f \circ H)(p) = \frac{\partial(f_1, \cdots, f_n)}{\partial(x^{j_1}, \cdots, x^{j_n})}(p)$ and $f \circ H$ is of the type considered previously. So using the notation of the previous paragraph, $f \circ (H \circ G)(x) = \pi(x)$ on W. □

By the lemma and using its notation, $\forall p \in M = \{x : f(x) = 0\}$, there is a C^r-diffeomorphism G between an open set W of $\mathbb{R}^{n+k}$ and an open set U of $\mathbb{R}^{n+k}$ containing p, such that $f \circ G = \pi$ on W. So $U \cap M = \{x \in U : f(x) = 0\} = G(W) \cap (f \circ G \circ G^{-1})^{-1}(\{0\}) = G(W) \cap G(\pi^{-1}(\{0\})) = G(W \cap \{0\} \times \mathbb{R}^k)$. Therefore $\alpha(x_1, \cdots, x_k) := G((0, x_1, \cdots, x_k))$ is a k-dimensional coordinate patch on M about p. Since p is arbitrarily chosen, we have proved M is a k-manifold without boundary in $\mathbb{R}^{n+k}$.

Now, $\forall p \in N = \{x : f_1(x) = \cdots = f_{n-1}(x), f_n(x) \geq 0\}$, there are two cases: $f_n(p) > 0$ and $f_n(p) = 0$. For the first case, by an argument similar to that of M, we can find a C^r-diffeomorphism G_1 between an open set W of $\mathbb{R}^{n+k}$ and an open set U of $\mathbb{R}^{n+k}$ containing p, such that $f \circ G_1 = \pi_1$ on W. Here π_1 is the projection mapping to the first $(n-1)$ coordinates. So $U \cap N = U \cap \{x : f_1(x) = \cdots = f_{n-1}(x) = 0\} \cap \{x : f_n(x) \geq 0\} = G_1(W \cap \{0\} \times \mathbb{R}^{k+1}) \cap \{x \in U : f_n(x) \geq 0\}$. When U is sufficiently small, by the continuity of f_n and the fact $f_n(p) > 0$, we can assume $f_n(x) > 0$, $\forall x \in U$. So

$$\begin{aligned} U \cap N &= U \cap \{x : f_1(x) = \cdots = f_n(x) = 0, f_n(x) > 0\} \\ &= G_1(W \cap \{0\} \times \mathbb{R}^{k+1}) \cap \{x \in U : f_n(x) > 0\} \\ &= G_1(W \cap \{0\} \times \mathbb{R}^{k+1} \cap G_1^{-1}(U \cap \{x : f_n(x) > 0\})) \\ &= G_1([W \cap G_1^{-1}(U \cap \{x : f_n(x) > 0\})] \cap \{0\} \times \mathbb{R}^{k+1}). \end{aligned}$$

This shows $\beta(x_1, \cdots, x_{k+1}) := G_1((0, x_1, \cdots, x_{k+1}))$ is a $(k+1)$-dimensional coordinate patch on N about p.

For the second case, we note p is necessarily in M. So $Df(p)$ is of rank n and there is a C^r-diffeomorphism G between an open set W of $\mathbb{R}^{n+k}$ and an open set U of $\mathbb{R}^{n+k}$ containing p, such that $f \circ G = \pi$ on W. So $U \cap N = \{x \in U : f_1(x) = \cdots = f_{n-1}(x) = 0, f_n(x) \geq 0\} = G(W) \cap (\pi \circ G^{-1})^{-1}(\{0\} \times [0, \infty)) = G(W \cap \pi^{-1}(\{0\} \times [0, \infty))) = G(W \cap \{0\} \times [0, \infty) \times \mathbb{R}^k)$. This shows $\gamma(x_1, \cdots, x_{k+1}) := G((0, x_{k+1}, x_1, \cdots, x_k))$ is a $(k+1)$-dimensional coordinate patch on N about p.

In summary, we have shown N is a $(k+1)$-manifold. Lemma 24.2 shows $\partial N = M$. $\square$

3.

Proof. Define $H : \mathbb{R}^3 \to \mathbb{R}^2$ by $H(x, y, z) = (f(x, y, z), g(x, y, z))$. By the theorem proved in Problem 2, if $DH(x, y, z) = \begin{bmatrix} D_x f(x,y,z) & D_y f(x,y,z) & D_z f(x,y,z) \\ D_x g(x,y,z) & D_y g(x,y,z) & D_z g(x,y,z) \end{bmatrix}$ has rank 2 for $(x, y, z) \in M := \{(x, y, z) : f(x, y, z) = g(x, y, z) = 0\}$, M is a 1-manifold without boundary in $\mathbb{R}^3$, i.e. a C^r curve without singularities. $\square$

4.

Proof. We define $f(x) = (f_1(x), f_2(x)) = (\|x\|^2 - a^2, x_n)$. Let $N = \{x : f_1(x) = 0, f_2(x) \geq 0\} = S^{n-1}(a) \cap \mathbb{H}^n$ and $M = \{x : f(x) = 0\}$. Since $Df(x) = \begin{bmatrix} 2x_1 & 2x_2 & \cdots & 2x_{n-1} & 2x_n \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2x_1 & 2x_2 & \cdots & 2x_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$ has rank 2 on M and $\partial f_1/\partial x = [2x_1, 2x_2, \cdots, 2x_n]$ has rank 1 on N, by the theorem proved in Problem 2, $E_+^{n-1}(a) = N$ is an $(n-1)$ manifold whose boundary is the $(n-2)$ manifold M. Geometrically, M is $S^{n-2}(a)$. $\square$

5. (a)

Proof. We write any point $x \in \mathbb{R}^9$ as $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, where $x_1 = [x_{11}, x_{12}, x_{13}]$, $x_2 = [x_{21}, x_{22}, x_{23}]$, and $x_3 = [x_{31}, x_{32}, x_{33}]$. Define $f_1(x) = \|x_1\|^2 - 1$, $f_2(x) = \|x_2\|^2 - 1$, $f_3(x) = \|x_3\|^2 - 1$, $f_4(x) = (x_1, x_2)$, $f_5(x) = (x_1, x_3)$, and $f_6(x) = (x_2, x_3)$. Then $\mathcal{O}(3)$ is the solution set of the equation $f(x) = 0$. $\square$

(b)

Proof. We note

$$\begin{aligned} Df(x) &= \frac{\partial(f_1, \cdots, f_6)}{\partial(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33})} \\ &= \begin{bmatrix} 2x_{11} & 2x_{12} & 2x_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2x_{21} & 2x_{22} & 2x_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2x_{31} & 2x_{32} & 2x_{33} \\ x_{21} & x_{22} & x_{23} & x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 0 & 0 & 0 & x_{11} & x_{12} & x_{13} \\ 0 & 0 & 0 & x_{31} & x_{32} & x_{33} & x_{21} & x_{22} & x_{23} \end{bmatrix} \end{aligned}$$

Since x_1, x_2, x_3 are pairwise orthogonal and are non-zero, we conclude x_1, x_2 and x_3 are independent. From the structure of Df, the row space of $Df(x)$ for $x \in \mathcal{O}(3)$ has rank 6. By the theorem proved in Problem 2, $\mathcal{O}(3)$ is a 3-manifold without boundary in $\mathbb{R}^9$. Finally, $\mathcal{O}(3) = \{x : f(x) = 0\}$ is clearly bounded and closed, hence compact. □

6.

Proof. The argument is similar to that of Problem 5, and the dimension $= n^2 - n - \frac{n(n-1)}{2} = \frac{n(n-1)}{2}$. □

25 Integrating a Scalar Function over a Manifold

1.

Proof. To see $\alpha(t, z)$ is a coordinate patch, we note that α is one-to-one and onto $S^2(a) - D$, where $D = \{(x, y, z) : (\sqrt{a^2 - z^2}, 0, z), |z| \le a\}$ is a closed set and has measure zero in $S^2(a)$ (note D is a parametrized 1-manifold, hence it has measure zero in $\mathbb{R}^2$). On the set $\{(t, z) : 0 < t < 2\pi, |z| < a\}$, α is smooth and $\alpha^{-1}(x, y, z) = (t, z)$ is continuous on $S^2(a) - D$. Finally, by the calculation done in the text, the rank of $D\alpha$ is 2 on $\{(t, z) : 0 < t < 2\pi, |z| < a\}$.

$$\begin{aligned} & (D\alpha)^{tr} D\alpha \\ = & \begin{bmatrix} -(a^2 - z^2)^{1/2} \sin t & (a^2 - z^2)^{1/2} \cos t & 0 \\ (-z \cos t)/(a^2 - z^2)^{1/2} & (-z \sin t)/(a^2 - z^2)^{1/2} & 1 \end{bmatrix} \begin{bmatrix} -(a^2 - z^2)^{1/2} \sin t & (-z \cos t)/(a^2 - z^2)^{1/2} \\ (a^2 - z^2)^{1/2} \cos t & (-z \sin t)/(a^2 - z^2)^{1/2} \\ 0 & 1 \end{bmatrix} \\ = & \begin{bmatrix} a^2 - z^2 & 0 \\ 0 & \frac{a^2}{a^2 - z^2} \end{bmatrix}. \end{aligned}$$

So $V(D\alpha) = a$ and $v(S^2(a)) = \int_{\{(t,z):0<t<2\pi,|z|<a\}} V(D\alpha) = 4\pi a^2$. □

4.

Proof. Let (α_j) be a family of coordinate patches that covers M. Then $(h \circ \alpha_j)$ is a family of coordinate patches that covers N. Suppose $\phi_1, \cdots, \phi_l$ is a partition of unity on M that is dominated by (α_j), then

$\phi_1 \circ h^{-1}, \cdots, \phi_l \circ h^{-1}$ is a partition of unity on N that is dominated by $(h \circ \alpha_j)$. Then

$$\begin{aligned}\int_N f dV &= \sum_{i=1}^{l} \int_N (\phi_i \circ h^{-1}) f dV \\ &= \sum_{i=1}^{l} \int_{IntU_i} (\phi_i \circ h^{-1} \circ h \circ \alpha_i)(f \circ h \circ \alpha_i) V(D(h \circ \alpha_i)) \\ &= \sum_{i=1}^{l} \int_{IntU_i} (\phi_i \circ \alpha_i)(f \circ h \circ \alpha_i) V(D\alpha_i) \\ &= \sum_{i=1}^{l} \int_M \phi_i (f \circ h) dV \\ &= \int_M f \circ h dV.\end{aligned}$$

In particular, by setting $f \equiv 1$, we get $v(N) = v(M)$. □

6.

Proof. Let $L_0 = \{x \in \mathbb{R}^n : x_i > 0\}$. Then $M \cap L_0$ is a manifold, for if $\alpha : U \to V$ is a coordinate patch on M, $\alpha : U \cap \alpha^{-1}(L_0) \to V \cap L_0$ is a coordinate patch on $M \cap L$. Similarly, if we let $L_1 = \{x \in \mathbb{R}^n : x_i < 0\}$, $M \cap L_1$ is a manifold. Theorem 25.4 implies

$$c_i(M) = \frac{1}{v(M)} \int_M \pi dV = \frac{1}{v(M)} \left[\int_{M \cap L_0} \pi dV + \int_{M \cap L_1} \pi dV \right].$$

Suppose (α_j) is a family of coordinate patches on $M \cap L_0$ and there is a partition of unity $\phi_1, \cdots, \phi_l$ on $M \cap L_0$ that is dominated by (α_j), then

$$\int_{M \cap L_0} \pi_i dV = \sum_{j=1}^{l} \int_M (\phi_j \pi_i) dV = \sum_{j=1}^{l} \int_{IntU_j} (\phi_j \circ \alpha_j)(\pi_i \circ \alpha_j) V(D\alpha_j)$$

Define $f : \mathbb{R}^n \to \mathbb{R}^n$ by $f(x) = (x_1, \cdots, -x_i, \cdots, x_n)$. It's easy to see $(f \circ \alpha_j)$ is a family of coordinate patches on $M \cap L_1$ and $\phi_1 \circ f, \cdots, \phi_l \circ f$ is a partition of unity on $M \cap L_1$ that is dominated by $(f \circ \alpha_j)$. Therefore

$$\int_{M \cap L_1} \pi_i dV = \sum_{j=1}^{l} \int_{IntU_j} (\phi_j \circ f \circ f \circ \alpha_j)(\pi_i \circ f \circ \alpha_j) V(D(f \circ \alpha_j)) = \sum_{j=1}^{l} \int_{IntU_j} (\phi_j \circ \alpha_j)(\pi_i \circ f \circ \alpha_j) V(D(f \circ \alpha_j))$$

In order to show $c_i(M) = 0$, it suffices to show $(\pi_i \circ \alpha_j) V(D\alpha_j) = -(\pi_i \circ f \circ \alpha_j) V(D(f \circ \alpha_j))$. Indeed,

$$\begin{aligned}V^2(D(f \circ \alpha_j))(x) &= V^2(Df(\alpha_j(x)) D\alpha_j(x)) \\ &= \det(D\alpha_j(x)^{tr} Df(\alpha_j(x))^{tr} Df(\alpha_j(x)) D\alpha_j(x)) \\ &= \det(D\alpha_j(x)^{tr} D\alpha_j(x)) \\ &= V^2(D\alpha)(x),\end{aligned}$$

and $\pi_i \circ f = -\pi_i$. Combined, we conclude $\int_{M \cap L_1} \pi_i dV = -\int_{M \cap L_0} \pi_i dV$. Hence $c_i(M) = 0$. □

8. (a)

Proof. Let (α_i) be a family of coordinate patches on M and $\phi_1, \cdots, \phi_l$ a partition of unity on M dominated by (α_i). Let (β_j) be a family of coordinate patches on N and $\psi_1, \cdots, \psi_k$ a partition of unity on N dominated by (β_j). Then it's easy to see $((\alpha_i, \beta_j))_{i,j}$ is a family of coordinate patches on $M\times N$ and $(\phi_m\psi_n)_{1\le m\le l, 1\le n\le k}$ is a partition of unity on $M \times N$ dominated by $((\alpha_i, \beta_j))_{i,j}$. Then

$$\begin{aligned}
\int_{M\times N} f\cdot g dV &= \sum_{1\le m\le l, 1\le n\le k} \int_{M\times N} (\phi_m f)(\psi_n g) dV \\
&= \sum_{1\le m\le l, 1\le n\le k} \int_{IntU_m\times IntV_n} (\phi_m\circ\alpha_m\cdot f\circ\alpha_m)V(D\alpha_m)(\psi_n\circ\beta_n\cdot g\circ\beta_n)V(D\beta_n) \\
&= \sum_{1\le m\le l, 1\le n\le k} \int_{IntU_m} (\phi_m\circ\alpha_m\cdot f\circ\alpha_m)V(D\alpha_m) \int_{IntV_n} (\psi_n\circ\beta_n\cdot g\circ\beta_n)V(D\beta_n) \\
&= [\int_M f dV][\int_N g dV].
\end{aligned}$$

□

(b)

Proof. Set $f = 1$ and $g = 1$ in (a). □

(c)

Proof. By (a), $v(S^1 \times S^1) = v(S^1)\cdot v(S^1) = 4\pi^2 a^2$. □

26 Multilinear Algebra

4.

Proof. By Example 1, it is easy to see f and g are not tensors on $\mathbb{R}^4$. h is a tensor: $h = \phi_{1,1} - 7\phi_{2,3}$. □

5.

Proof. f and h are not tensors. g is a tensor and $g = 5\phi_{3,2,3,4,4}$. □

6. (a)

Proof. $f = 2\phi_{1,2,2} - \phi_{2,3,1}$, $g = \phi_{2,1} - 5\phi_{3,1}$. So $f\otimes g = 2\phi_{1,2,2,2,1} - 10\phi_{1,2,2,3,1} - \phi_{2,3,1,2,1} + 5\phi_{2,3,1,3,1}$. □

(b)

Proof. $f\otimes g(x,y,z,u,v) = 2x_1y_2z_2u_2v_1 - 10x_1y_2z_2u_3v_1 - x_2y_3z_1u_2v_1 + 5x_2y_3z_1u_3v_1$. □

7.

Proof. Suppose $f = \sum_I d_I\phi_I$ and $g = \sum_J d_J\phi_J$. Then $f\otimes g = (\sum_I d_I\phi_I)\otimes(\sum_J d_J\phi_J) = \sum_{I,J} d_I d_J \phi_I\otimes\phi_J = \sum_{I,J} d_I d_J \phi_{I,J}$. This shows the four properties stated in Theorem 26.4 characterize the tensor product uniquely. □

8.

Proof. For any $x\in\mathbb{R}^m$, $T^*f(x) = f(T(x)) = f(B\cdot x) = (AB)\cdot x$. So the matrix of the 1-tensor T^*f on $\mathbb{R}^m$ is AB. □

27 Alternating Tensors

1.

Proof. Since h is not multilinear, h is not an alternating tensor. $f = \phi_{1,2} - \phi_{2,1} + \phi_{1,1}$ is a tensor. The only permutation of $\{1, 2\}$ are the identity mapping id and $\sigma : \sigma(1) = 2, \sigma(2) = 1$. So f is alternating if and only if $f^\sigma(x, y) = -f(x, y)$. Since $f^\sigma(x, y) = f(y, x) = y_1x_2 - y_2x_1 + y_1x_1 \neq -f(x, y)$, we conclude f is not alternating.

Similarly, $g = \phi_{1,3} - \phi_{3,2}$ is a tensor. And $g^\sigma = \phi_{2,1} - \phi_{2,3} \neq -g$. So g is not alternating. □

3.

Proof. Suppose $I = (i_1, \cdots, i_k)$. If $\{i_1, \cdots, i_k\} \neq \{j_1, \cdots, j_k\}$ (set equality), then $\phi_I(a_{j_1}, \cdots, a_{j_k}) = 0$. If $\{i_1, \cdots, i_k\} = \{j_1, \cdots, j_k\}$, there must exist a permutation σ of $\{1, 2, \cdots, k\}$, such that $I = (i_1, \cdots, i_k) = (j_{\sigma(1)}, \cdots, j_{\sigma(k)})$. Then $\phi_I(a_{j_1}, \cdots, a_{j_k}) = (\text{sgn}\sigma)(\phi_I)^\sigma(a_{j_1}, \cdots, a_{j_k}) = (\text{sgn}\sigma)\phi_I(a_{j_{\sigma(1)}}, \cdots, a_{j_{\sigma(k)}}) = \text{sgn}\sigma$. In summary, we have

$$\phi_I(a_{j_1}, \cdots, a_{j_k}) = \begin{cases} \text{sgn}\sigma & \text{if there is a permutation } \sigma \text{ of } \{1, 2, \cdots, k\} \text{ such that } I = J_\sigma = (j_{\sigma(1)}, \cdots, j_{\sigma(k)}) \\ 0 & \text{otherwise.} \end{cases}$$

□

4.

Proof. For any $v_1, \cdots, v_k \in V$ and a permutation σ of $\{1, \cdots, k\}$.

$$\begin{aligned} (T^*f)^\sigma(v_1, \cdots, v_k) &= T^*f(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = f(T(v_{\sigma(1)}), \cdots, T(v_{\sigma(k)})) = f^\sigma(T(v_1), \cdots, T(v_k)) \\ &= (\text{sgn}\sigma)f(T(v_1), \cdots, T(v_k)) = (\text{sgn}\sigma)T^*f(v_1, \cdots, v_k). \end{aligned}$$

So $(T^*f)^\sigma = (\text{sgn}\sigma)T^*f$, which implies $T^*f \in \mathcal{A}^k(V)$. □

5.

Proof. We follow the hint and prove $\phi_{I_\sigma} = (\phi_I)^{\sigma^{-1}}$. Indeed, suppose $a_1, \cdots, a_n$ is a basis of the underlying vector space V, then

$$\begin{aligned} (\phi_I)^{\sigma^{-1}}(a_{j_1}, \cdots, a_{j_k}) &= (\phi_I)(a_{j_{\sigma^{-1}(1)}}, \cdots, a_{j_{\sigma^{-1}(k)}}) = \begin{cases} 0 & \text{if } I \neq (j_{\sigma^{-1}(1)}, \cdots, j_{\sigma^{-1}(k)}) \\ 1 & \text{if } I = (j_{\sigma^{-1}(1)}, \cdots, j_{\sigma^{-1}(k)}) \end{cases} \\ &= \begin{cases} 0 & \text{if } I_\sigma \neq (j_{\sigma\circ\sigma^{-1}(1)}, \cdots, j_{\sigma\circ\sigma^{-1}(k)}) = J \\ 1 & \text{if } I_\sigma = (j_{\sigma\circ\sigma^{-1}(1)}, \cdots, j_{\sigma\circ\sigma^{-1}(k)}) = J \end{cases} \\ &= \phi_{I_\sigma}(a_{j_1}, \cdots, a_{j_k}). \end{aligned}$$

Thus, $\phi_I = \sum_\sigma (\text{sgn}\sigma)(\phi_I)^\sigma = \sum_{\sigma^{-1}}(\text{sgn}\sigma^{-1})(\phi_I)^{\sigma^{-1}} = \sum_{\sigma^{-1}}(\text{sgn}\sigma)\phi_{I_\sigma} = \sum_\sigma (\text{sgn}\sigma)\phi_{I_\sigma}$. □

28 The Wedge Product

1. (a)

Proof. $F = 2\phi_2 \otimes \phi_2 \otimes \phi_1 + \phi_1 \otimes \phi_5 \otimes \phi_4$, $G = \phi_1 \otimes \phi_3 + \phi_3 \otimes \phi_1$. So $AF = 2\phi_2 \wedge \phi_2 \wedge \phi_1 + \phi_1 \wedge \phi_5 \wedge \phi_4 = -\phi_1 \wedge \phi_4 \wedge \phi_5$ and $AG = \phi_1 \wedge \phi_3 - \phi_1 \wedge \phi_3 = 0$, by Step 9 of the proof of Theorem 28.1. □

(b)

Proof. $(AF) \wedge h = -\phi_1 \wedge \phi_4 \wedge \phi_5 \wedge (\phi_1 - 2\phi_3) = 2\phi_1 \wedge \phi_4 \wedge \phi_5 \wedge \phi_3 = 2\phi_1 \wedge \phi_3 \wedge \phi_4 \wedge \phi_5$. □

(c)

Proof. $(AF)(x,y,z) = -\phi_1 \wedge \phi_4 \wedge \phi_5(x,y,z) = -\det \begin{bmatrix} x_1 & y_1 & z_1 \\ x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \end{bmatrix} = -x_1y_4z_5 + x_1y_5z_4 + x_4y_1z_5 - x_4y_5z_1 - x_5y_1z_4 + x_5y_4z_1$. □

2.

Proof. Suppose G is a k-tensor, then $AG(v_1, \cdots, v_k) = \sum_\sigma (\text{sgn}\sigma) G^\sigma(v_1, \cdots, v_k) = \sum_\sigma (\text{sgn}\sigma) G(v_1, \cdots, v_k) = [\sum_\sigma (\text{sgn}\sigma)] G(v_1, \cdots, v_k)$. Let e be an elementary permutation. Then $e : \sigma \to e \circ \sigma$ is an isomorphism on the permutation group S_k of $\{1, 2, \cdots, k\}$. So S_k can be divided into two disjoint subsets U_1 and U_2 so that e establishes a one-to-one correspondence between U_1 and U_2. By the fact sgn$e \circ \sigma = -$sgnσ, we conclude $\sum_\sigma (\text{sgn}\sigma) = 0$. This implies $AG = 0$. □

3.

Proof. We work by induction. For $k = 2$, $\frac{1}{l_1!l_2!} A(f_1 \otimes f_2) = f_1 \wedge f_2$ by the definition of $\wedge$. Assume for $k = n$, the claim is true. Then for $k = n+1$,

$$\frac{1}{l_1! \cdots l_n! l_{n+1}!} A(f_1 \otimes \cdots \otimes f_n \otimes f_{n+1}) = \frac{1}{l_1! \cdots l_n!} \frac{1}{l_{n+1}!} A((f_1 \otimes \cdots \times f_n) \otimes f_{n+1}) = \frac{1}{l_1! \cdots l_n!} A(f_1 \otimes \cdots \otimes f_n) \wedge f_{n+1}$$

by Step 6 of the proof of Theorem 28.1. By induction, $\frac{1}{l_1! \cdots l_n!} A(f_1 \otimes \cdots \otimes f_n) = f_1 \wedge \cdots \wedge f_n$. So $\frac{1}{l_1! \cdots l_n! l_{n+1}!} A(f_1 \otimes \cdots \otimes f_n \otimes f_{n+1}) = f_1 \wedge \cdots \wedge f_n \wedge f_{n+1}$. By the principle of mathematical induction,

$$\frac{1}{l_1! \cdots l_k!} A(f_1 \otimes \cdots \otimes f_k) = f_1 \wedge \cdots \wedge f_k$$

for any k. □

4.

Proof. $\phi_{i_1} \wedge \cdots \phi_{i_k}(x_1, \cdots, x_k) = A(\phi_{i_1} \otimes \cdots \otimes \phi_{i_k})(x_1, \cdots, x_k) = \sum_\sigma (\text{sgn}\sigma)(\phi_{i_1} \otimes \cdots \otimes \phi_{i_k})^\sigma (x_1, \cdots, x_k) = \sum_\sigma (\text{sgn}\sigma)(\phi_{i_1} \otimes \cdots \otimes \phi_{i_k})(x_{\sigma(1)}, \cdots, x_{\sigma(k)}) = \sum_\sigma (\text{sgn}\sigma) x_{i_1, \sigma(1)}, \cdots, x_{i_k, \sigma(k)} = \det X_I$. □

5.

Proof. Suppose F is a k-tensor. Then

$$\begin{aligned} T^*(F^\sigma)(v_1, \cdots, v_k) &= F^\sigma(T(v_1), \cdots, T(v_k)) \\ &= F(T(v_{\sigma(1)}), \cdots, T(v_{\sigma(k)})) \\ &= T^*F(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= (T^*F)^\sigma(v_1, \cdots, v_k). \end{aligned}$$

□

6. (a)

Proof. $T^*\psi_I(v_1, \cdots, v_k) = \psi_I(T(v_1), \cdots, T(v_k)) = \psi_I(B \cdot v_1, \cdots, B \cdot v_k)$. In particular, for $\bar{J} = (\bar{j}_1, \cdots, \bar{j}_k)$, $c_{\bar{J}} = \sum_{[J]} c_J \psi_J(e_{\bar{j}_1}, \cdots, e_{\bar{j}_k}) = T^*\psi_I(e_{\bar{j}_1}, \cdots, e_{\bar{j}_k}) = \psi_I(B \cdot e_{\bar{j}_1}, \cdots, B \cdot e_{\bar{j}_k}) = \psi_I(\beta_{\bar{j}_1}, \cdots, \beta_{\bar{j}_k})$ where β_i is the i-th column of B. So $c_{\bar{J}} = \det[\beta_{\bar{j}_1}, \cdots, \beta_{\bar{j}_k}]_I$. Therefore, c_J is the determinant of the matrix consisting of the i_1, $\cdots$, i_k rows and the j_1, $\cdots$, j_k columns of B, where $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_k)$. □

(b)

Proof. $T^*f = \sum_{[I]} d_I T^*(\psi_I) = \sum_{[I]} d_I \sum_{[I]} \det B_{I,J} \psi_J = \sum_{[J]} (\sum_{[I]} d_I \det B_{I,J}) \psi_J$ where $B_{I,J}$ is the matrix consisting of the i_1, $\cdots$, i_k rows and the j_1, $\cdots$, j_k columns of B ($I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_k)$). □

29 Tangent Vectors and Differential Forms

1.

Proof. $\gamma_*(t; e_1) = (\gamma(t); D\gamma(t)\cdot e_1) = (\gamma(t); \begin{bmatrix}\gamma_1'(t)\\ \cdots \\ \gamma_n'(t)\end{bmatrix})$, which is the velocity vector of γ corresponding to the parameter value t. □

2.

Proof. The velocity vector of the curve $\gamma(t) = \alpha(x+tv)$ corresponding to parameter value $t = 0$ is calculated by $\frac{d}{dt}\gamma(t)|_{t=0} = \lim_{t\to 0}\frac{\alpha(x+tv)-\alpha(x)}{t} = D\alpha(x)\cdot v$. So $\alpha_*(x; v) = (\alpha(x); D\alpha(x)\cdot v) = (\alpha(x); \frac{d}{dt}\gamma(t)|_{t=0})$. □

3.

Proof. Suppose $\alpha : U_\alpha \to V_\alpha$ and $\beta : U_\beta \to V_\beta$ are two coordinate patches about p, with $\alpha(x) = \beta(y) = p$. Because $\mathbb{R}^k$ is spanned by the vectors $e_1, \cdots, e_k$, the space $\mathcal{T}_p^\alpha(M)$ obtained by using α is spanned by the vectors $(p; \frac{\partial\alpha(x)}{\partial x_j})_{j=1}^k$ and the space $\mathcal{T}_p^\beta(M)$ obtained by using β is spanned by the vectors $(p; \frac{\partial\beta(y)}{\partial y_i})_{i=1}^k$. Let $W = V_\alpha \cap V_\beta$, $U_\alpha' = \alpha^{-1}(W)$, and $U_\beta' = \beta^{-1}(W)$. Then $\beta^{-1}\circ\alpha : U_\alpha' \to U_\beta'$ is a C^r-diffeomorphism by Theorem 24.1. By chain rule,

$$D\alpha(x) = D(\beta\circ\beta^{-1}\circ\alpha)(x) = D\beta(y)\cdot D(\beta^{-1}\circ\alpha)(x).$$

Since $D(\beta^{-1}\circ\alpha)(x)$ is of rank k, the linear space spanned by $(\partial\alpha(x)/\partial x_j)_{j=1}^k$ agrees with the linear space spanned by $(\partial\beta(y)/\partial y_i)_{i=1}^k$. □

4. (a)

Proof. Suppose $\alpha : U \to V$ is a coordinate patch about p, with $\alpha(x) = p$. Since $p \in M - \partial M$, we can without loss of generality assume U is an open subset of $\mathbb{R}^k$. By the definition of tangent vector, there exists $u \in \mathbb{R}^k$ such that $v = D\alpha(x)\cdot u$. For ε sufficiently small, $\{x + tu : |t| \le \varepsilon\} \subset U$ and $\gamma(t) := \alpha(x + tu)$ ($|t| \le \varepsilon$) has its image in M. Clearly $\frac{d}{dt}\gamma(t)|_{t=0} = D\alpha(x)\cdot u = v$. □

(b)

Proof. Suppose $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is a parametrized-curve whose image set lies in M. Denote $\gamma(0)$ by p and assume $\alpha : U \to V$ is a coordinate patch about p. For $v := \frac{d}{dt}\gamma(t)|_{t=0}$, we define $u = D\alpha^{-1}(p)\cdot v$. Then

$$\alpha_*(x; u) = (p\,; D\alpha(x)\cdot u) = (p\,; D\alpha(x)\cdot D\alpha^{-1}(p)\cdot v) = (p\,; D(\alpha\circ\alpha^{-1})(p)\cdot v) = (p\,; v).$$

So the velocity vector of γ corresponding to parameter value $t = 0$ is a tangent vector. □

5.

Proof. Similar to the proof of Problem 4, with $(-\varepsilon, \varepsilon)$ changed to $[0, \varepsilon)$ or $(-\varepsilon, 0]$. We omit the details. □

30 The Differential Operator

2.

Proof. $d\omega = -xdx \wedge dy - zdy \wedge dz$. So $d(d\omega) = -dx \wedge dx \wedge dy - dz \wedge dy \wedge dz = 0$. Meanwhile,

$$d\eta = -2yzdz \wedge dy + 2dx \wedge dz = 2yzdy \wedge dz + 2dx \wedge dz$$

and

$$\omega \wedge \eta = (-xy^2z^2 - 3x)dx \wedge dy + (2x^2y + xyz)dx \wedge dz + (6x - y^2z^3)dy \wedge dz.$$

So

$$d(\omega \wedge \eta) = (-2xy^2z - 2x^2 - xz + 6)dx \wedge dy \wedge dz,$$

$$(d\omega) \wedge \eta = -2x^2dx \wedge dy \wedge dz - xzdx \wedge dy \wedge dz,$$

and

$$\omega \wedge d\eta = 2xy^2zdx \wedge dy \wedge dz - 6dx \wedge dy \wedge dz.$$

Therefore, $(d\omega) \wedge \eta - \omega \wedge d\eta = (-2xy^2z - 2x^2 - xz + 6)dx \wedge dy \wedge dz = d(\omega \wedge \eta)$. □

3.

Proof. In $\mathbb{R}^2$, $\omega = ydx - xdy$ vanishes at $x_0 = (0, 0)$, but $d\omega = -2dx \wedge dy$ does not vanish at x_0. In general, suppose ω is a k-form defined in an open set A of $\mathbb{R}^n$, and it has the general form $\omega = \sum_{[I]} f_I dx_I$. If it vanishes at each x in a neighborhood of x_0, we must have $f_I = 0$ in a neighborhood of x_0 for each I. By continuity, we conclude $f_I \equiv 0$ in a neighborhood of x_0, including x_0. So $d\omega = \sum_{[I]} df_I \wedge dx_I = \sum_{[I]}(\sum_i D_i f dx_i) \wedge dx_I$ vanishes at x_0. □

4.

Proof. $d\omega = d\left(\frac{x}{x^2+y^2}dx\right) + d\left(\frac{y}{x^2+y^2}dy\right) = \frac{2xy}{(x^2+y^2)^2}dx \wedge dy + \frac{-2xy}{(x^2+y^2)^2}dx \wedge dy = 0$. So ω is closed. Define $\theta = \frac{1}{2}\log(x^2 + y^2)$, then $d\theta = \omega$. So ω is exact on A. □

5. (a)

Proof. $d\omega = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2}dy \wedge dx + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}dx \wedge dy = 0$. So ω is closed. □

(c)

Proof. We consider the following transformation from $(0, \infty) \times (0, 2\pi)$ to B:

$$\begin{cases} x = r\cos t \\ y = r\sin t. \end{cases}$$

Then

$$\det\frac{\partial(x, y)}{\partial(r, t)} = \det\begin{bmatrix} \cos t & -r\sin t \\ \sin t & r\cos t \end{bmatrix} = r \neq 0.$$

By part (b) and the inverse function theorem (Theorem 8.2, the global version), we conclude ϕ is of class C^∞. □

(d)

Proof. Using the transformation given in part (c), we have $dx = \cos tdr - r\sin tdt$ and $dy = \sin tdr + r\cos tdt$. So $\omega = [-r\sin t(\cos tdr - r\sin tdt) + r\cos t(\sin tdr + r\cos tdt)]/r^2 = dt = d\phi$. □

(e)

Proof. We follow the hint. Suppose g is a closed 0-form in B. Denote by a the point $(-1,0)$ of $\mathbb{R}^2$. For any $x \in B$, let $\gamma(t) : [0,1] \to B$ be the segment connecting a and x, with $\gamma(0) = a$ and $\gamma(1) = x$. Then by mean-value theorem (Theorem 7.3), there exists $t_0 \in (0,1)$, such that $g(a) - g(x) = Dg(a + t_0(x-a)) \cdot (a-x)$. Since g is closed in B, $Dg = 0$ in B. This implies $g(x) = g(a)$ for any $x \in B$. □

(f)

Proof. First, we note ϕ is not well-defined in all of A, so part (d) can not be used to prove ω is exact in A. Assume $\omega = df$ in A for some 0-form f. Then $d(f - \phi) = df - d\phi = \omega - \omega = 0$ in B. By part (e), $f - \phi$ is a constant in B. Since $\lim_{y\downarrow 0} \phi(1,y) = 0$ and $\lim_{y\uparrow 0} \phi(1,y) = 2\pi$, $f(1,y)$ has different limits when y approaches 0 through positive and negative values. This is a contradiction since f is C^1 function defined everywhere in A. □

6.

Proof. $d\eta = \sum_{i=1}^n (-1)^{i-1} D_i f_i dx_i \wedge dx_1 \wedge \cdots \widehat{dx_i} \wedge \cdots \wedge dx_n = \sum_{i=1}^n D_i f_i dx_1 \wedge \cdots \wedge dx_n$. So $d\eta = 0$ if and only if $\sum_{i=1}^n D_i f_i = 0$. Since $D_i f_i(x) = \frac{\|x\|^2 - m x_i^2}{\|x\|^{m+2}}$, $\sum_{i=1}^n D_i f_i(x) = \frac{n-m}{\|x\|^m}$. So $d\eta = 0$ if and only if $m = n$. □

7.

Proof. By linearity, it suffices to prove the theorem for $\omega = f dx_I$, where $I = (i_1, \cdots, i_{k-1})$ is a k-tuple from $\{1, \cdots, n\}$ in ascending order. Indeed, in this case, $h(x) = d(f dx_I)(x)((x; v_1), \cdots, (x; v_k)) = (\sum_{i=1}^n D_i f(x) dx_i \wedge dx_I)((x; v_1), \cdots, (x; v_k))$. Let $X = [v_1 \cdots v_k]$. For each $j \in \{1, \cdots, k\}$, let $Y_j = [v_1 \cdots \widehat{v}_j \cdots v_k]$. Then by Theorem 2.15 and Problem 4 of §28,

$$\det X(i, i_1, \cdots, i_{k-1}) = \sum_{j=1}^k (-1)^{j-1} v_{ij} \det Y_j(i_1, \cdots, i_{k-1}).$$

Therefore

$$\begin{aligned} h(x) &= \sum_{i=1}^n D_i f(x) \det X(i, i_1, \cdots, i_{k-1}) \\ &= \sum_{i=1}^n \sum_{j=1}^k D_i f(x) (-1)^{j-1} v_{ij} \det Y_j(i_1, \cdots, i_{k-1}) \\ &= \sum_{j=1}^k (-1)^{j-1} Df(x) \cdot v_j \det Y_j(i_1, \cdots, i_{k-1}). \end{aligned}$$

Meanwhile, $g_j(x) = \omega(x)((x; v_1), \cdots, \widehat{(x; v_j)}, \cdots, (x; v_k)) = f(x) \det Y_j(i_1, \cdots, i_{k-1})$. So

$$Dg_j(x) = Df(x) \det Y_j(i_1, \cdots, i_{k-1})$$

and consequently, $h(x) = \sum_{j=1}^k (-1)^{j-1} Dg_j(x) \cdot v_j$. In particular, for $k = 1$, $h(x) = Df(x) \cdot v$, which is a directional derivative. □

31 Application to Vector and Scalar Fields

1.

Proof. (Proof of Theorem 31.1) It is straightforward to check that α_i and β_j are isomorphisms. Moreover, $d \circ \alpha_0(f) = df = \sum_{i=1}^n D_i f dx_i$ and $\alpha_1 \circ \text{grad}(f) = \alpha_1((x; \sum_{i=1}^n D_i f(x) e_i)) = \sum_{i=1}^n D_i f(x) dx_i$. So $d \circ \alpha_0 = \alpha_1 \circ \text{grad}$.

Also, $d\circ\beta_{n-1}(G) = d(\sum_{i=1}^n(-1)^{i-1}g_i dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n) = \sum_{i=1}^n(-1)^{i-1}D_i g_i dx_i\wedge dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n = (\sum_{i=1}^n D_i g_i)dx_1\wedge\cdots\wedge dx_n$, and $\beta_n\circ\text{div}(G) = \beta_n(\sum_{i=1}^n D_i g_i) = (\sum_{i=1}^n D_i g_i)dx_1\wedge\cdots\wedge dx_n$. So $d\circ\beta_{n-1} = \beta_n\circ\text{div}$.

(Proof of Theorem 31.2) We only need to check $d\circ\alpha_1 = \beta_2\circ\text{curl}$. Indeed, $d\circ\alpha_1(F) = d(\sum_{i=1}^3 f_i dx_i) = (D_2f_1dx_2+D_3f_1dx_3)\wedge dx_1+(D_1f_2dx_1+D_3f_2dx_3)\wedge dx_2+(D_1f_3dx_1+D_2f_3dx_2)\wedge dx_3 = (D_2f_3-D_3f_2)dx_2\wedge dx_3+(D_3f_1-D_1f_3)dx_3\wedge dx_1+(D_1f_2-D_2f_1)dx_1\wedge dx_2$, and $\beta_2\circ\text{curl}(F) = \beta_2((x;(D_2f_3-D_3f_2)e_1+(D_3f_1-D_1f_3)e_2+(D_1f_2-D_2f_1)e_3)) = (D_2f_3-D_3f_2)dx_2\wedge dx_3-(D_3f_1-D_1f_3)dx_1\wedge dx_3+(D_1f_2-D_2f_1)dx_1\wedge dx_2$. So $d\circ\alpha_1 = \beta_2\circ\text{curl}$. □

2.

Proof. $\alpha_1 F = f_1dx_1 + f_2dx_2$ and $\beta_1 F = f_1dx_2 - f_2dx_1$. □

3. (a)

Proof. Let f be a scalar field in A and $F(x) = (x;[f_1(x), f_2(x), f_3(x)])$ be a vector field in A. Define $\omega_F^1 = f_1dx_1 + f_2dx_2 + f_3dx_3$ and $\omega_F^2 = f_1dx_2\wedge dx_3 + f_2dx_3\wedge dx_1 + f_3dx_1\wedge dx_2$. Then it is straightforward to check that $d\omega_F^1 = w^2_{\text{curl}F}$ and $d\omega_F^2 = (\text{div}F)dx_1\wedge dx_2\wedge dx_3$. So by the general principle $d(d\omega) = 0$, we have

$$0 = d(df) = d\left(\omega^1_{\text{grad}f}\right) = \omega^2_{\text{curl grad}f}$$

and

$$0 = d(d\omega_F^1) = d\left(\omega^2_{\text{curl }F}\right) = (\text{div curl}F)dx_1\wedge dx_2\wedge dx_3.$$

These two equations imply that curl grad$f = 0$ and div curl$F = 0$. □

4. (a)

Proof. $\gamma_2(\alpha H+\beta G) = \sum_{i<j}[\alpha h_{ij}(x)+\beta g_{ij}(x)]dx_i\wedge dx_j = \alpha\sum_{i<j}h_{ij}(x)dx_i\wedge dx_j+\beta\sum_{i<j}g_{ij}(x)dx_i\wedge dx_j = \alpha\gamma_2(H)+\beta\gamma_2(G)$. So γ_2 is a linear mapping. It is also easy to see γ_2 is one-to-one and onto as the skew-symmetry of H implies $h_{ii} = 0$ and $h_{ij} + h_{ji} = 0$. □

(b)

Proof. Suppose F is a vector field in A and $H\in\mathcal{S}(A)$. We define *twist* : {vector fields in A} $\to\mathcal{S}(A)$ by $\mathit{twist}(F)_{ij} = D_if_j - D_jf_i$, and *spin* : $\mathcal{S}(A)\to$ {vector fields in A} by $\mathit{spin}(H) = (x;(D_4h_{23} - D_3h_{24} + D_2h_{34}, -D_4h_{13} + D_3h_{14} - D_1h_{34}, D_4h_{12} - D_2h_{14} + D_1h_{24}, -D_3h_{12} + D_2h_{13} - D_1h_{23}))$. □

5. (a)

Proof. Suppose $\omega = \sum_{i=1}^n a_idx_i$ is a 1-form such that $\omega(x)(x;v) = \langle f(x), v\rangle$. Then $\sum_{i=1}^n a_i(x)v_i = \sum_{i=1}^n f_i(x)v_i$. Choose $v = e_i$, we conclude $a_i = f_i$. So $\omega = \alpha_1 F$. □

(b)

Proof. Suppose ω is an $(n-1)$ form such that $\omega(x)((x;v_1),\cdots,(x;v_{n-1})) = \varepsilon V(g(x), v_1,\cdots,v_{n-1})$. Assume ω has the representation $\sum_{i=1}^n a_idx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n$, then

$$\begin{aligned}
\omega(x)((x;v_1),\cdots,(x;v_{n-1})) &= \sum_{i=1}^n a_i(x)\det[v_1,\cdots,v_{n-1}]_{(1,\cdots,\widehat{i},\cdots,n)}\\
&= \sum_{i=1}^n(-1)^{i-1}[(-1)^{i-1}a_i(x)]\det[v_1,\cdots,v_{n-1}]_{(1,\cdots,\widehat{i},\cdots,n)}\\
&= \det[a(x), v_1,\cdots,v_{n-1}],
\end{aligned}$$

where $a(x) = [a_1(x),\cdots,(-1)^{i-1}a_i(x),\cdots,(-1)^{n-1}a_n(x)]^{Tr}$. Since

$$\varepsilon V(g(x), v_1,\cdots,v_{n-1}) = \det[g(x), v_1,\cdots,v_{n-1}],$$

we can conclude $\det[a(x), v_1, \cdots, v_{n-1}] = \det[g(x), v_1, \cdots, v_{n-1}]$, or equivalently,

$$\det[a(x) - g(x), v_1, \cdots, v_{n-1}] = 0.$$

Since $v_1, \cdots, v_{n-1}$ can be arbitrary, we must have $g(x) = a(x)$, i.e. $\omega = \sum_{i=1}^n (-1)^{i-1} g_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n = \beta_{n-1} G$. □

(c)

Proof. Suppose $\omega = f dx_1 \wedge \cdots \wedge dx_n$ is an n-form such that $\omega(x)((x; v_1), \cdots, (x; v_n)) = \varepsilon \cdot h(x) \cdot V(v_1, \cdots, v_n)$. This is equivalent to $f(x)\det[v_1, \cdots, v_n] = h(x)\det[v_1, \cdots, v_n]$. So $f = h$ and $\omega = \beta_n h$. □

32 The Action of a Differentiable Map

1.

Proof. Let ω, η and θ be 0-forms. Then
(1) $\beta^*(a\omega + b\eta) = a\omega \circ \beta + b\eta \circ \beta = a\beta^*(\omega) + b\beta^*(\eta)$.
(2) $\beta^*(\omega \wedge \theta) = \beta^*(\omega \cdot \theta) = \omega \circ \beta \cdot \theta \circ \beta = \beta^*(\omega) \cdot \beta^*(\theta) = \beta^*(\omega) \wedge \beta^*(\theta)$.
(3) $(\beta \circ \alpha)^* \omega = \omega \circ \beta \circ \alpha = \alpha^*(\omega \circ \beta) = \alpha^*(\beta^* \omega)$. □

2.

Proof.

$$\begin{aligned}
& d\alpha_1 \wedge d\alpha_3 \wedge d\alpha_5 \\
= \; & (D_1\alpha_1 dx_1 + D_2\alpha_1 dx_2 + D_3\alpha_1 dx_3) \wedge (D_1\alpha_3 dx_1 + D_2\alpha_3 dx_2 + D_3\alpha_3 dx_3) \\
& \wedge (D_1\alpha_5 dx_1 + D_2\alpha_5 dx_2 + D_3\alpha_5 dx_3) \\
= \; & (D_1\alpha_1 D_2\alpha_3 dx_1 \wedge dx_2 + D_1\alpha_1 D_3\alpha_3 dx_1 \wedge dx_3 + D_2\alpha_1 D_1\alpha_3 dx_2 \wedge dx_1 + D_2\alpha_1 D_3\alpha_3 dx_2 \wedge dx_3 \\
& + D_3\alpha_1 D_1\alpha_3 dx_3 \wedge dx_1 + D_3\alpha_1 D_2\alpha_3 dx_3 \wedge dx_2) \wedge (D_1\alpha_5 dx_1 + D_2\alpha_5 dx_2 + D_3\alpha_5 dx_3) \\
= \; & D_2\alpha_1 D_3\alpha_3 D_1\alpha_5 dx_2 \wedge dx_3 \wedge dx_1 + D_3\alpha_1 D_2\alpha_3 D_1\alpha_5 dx_3 \wedge dx_2 \wedge dx_1 + D_1\alpha_1 D_3\alpha_3 D_2\alpha_5 dx_1 \wedge dx_3 \wedge dx_2 \\
& + D_3\alpha_1 D_1\alpha_3 D_2\alpha_5 dx_3 \wedge dx_1 \wedge dx_2 + D_1\alpha_1 D_2\alpha_3 D_3\alpha_5 dx_1 \wedge dx_2 \wedge dx_3 + D_2\alpha_1 D_1\alpha_3 D_3\alpha_5 dx_2 \wedge dx_1 \wedge dx_3 \\
= \; & (D_2\alpha_1 D_3\alpha_3 D_1\alpha_5 - D_3\alpha_1 D_2\alpha_3 D_1\alpha_5 - D_1\alpha_1 D_3\alpha_3 D_2\alpha_5 + D_3\alpha_1 D_1\alpha_3 D_2\alpha_5 + D_1\alpha_1 D_2\alpha_3 D_3\alpha_5 \\
& - D_2\alpha_1 D_1\alpha_3 D_3\alpha_5) dx_1 \wedge dx_2 \wedge dx_3 \\
= \; & \det \begin{bmatrix} D_1\alpha_1 & D_2\alpha_1 & D_3\alpha_1 \\ D_1\alpha_3 & D_2\alpha_3 & D_3\alpha_3 \\ D_1\alpha_5 & D_2\alpha_5 & D_3\alpha_5 \end{bmatrix} dx_1 \wedge dx_2 \wedge dx_3 \\
= \; & \det D\alpha(1,3,5) dx_1 \wedge dx_2 \wedge dx_3.
\end{aligned}$$

So $\alpha^*(dy_{(1,3,5)} = \alpha^*(dy_1 \wedge dy_3 \wedge dy_5) = \alpha^*(dy_1) \wedge \alpha^*(dy_3) \wedge \alpha^*(dy_5) = d\alpha_1 \wedge d\alpha_3 \wedge d\alpha_5 = \det \frac{\partial \alpha_{(1,3,5)}}{\partial x} dx_1 \wedge dx_2 \wedge dx_3$. This confirms Theorem 32.2. □

3.

Proof. $d\omega = -xdx \wedge dy - 3dy \wedge dz$, $\alpha^*(\omega) = x \circ \alpha \cdot y \circ \alpha d\alpha_1 + 2z \circ \alpha d\alpha_2 - y \circ \alpha d\alpha_3 = u^3 v(udv + vdu) + 2(3u + v) \cdot (2udu) - u^2(3du + dv) = (u^3v^2 + 9u^2 + 4uv)du + (u^4 v - u^2)dv$. Therefore

$$\begin{aligned}
\alpha^*(d\omega) &= -x \circ \alpha d\alpha_1 \wedge d\alpha_2 - 3d\alpha_2 \wedge d\alpha_3 \\
&= -uv(udv + vdu) \wedge (2udu) - 2(2udu) \wedge (3du + dv) - (2udu) \wedge (3du + dv) \\
&= (2u^3 v - 6u) du \wedge dv,
\end{aligned}$$

and

$$\begin{aligned}
d(\alpha^*\omega) &= (2u^3vdv + 4udv) \wedge du + (4u^3vdu - 2udu) \wedge dv \\
&= (-2u^3v - 4u + 4u^3v - 2u)du \wedge dv \\
&= (2u^3v - 6u)du \wedge dv.
\end{aligned}$$

So $\alpha^*(d\omega) = d(\alpha^*\omega)$. $\square$

4.

Proof. Note $\alpha^* y_i = y_i \circ \alpha = \alpha_i$. $\square$

5.

Proof. $\alpha^*(dy_I)$ is an l-form in A, so we can write it as $\alpha^*(dy_I) = \sum_{[J]} h_J dx_J$, where J is an ascending l-tuple form the set $\{1, \cdots, k\}$. Fix $J = (j_1, \cdots, j_l)$, we have

$$\begin{aligned}
h_J(x) &= \alpha^*(dy_I)(x)((x; e_{j_1}), \cdots, (x; e_{j_l})) \\
&= (dy_I)(x)(\alpha_*(x; e_{j_1}), \cdots, \alpha_*(x; e_{j_l})) \\
&= (dy_I)(x)((\alpha(x); D_{j_1}\alpha(x)), \cdots, (\alpha(x); D_{j_l}\alpha(x))) \\
&= \det[D_{j_1}\alpha(x), \cdots, D_{j_l}\alpha(x)]_I \\
&= \det \frac{\partial \alpha_I}{\partial x_J}(x).
\end{aligned}$$

Therefore $\alpha^*(dy_I) = \sum_{[J]} \left(\det \frac{\partial \alpha_I}{\partial x_J}\right) dx_J$. $\square$

6. (a)

Proof. We fix $x \in A$ and denote $\alpha(x)$ by y. Then $G(y) = \alpha_*(F(x)) = (y; D\alpha(x) \cdot f(x))$. Define $g(y) = D\alpha(x) \cdot f(x) = (D\alpha \cdot f)(\alpha^{-1}(y))$. Then $g_i(y) = (\sum_{j=1}^n D_j\alpha_i f_j)(\alpha^{-1}(y))$ and we have

$$\alpha^*(\alpha_1 G) = \alpha^*(\sum_{i=1}^n g_i dy_i) = \sum_{i=1}^n g_i \circ \alpha d\alpha_i = \sum_{i=1}^n g_i \circ \alpha \sum_{j=1}^n D_j\alpha_j dx_j = \sum_{j=1}^n (\sum_{i=1}^n D_j\alpha_i g_i \circ \alpha) dx_j.$$

Therefore $\alpha^*(\alpha_1 G) = \alpha_1 F$ if and only if

$$f_j = \sum_{i=1}^n D_j\alpha_i g_i \circ \alpha = \sum_{i=1}^n D_j\alpha_i \sum_{k=1}^n D_k\alpha_i f_k = [D_j\alpha_1 D_j\alpha_2 \cdots D_j\alpha_n] \cdot D\alpha \cdot f,$$

that is, $D\alpha(x)^{tr} \cdot D\alpha(x) \cdot f(x) = f(x)$. So $\alpha^*(\alpha_1 G) = \alpha_1 F$ if and only if $D\alpha(x)$ is an orthogonal matrix for each x. $\square$

(b)

Proof. $\beta_{n-1}F = \sum_{i=1}^n (-1)^{i-1} f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ and

$$\begin{aligned}
\alpha^*(\beta_{n-1}G) &= \alpha^*(\sum_{i=1}^n (-1)^{i-1} g_i dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dy_n) \\
&= \sum_{i=1}^n (-1)^{i-1} (g_i \circ \alpha) \alpha^*(dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dy_n) \\
&= \sum_{i=1}^n (-1)^{i-1} (\sum_{j=1}^n D_j\alpha_i f_j) \left(\sum_{k=1}^n \det \frac{\partial(\alpha_1, \cdots, \widehat{\alpha_i}, \cdots, \alpha_n)}{\partial(x_1, \cdots, \widehat{x_k}, \cdots, x_n)} dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n \right).
\end{aligned}$$

So $\alpha^*(\beta_{n-1}F) = \beta_{n-1}F$ if and only if for any $k \in \{1, \cdots, n\}$,

$$\begin{aligned} f_k &= \sum_{i,j=1}^{n} (-1)^{k+i} D_j\alpha_i f_j \det \frac{\partial(\alpha_1, \cdots, \widehat{\alpha_i}, \cdots, \alpha_n)}{\partial(x_1, \cdots, \widehat{x_k}, \cdots, x_n)} \\ &= \sum_{j=1}^{n} f_j \sum_{i=1}^{n} (-1)^{k+i} D_j\alpha_i \det \frac{\partial(\alpha_1, \cdots, \widehat{\alpha_i}, \cdots, \alpha_n)}{\partial(x_1, \cdots, \widehat{x_k}, \cdots, x_n)} \\ &= \sum_{j=1}^{n} f_j \delta_{kj} \det D\alpha \\ &= f_k \det D\alpha. \end{aligned}$$

Since F can be arbitrary, $\alpha^*(\beta_{n-1}F) = \beta_{n-1}F$ if and only if $\det D\alpha = 1$. □

(c)

Proof. $\alpha^*(\beta_n k) = \alpha^*(k dy_1 \wedge \cdots \wedge dy_n) = k \circ \alpha \cdot \alpha^*(dy_1 \wedge \cdots \wedge dy_n) = h \cdot \det D\alpha \cdot dx_1 \wedge \cdots \wedge dx_n$ and $\beta_n h = h dx_1 \wedge \cdots \wedge dx_n$. So $\alpha^*(\beta_n k) = \beta_n h$ for all h if and only if $\det D\alpha = 1$. □

7.

Proof. If α is an orientation-preserving isometry of $\mathbb{R}^n$, Exercise 6 implies $\alpha^*(\alpha_1 G) = \alpha_1 F$, $\alpha^*(\beta_{n-1}G) = \beta_{n-1}F$, and $\alpha^*(\beta_n k) = \beta_n h$, where F, G, h and k are as defined in Exercise 6. Fix $x \in A$ and let $y = \alpha(x)$. We need to show
(1) $\widetilde{\alpha}_*(\text{div}F)(y) = \text{div}(\widetilde{\alpha}_*(F))(y)$. Indeed, $\text{div}(\widetilde{\alpha}_*(F))(y) = \text{div}G(y)$, and

$$\begin{aligned} \widetilde{\alpha}_*(\text{div}F)(y) &= \text{div}F(x) = \beta_n^{-1} \circ \beta(\text{div}F)(x) = \beta_n^{-1} \circ d(\beta_{n-1}F)(x) = \beta_n^{-1} \circ d(\alpha^*(\beta_{n-1}G))(x) \\ &= \beta_n^{-1} \circ \alpha^* \circ d(\beta_{n-1}G)(x) = \beta_n^{-1} \circ \alpha^* \circ \beta_n(\text{div}G)(x). \end{aligned}$$

For any function $g \in C^\infty(B)$,

$$\beta_n^{-1} \circ \alpha^* \circ \beta_n(g) = \beta_n^{-1} \circ \alpha^*(g dy_1 \wedge \cdots \wedge dy_n) = \beta_n^{-1}(g \circ \alpha \cdot \det D\alpha \cdot dx_1 \wedge \cdots \wedge dx_n) = g \circ \alpha.$$

So

$$\widetilde{\alpha}_*(\text{div}F)(y) = \beta_n^{-1} \circ \alpha^* \circ \beta_n(\text{div}G)(x) = \text{div}G(\alpha(x)) = \text{div}G(y) = \text{div}(\widetilde{\alpha}_*(F))(y).$$

(2) $\widetilde{\alpha}_*(\text{grad}h) = \text{grad} \circ \widetilde{\alpha}_*(h)$. Indeed,

$$\widetilde{\alpha}_*(\text{grad}h)(y) = \alpha_*(\text{grad}h \circ \alpha^{-1}(y)) = \alpha_*(\text{grad}h(x)) = (y; D\alpha(x) \cdot \begin{bmatrix} D_1h(x) \\ \cdots \\ D_nh(x) \end{bmatrix}) = (y; D\alpha(x) \cdot (Dh(x))^{tr}),$$

and

$$\begin{aligned} \text{grad} \circ \widetilde{\alpha}_*(h)(y) &= \text{grad}(h \circ \alpha^{-1})(y) \\ &= (y; [D(h \circ \alpha^{-1})(y)]^{tr}) \\ &= (y; [Dh(\alpha^{-1}(y)) \cdot D\alpha^{-1}(y)]^{tr}) \\ &= (y; [Dh(x) \cdot (D\alpha(x))^{-1}]^{tr}). \end{aligned}$$

Since $D\alpha$ is orthogonal, we have

$$\text{grad} \circ \widetilde{\alpha}_*(h)(y) = (y; [Dh(x) \cdot (D\alpha(x))^{tr}]^{tr}) = (y; D\alpha(x) \cdot (Dh(x))^{tr}) = \widetilde{\alpha}_*(\text{grad}h)(y).$$

(3) For $n=3$, $\widetilde{\alpha}_*(\text{curl}F) = \text{curl}(\widetilde{\alpha}_* F)$. Indeed, $\text{curl}(\widetilde{\alpha}_* F)(y) = \text{curl}G(y)$, and

$$\begin{aligned}
\widetilde{\alpha}_*(\text{curl}F)(y) &= \alpha_*(\text{curl}F(\alpha^{-1}(y))) \\
&= \alpha_*(\beta_2^{-1}\circ\beta_2\circ\text{curl}F(x)) \\
&= \alpha_*(\beta_2^{-1}\circ d\circ\alpha_1 F(x)) \\
&= \alpha_*(\beta_2^{-1}\circ d\circ\alpha^*\circ\alpha_1 G(x)) \\
&= \alpha_*(\beta_2^{-1}\circ\alpha^*\circ d\circ\alpha_1 G(x)) \\
&= \alpha_*(\beta_2^{-1}\circ\alpha^*\circ\beta_2\circ\text{curl}G(x))
\end{aligned}$$

Let H be a vector field in B, we show $\alpha_*(\beta_2^{-1}\circ\alpha^*\circ\beta_2(H)(x)) = H(\alpha(x)) = H(y)$. Indeed,

$$\begin{aligned}
&\alpha_*(\beta_2^{-1}\circ\alpha^*\circ\beta_2(H)(x)) \\
=\ &\alpha_*(\beta_2^{-1}\circ\alpha^*(\sum_{i=1}^n(-1)^{i-1}h_i dy_1\wedge\cdots\wedge\widehat{dy_i}\wedge\cdots\wedge dy_n)) \\
=\ &\alpha_*\circ\beta_2^{-1}\left(\sum_{i=1}^n(-1)^{i-1}h_i\circ\alpha\sum_{j=1}^n\det\frac{\partial(\alpha_1,\cdots,\widehat{\alpha_i},\cdots,\alpha_n)}{\partial(x_1,\cdots,\widehat{x_j},\cdots,x_n)}dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n\right) \\
=\ &\alpha_*\circ\beta_2^{-1}\left(\sum_{j=1}^n\left(\sum_{i=1}^n(-1)^{i-1}h_i\circ\alpha\cdot\det\frac{\partial(\alpha_1,\cdots,\widehat{\alpha_i},\cdots,\alpha_n)}{\partial(x_1,\cdots,\widehat{x_j},\cdots,x_n)}\right)dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n\right) \\
=\ &\alpha_*\left(\sum_{j=1}^n\left(\sum_{i=1}^n(-1)^{i+j}h_i\circ\alpha\cdot\det\frac{\partial(\alpha_1,\cdots,\widehat{\alpha_i},\cdots,\alpha_n)}{\partial(x_1,\cdots,\widehat{x_j},\cdots,x_n)}\right)e_j\right).
\end{aligned}$$

Using the definition of α_* and the fact that $\det D\alpha = 1$, we have

$$\begin{aligned}
&\alpha_*\left(\sum_{j=1}^n\left(\sum_{i=1}^n(-1)^{i+j}h_i\circ\alpha\cdot\det\frac{\partial(\alpha_1,\cdots,\widehat{\alpha_i},\cdots,\alpha_n)}{\partial(x_1,\cdots,\widehat{x_j},\cdots,x_n)}\right)e_j\right) \\
=\ &D\alpha(x)\cdot\begin{bmatrix}\sum_{i=1}^n(-1)^{i+1}h_i\circ\alpha\cdot\det\frac{\partial(\alpha_1,\cdots,\widehat{\alpha_i},\cdots,\alpha_n)}{\partial(\widehat{x_1},\cdots,x_n)} \\ \cdots \\ \sum_{i=1}^n(-1)^{i+j}h_i\circ\alpha\cdot\det\frac{\partial(\alpha_1,\cdots,\widehat{\alpha_i},\cdots,\alpha_n)}{\partial(x_1,\cdots,\widehat{x_j},\cdots,x_n)} \\ \cdots \\ \sum_{i=1}^n(-1)^{i+n}h_i\circ\alpha\cdot\det\frac{\partial(\alpha_1,\cdots,\widehat{\alpha_i},\cdots,\alpha_n)}{\partial(x_1,\cdots,\widehat{x_n})}\end{bmatrix}.
\end{aligned}$$

So the k-th component of the above column vector is

$$\begin{aligned}
&\sum_{j=1}^n D_j\alpha_k\sum_{i=1}^n(-1)^{i+j}h_i\circ\alpha\cdot\det\frac{\partial(\alpha_1,\cdots,\widehat{\alpha_i},\cdots,\alpha_n)}{\partial(x_1,\cdots,\widehat{x_j},\cdots,x_n)} \\
=\ &\sum_{i=1}^n h_i\circ\alpha\sum_{j=1}^n(-1)^{i+j}D_j\alpha_k\det\frac{\partial(\alpha_1,\cdots,\widehat{\alpha_i},\cdots,\alpha_n)}{\partial(x_1,\cdots,\widehat{x_j},\cdots,x_n)} \\
=\ &h_k\circ\alpha\det D\alpha \\
=\ &h_k\circ\alpha.
\end{aligned}$$

Thus, we have proved $\alpha_*(\beta_2^{-1}\circ\alpha^*\circ\beta_2(H)(x)) = H(y)$. Replace H with $\text{curl}G$, we have

$$\widetilde{\alpha}_*(\text{curl}F)(y) = \text{curl}G(y) = \text{curl}(\widetilde{\alpha}_* F)(y).$$

□

33 Integrating Forms over Parametrized-Manifolds

1.

Proof. $\int_{Y_\alpha}(x_2dx_2\wedge dx_3 + x_1x_3dx_1\wedge dx_3) = \int_A v\text{det}\begin{bmatrix}0 & 1\\ 2u & 2v\end{bmatrix} + u(u^2+v^2+1)\text{det}\begin{bmatrix}1 & 0\\ 2u & 2v\end{bmatrix} = \int_A -2uv + 2uv(u^2+v^2+1) = 1.$ □

2.

Proof.

$$\begin{aligned}
&\int_{Y_\alpha} x_1dx_1\wedge dx_4\wedge dx_3 + 2x_2x_3dx_1\wedge dx_2\wedge dx_3\\
=&\int_A \alpha^*(-x_1dx_1\wedge dx_3\wedge dx_4 + 2x_2x_3dx_1\wedge dx_2\wedge dx_3)\\
=&\int_A\left[-s\text{det}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 4(2u-t) & 2(t-2u)\end{bmatrix} + 2ut\text{det}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right] ds\wedge du\wedge dt\\
=&\int_A 4s(2u-t)+2ut\\
=&\ 6.
\end{aligned}$$

□

3. (a)

Proof.

$$\begin{aligned}
&\int_{Y_\alpha}\frac{1}{\|x\|^m}(x_1dx_2\wedge dx_3 - x_2dx_1\wedge dx_3 + x_3dx_1\wedge dx_2)\\
=&\int_A\frac{1}{\|(u,v,(1-u^2-v^2)^{1/2})\|^m}\left[u\text{det}\frac{\partial(x_2,x_3)}{\partial(u,v)} - v\text{det}\frac{\partial(x_1,x_3)}{\partial(u,v)} + (1-u^2-v^2)^{1/2}\text{det}\frac{\partial(x_1,x_2)}{\partial(u,v)}\right]\\
=&\int_A u\text{det}\begin{bmatrix}0 & 1\\ -\frac{u}{1-u^2-v^2} & -\frac{v}{\sqrt{1-u^2-v^2}}\end{bmatrix} - v\text{det}\begin{bmatrix}1 & 0\\ -\frac{u}{1-u^2-v^2} & -\frac{v}{\sqrt{1-u^2-v^2}}\end{bmatrix} + (1-u^2-v^2)^{1/2}\text{det}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\
=&\int_A\frac{u^2}{\sqrt{1-u^2-v^2}}+\frac{v^2}{\sqrt{1-u^2-v^2}}+\sqrt{1-u^2-v^2}\\
=&\int_A\frac{1}{\sqrt{1-u^2-v^2}}.
\end{aligned}$$

Apply change-of-variable, $\begin{cases}u = r\cos\theta\\ v = r\sin\theta\end{cases}$ $(0\le r\le 1, 0\le\theta<2\pi)$, we have

$$\int_A\frac{1}{\sqrt{1-u^2-v^2}} = \int_{[0,1]^2}\frac{1}{\sqrt{1-r^2}}\text{det}\begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix} = 2\pi.$$

□

(b)

Proof. -2π. □

4.

Proof. Suppose η has the representation $\eta = f dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$, where dx_i is the standard elementary 1-form depending on the standard basis $e_1, \cdots, e_k$ in $\mathbb{R}^k$. Let $a_1,\cdots,a_k$ be another basis for $\mathbb{R}^k$ and define $A = [a_1, \cdots, a_k]$. Then

$$\eta(x)((x; a_1), \cdots, (x; a_k)) = f(x)\det A.$$

If the frame $(a_1, \cdots, a_k)$ is orthonormal and right-handed, $\det A = 1$. We consequently have

$$\int_A \eta = \int_A f = \int_{x\in A} \eta(x)((x; a_1), \cdots, (x; a_k)).$$

□

34 Orientable Manifolds

1.

Proof. Let $\alpha : U_\alpha \to V_\alpha$ and $\beta : U_\beta \to V_\beta$ be two coordinate patches and suppose $W:V_\alpha \cap V_\beta$ is non-empty. $\forall p \in W$, denote by x and y the points in $\alpha^{-1}(W)$ and $\beta^{-1}(W)$ such that $\alpha(x) = p = \beta(y)$, respectively. Then

$$D\alpha^{-1} \circ \beta(y) = D\alpha^{-1}(p) \cdot D\beta(y) = [D\alpha(x)]^{-1} \cdot D\beta(y).$$

So $\det D\alpha^{-1} \circ \beta(y) = [\det D\alpha(x)]^{-1} \det D\beta(y) > 0$. Since p is arbitrarily chosen, we conclude α and β overlap positively. □

2.

Proof. Let $\alpha : U_\alpha \to V_\alpha$ and $\beta : U_\beta \to V_\beta$ be two coordinate patches and suppose $W := V_\alpha \cap V_\beta$ is non-empty. $\forall p \in W$, denote by x and y the points in $\alpha^{-1}(W)$ and $\beta^{-1}(W)$ such that $\alpha(x) = p = \beta(y)$, respectively. Then

$$\begin{aligned} D(\alpha \circ r)^{-1} \circ (\beta \circ r)(r^{-1}(y)) &= D(\alpha \circ r)^{-1}(p) \cdot D(\beta \circ r)(r^{-1}(y)) \\ &= D(r^{-1} \circ \alpha^{-1})(p) \cdot D(\beta \circ r)(r^{-1}(y)) \\ &= Dr^{-1}(x) D\alpha^{-1}(p) \cdot D\beta(y) \cdot Dr(r^{-1}(y)). \end{aligned}$$

Note $r^{-1} = r$ and $\det Dr = \det Dr^{-1} = -1$, we have

$$\det(D(\alpha \circ r)^{-1} \circ (\beta \circ r)(r^{-1}(y))) = [\det D\alpha(x)]^{-1} \det D\beta(y).$$

So if α and β overlap positively, so do $\alpha \circ r$ and $\beta \circ r$. □

3.

Proof. Denote by n the unit normal field corresponding to the orientation of M. Then $[n, T]$ is right-handed, i.e. $\det[n, T] > 0$. □

4.

Proof. $\frac{\partial\alpha}{\partial u} = \begin{bmatrix} -2\pi\sin(2\pi u) \\ 2\pi\cos(2\pi u) \\ 0 \end{bmatrix}$, $\frac{\partial\alpha}{\partial v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. We need to find $n = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$, such that $\det\left[n, \frac{\partial\alpha}{\partial u}, \frac{\partial\alpha}{\partial v}\right] > 0$, $\|n\| = 1$, and $n \perp \text{span}\{\frac{\partial\alpha}{\partial u}, \frac{\partial\alpha}{\partial v}\}$. Indeed, $\langle n, \frac{\partial\alpha}{\partial v}\rangle = 0$ implies $n_3 = 0$, $\langle n, \frac{\partial\alpha}{\partial u}\rangle = 0$ implies $-n_1\sin(2\pi u) + n_2\cos(2\pi u) = 0$. Combined with the condition $n_1^2 + n_2^2 + n_3^2 = n_1^2 + n_2^2 = 1$ and $\det \begin{bmatrix} n_1 & -2\pi\sin(2\pi u) & 0 \\ n_2 & 2\pi\cos(2\pi u) & 0 \\ 0 & 0 & 1 \end{bmatrix} = (n_1\cos(2\pi u) + n_2\sin(2\pi u)) \cdot 2\pi > 0$, we can solve for n_1 and n_2: $\begin{cases} n_1 = \cos(2\pi u) \\ n_2 = \sin(2\pi u) \end{cases}$. So the unit normal field corresponding to this orientation of C is given by $n = \begin{bmatrix} \cos(2\pi u) \\ \sin(2\pi u) \\ 0 \end{bmatrix}$. In particular, for $u = 0$, $\alpha(0, v) = (1, 0, v)$ and $n = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. So n points outwards.

By Example 5, the orientation of $\{(x, y, z) : x^2 + y^2 = 1, z = 0\}$ is counter-clockwise and the orientation of $\{(x, y, z) : x^2 + y^2 = 1, z = 0\}$ is clockwise. $\square$

5.

Proof. We can regard M as a 2-manifold in $\mathbb{R}^3$ and apply Example 5. The unit normal vector of M as a 2-manifold is perpendicular to the plane where M lies on and points towards us. Example 5 then gives the unit tangent vector field corresponding to the induced orientation of ∂M. Denote by n the unit normal field corresponding to ∂M. If α is a coordinate patch of M, $[n, \frac{\partial \alpha}{\partial x_1}]$ is right-handed. Since $[\frac{\partial \alpha}{\partial x_1}, \frac{\partial \alpha}{\partial x_2}]$ is right-handed and $\frac{\partial \alpha}{\partial x_2}$ points into M, n points outwards from M.

Alternatively, we can apply Lemma 38.7. $\square$

6. (a)

Proof. The meaning of "well-defined" is that if x is covered by more than one coordinate patch of the same coordinate system, the definition of $\lambda(x)$ is unchanged. More precisely, assume x is both covered by α_{i_1} and α_{i_2}, as well as β_{j_1} and β_{j_2}, $\det D(\alpha_{i_1}^{-1} \circ \beta_{j_1})(\beta_{j_1}^{-1}(x))$ and $\det D(\alpha_{i_2}^{-1} \circ \beta_{j_2})(\beta_{j_2}^{-1}(x))$ have the same sign. Indeed,

$$\begin{aligned} & \det D(\alpha_{i_1}^{-1} \circ \beta_{j_1})(\beta_{j_1}^{-1}(x)) \\ = \; & \det D(\alpha_{i_1}^{-1} \circ \alpha_{i_2} \circ \alpha_{i_2}^{-1} \circ \beta_{j_2} \circ \beta_{j_2}^{-1} \circ \beta_{j_1})(\beta_{j_1}^{-1}(x)) \\ = \; & \det D(\alpha_{i_1}^{-1} \circ \alpha_{i_2})(\alpha_{i_2}^{-1}(x)) \cdot \det D(\alpha_{i_2}^{-1} \circ \beta_{j_2})(\beta_{j_2}^{-1}(x)) \cdot \det D(\beta_{j_2}^{-1} \circ \beta_{j_1})(\beta_{j_1}^{-1}(x)). \end{aligned}$$

Since $\det D(\alpha_{i_1}^{-1} \circ \alpha_{i_2}) > 0$ and $\det D(\beta_{j_2}^{-1} \circ \beta_{j_1}) > 0$, we can conclude $\det D(\alpha_{i_1}^{-1} \circ \beta_{j_1})(\beta_{j_1}^{-1}(x))$ and $\det D(\alpha_{i_2}^{-1} \circ \beta_{j_2})(\beta_{j_2}^{-1}(x))$ have the same sign. $\square$

(b)

Proof. $\forall x, y \in M$. When x and y are sufficiently close, they can be covered by the same coordinate patch α_i and β_j. Since $\det D\alpha_i^{-1} \circ \beta_j$ does not change sign in the place where α_i and β_j overlap (recall $\alpha_i^{-1} \circ \beta_j$ is a diffeomorphism from an open subset of $\mathbb{R}^k$ to an open subset of $\mathbb{R}^k$), we conclude λ is a constant, in the place where α_i and β_j overlap. In particular, λ is continuous. $\square$

(c)

Proof. Since λ is continuous and λ is either 1 or -1, by the connectedness of M, λ must be a constant. More precisely, as the proof of part (b) has shown, $\{x \in M : \lambda(x) = 1\}$ and $\{x \in M : \lambda(x) = -1\}$ are both open sets. Since M is connected, exactly one of them is empty. $\square$

(d)

Proof. This is straightforward from part (a)-(c). $\square$

7.

Proof. By Example 4, the unit normal vector corresponding to the induced orientation of ∂M points outwards from M. This is a special case of Lemma 38.7. $\square$

8.

Proof. We consider a general problem similar to that of Example 4: *Let M be an n-manifold in $\mathbb{R}^n$, oriented naturally, what is the induced orientation of ∂M?*

Suppose $h : U \to V$ is a coordinate patch on M belonging to the natural orientation of M, about the point p of ∂M. Then the map

$$h \circ b(x) = h(x_1, \cdots, x_{n-1}, 0)$$

gives the restricted coordinate patch on ∂M about p. The normal field $N = (p\,; T)$ to ∂M corresponding to the induced orientation satisfies the condition that the frame

$$\left[(-1)^n T(p), \frac{\partial h(h^{-1}(p))}{\partial x_1}, \cdots, \frac{\partial h(h^{-1}(p))}{\partial x_{n-1}}\right]$$

is right-handed. Since Dh is right-handed, $(-1)^n T$ and $(-1)^{n-1}\frac{\partial h}{\partial x_n}$ lie on the same side of the tangent plane of M at p. Since $\frac{\partial h}{\partial x_n}$ points into M, T points outwards from M. Thus, the induced orientation of ∂M is characterized by the normal vector field to M pointing outwards from M. This is essentially Lemma 38.7.

To determine whether or not a coordinate patch on ∂M belongs to the induced orientation of ∂M, we suppose α is a coordinate patch on ∂M about p. Define $A(p) = D(h^{-1} \circ \alpha)(\alpha^{-1}(p))$. Then α belongs to the induced orientation if and only if $\text{sgn}(\det A(p)) = (-1)^n$. Since $D\alpha(\alpha^{-1}(p)) = Dh((h^{-1}(p)) \cdot A(p)$, we have

$$[(-1)^n T(p), D\alpha(\alpha^{-1}(p))] = \left[(-1)^n T(p), \frac{\partial h(h^{-1}(p))}{\partial x_1}, \cdots, \frac{\partial h(h^{-1}(p))}{\partial x_{n-1}}\right] \begin{bmatrix} 1 & 0 \\ 0 & A(p) \end{bmatrix}.$$

Therefore, α belongs to the induced orientation if and only if $[T(p), D\alpha(\alpha^{-1}(p))]$ is right-handed.

Back to our particular problem, the unit normal vector to S^{n-1} at p is $\frac{p}{\|p\|}$. So α belongs to the orientation of S^{n-1} if and only if $\left[p, D\alpha(\alpha^{-1}(p))\right]$ is right-handed. If $\alpha(u) = p$, we have

$$\left[p, D\alpha(\alpha^{-1}(p))\right] = \begin{bmatrix} u_1 & 1 & 0 & \cdots & 0 & 0 \\ u_2 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{n-1} & 0 & 0 & \cdots & 0 & 1 \\ \sqrt{1-\|u\|^2} & \frac{-u_1}{\sqrt{1-\|u\|^2}} & \frac{-u_2}{\sqrt{1-\|u\|^2}} & \cdots & \frac{-u_{n-2}}{\sqrt{1-\|u\|^2}} & \frac{-u_{n-1}}{\sqrt{1-\|u\|^2}} \end{bmatrix}.$$

Plain calculation yields $\det[p, D\alpha(\alpha^{-1}(p))] = (-1)^{n+1}/\sqrt{1-\|u\|^2}$. So α belongs to the orientation of S^{n-1} if and only if n is odd. Similarly, we can show β belongs to the orientation of S^{n-1} if and only if n is even. □

35 Integrating Forms over Oriented Manifolds

Notes. We view Theorem 17.1 (Substitution rule) in the light of integration of a form over an oriented manifold. The theorem states that, under certain conditions, $\int_{g((a,b))} f = \int_{(a,b)} (f \circ g)|g'|$. Throughout this note, we assume $a < b$. We also assume that when dx or dy appears in the integration formula, the formula means integration of a differential form over a manifold; when dx or dy is missing, the formula means Riemann integration over a domain.

First, as a general principle, $\int_a^b f(x)dx$ is regarded as the integration of the 1-form $f(x)dx$ over the naturally oriented manifold (a,b), and is therefore equal to $\int_{(a,b)} f$ by definition. Similarly, $\int_b^a f(x)dx$ is regarded as the integration of $f(x)dx$ over the manifold (a,b) whose orientation is reverse to the natural orientation, and is therefore equal to $-\int_a^b f(x)dx = -\int_{(a,b)} f$.

Second, if $g' > 0$, then $g(a) < g(b)$ and $\int_{g(a)}^{g(b)} f(y)dy$ is the integration of the 1-form $f(y)dy$ over the naturally oriented manifold $(g(a), g(b))$ with g a coordinate patch. So $\int_{g((a,b))} f = \int_{g(a)}^{g(b)} f(y)dy = \int_{(a,b)} g^*(f(y)dy) = \int_{(a,b)} f(g(x))g'(x)dx = \int_{(a,b)} f(g)g'$. If $g' < 0$, then $g(a) > g(b)$ and $\int_{g(a)}^{g(b)} f(y)dy$ is the integration of the 1-form $f(y)dy$ over the manifold $(g(b), g(a))$ whose orientation is reverse to the natural orientation. So $\int_{g((a,b))} f = -\int_{g(a)}^{g(b)} f(y)dy = -\int_{(a,b)} g^*(f(y)dy) = -\int_{(a,b)} f(g(x))g'(x)dx = \int_{(a,b)} f(g)(-g')$. Combined, we can conclude $\int_{g((a,b))} f = \int_{(a,b)} (f \circ g)|g'|$.

3. (a)

Proof. By Exercise 8 of §34, α and β always belong to different orientations of S^{n-1}. By Exercise 6 of §34, α and β belong to opposite orientations of S^{n-1}. □

(b)

Proof. Assume $\beta^*\eta = -\alpha^*\eta$, then by Theorem 35.2 and part (a)

$$\int_{S^{n-1}} \eta = \int_{S^{n-1}\cap\{x\in\mathbb{R}^n: x_n>0\}} \eta + \int_{S^{n-1}\cap\{x\in\mathbb{R}^n: x_n<0\}} \eta = \int_A \alpha^*\eta + (-1)\int_A \beta^*\eta = 2\int_A \alpha^*\eta.$$

Now we show $\beta^*\eta = -\alpha^*\eta$. Indeed, using our calculation in Exercise 8 of §34, we have

$$D\alpha(u) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \\ 0 & 1 & \cdots & 0 & 0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 & \\ \frac{-u_1}{\sqrt{1-\|u\|^2}} & \frac{-u_2}{\sqrt{1-\|u\|^2}} & \cdots & \frac{-u_{n-2}}{\sqrt{1-\|u\|^2}} & \frac{-u_{n-1}}{\sqrt{1-\|u\|^2}} & \end{bmatrix},$$

and

$$D\beta(u) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \\ 0 & 1 & \cdots & 0 & 0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 & \\ \frac{u_1}{\sqrt{1-\|u\|^2}} & \frac{u_2}{\sqrt{1-\|u\|^2}} & \cdots & \frac{u_{n-2}}{\sqrt{1-\|u\|^2}} & \frac{u_{n-1}}{\sqrt{1-\|u\|^2}} & \end{bmatrix}.$$

So for any $x \in A$,

$$\begin{aligned}
\alpha^*\eta(x) &= \sum_{i=1}^{n}(-1)^{i-1} f_i\circ\alpha(u)\det D\alpha(1,\cdots,\widehat{i},\cdots,n)du_1\wedge\cdots\wedge du_{n-1} \\
&= \left\{\sum_{i=1}^{n-1} u_i(-1)^{n-1-i}\frac{-u_i}{\sqrt{1-\|u\|^2}} + (-1)^{n-1}\sqrt{1-\|u\|^2}\right\} du_1\wedge\cdots\wedge du_{n-1} \\
&= -\left\{\sum_{i=1}^{n-1} u_i(-1)^{n-1-i}\frac{u_i}{\sqrt{1-\|u\|^2}} + (-1)^{n-1}(-1)\sqrt{1-\|u\|^2}\right\} du_1\wedge\cdots\wedge du_{n-1} \\
&= -\sum_{i=1}^{n}(-1)^{i-1} f_i\circ\beta(u)\det D\beta(1,\cdots,\widehat{i},\cdots,n)du_1\wedge\cdots\wedge du_{n-1} \\
&= -\beta^*\eta(x).
\end{aligned}$$

□

(c)

Proof. By our calculation in part (b), we have

$$\begin{aligned}
\int_A \alpha^*\eta &= \int_A \sum_{i=1}^{n-1}(-1)^{i-1}u_i(-1)^{n-i}\frac{u_i}{\sqrt{1-\|u\|^2}} + (-1)^{n-1}\sqrt{1-\|u\|^2} \\
&= (-1)^{n-1}\int_A \frac{\sum_{i=1}^{n-1} u_i^2}{\sqrt{1-\|u\|^2}} + \sqrt{1-\|u\|^2} \\
&= \pm\int_A \frac{1}{\sqrt{1-\|u\|^2}} \neq 0.
\end{aligned}$$

□

36 A Geometric Interpretation of Forms and Integrals

1.

Proof. Define $b_i = [D(\alpha^{-1}\circ\beta)(y)]^{-1}a_i = D(\beta^{-1}\circ\alpha)(x)a_i$. Then

$$\begin{aligned}\beta_*(y;b_i) &= (p\,;D\beta(y)b_i)\\ &= (p\,;D\beta(y)[D(\alpha^{-1}\circ\beta)(y)]^{-1}a_i)\\ &= (p\,;D\beta(y)D(\beta^{-1}\circ\alpha)(x)a_i)\\ &= (p\,;D\alpha(x)a_i)\\ &= \alpha_*(x;a_i).\end{aligned}$$

Moreover, $[b_1,\cdots,b_k] = D(\beta^{-1}\circ\alpha)(x)[a_1,\cdots,a_k]$. Since $\det D(\beta^{-1}\circ\alpha)(x) > 0$, $[b_1,\cdots,b_k]$ is right-handed if and only if $[a_1,\cdots,a_k]$ is right-handed. □

37 The Generalized Stokes' Theorem

2.

Proof. Assume $\eta = d\omega$ for some form. Since $\partial S^{n-1} = \emptyset$, Stokes' Theorem implies $\int_{S^{n-1}}\eta = \int_{S^{n-1}}d\omega = \int_{\partial S^{n-1}}\omega = 0$. Contradiction. □

3.

Proof. Apply Stokes' Theorem to $\omega = Pdx + Qdy$. □

4. (a)

Proof. $D\alpha(u,v) = \begin{bmatrix}1 & 0\\ -\frac{2u}{\sqrt{1-u^2-v^2}} & -\frac{2v}{\sqrt{1-u^2-v^2}}\\ 0 & 1\end{bmatrix}$. By Lemma 38.3, the normal vector n corresponding to the orientation of M satisfies $n = \frac{c}{\|c\|}$, where

$$c = \begin{bmatrix}\det D\alpha(u,v)(2,3)\\ -\det D\alpha(u,v)(1,3)\\ \det D\alpha(u,v)(1,2)\end{bmatrix} = \begin{bmatrix}-\frac{2u}{\sqrt{1-u^2-v^2}}\\ -1\\ -\frac{2v}{\sqrt{1-u^2-v^2}}\end{bmatrix}.$$

Plain calculation shows $\|c\| = \sqrt{\frac{1+3u^2+3v^2}{1-u^2-v^2}}$, so

$$n = \begin{bmatrix}-\frac{2u}{\sqrt{1+3u^2+3v^2}}\\ -\frac{\sqrt{1-u^2-v^2}}{\sqrt{1+3u^2+3v^2}}\\ -\frac{2v}{\sqrt{1+3u^2+3v^2}}\end{bmatrix}.$$

In particular, at the point $\alpha(0,0) = (0,2,0)$, $n = \begin{bmatrix}0\\ -1\\ 0\end{bmatrix}$, which points inwards into $\{(x_1,x_2,x_3) : 4(x_1)^2 + (x_2)^2 + 4(x_3)^2 \leq 4, x_2 \geq 0\}$. By Example 5 of §34, the tangent vector corresponding to the induced orientation of ∂M is easy to determine. □

(b)

Proof. According to the result of part (a), we can choose the following coordinate patch which belongs to the induced orientation of ∂M: $\beta(\theta) = (\cos\theta, 0, \sin\theta)$ $(0 \leq \theta 2\pi)$. By Theorem 35.2, we have

$$\int_{\partial M} x_2 dx_1 + 3x_1 dx_3 = \int_{[0,2\pi)} 3\cos\theta \cdot \cos\theta = 3\pi.$$

□

(c)

Proof. $d\omega = -dx_1 \wedge dx_2 + 3dx_1 \wedge dx_3$. So

$$\begin{aligned}
\int_M d\omega &= \int_M -dx_1 \wedge dx_2 + 3dx_1 \wedge dx_3 \\
&= \int_{\{(u,v):u^2+v^2<1\}} -\det D\alpha(u,v)(1,2) + 3\det D\alpha(u,v)(1,3) \\
&= \int_{\{(u,v):u^2+v^2<1\}} \left[\frac{2v}{\sqrt{1-u^2-v^2}} + 3\right] \\
&= \int_{\{(\theta,r):0\leq r<1, 0\leq\theta<2\pi\}} \left[\frac{2r\sin\theta}{\sqrt{1-r^2}} + 3\right] r \\
&= 3\pi.
\end{aligned}$$

□

5. (a)

Proof. By Stokes' Theorem, we have

$$\int_M d\omega = \int_{\partial M} \omega = \int_{S^2(d)} \omega + \int_{-S^2(c)} \omega = \int_{S^2(d)} \omega - \int_{S^2(c)} \omega = \frac{b}{d} - \frac{b}{c}.$$

□

(b)

Proof. If $d\omega = 0$, we conclude from part (a) that $b = 0$. This implies $\int_{S^2(r)} \omega = a$. To be continued ... □

(c)

Proof. If $\omega = d\eta$, by part (b) we conclude $b = 0$. Moreover, Stokes' Theorem implies $a = \int_{S^2(r)} \omega = \int_{S^2(r)} d\eta = 0$. □

6.

Proof. $\int_M d(\omega \wedge \eta) = \int_{\partial M} \omega \wedge \eta = 0$. Since $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$, we conclude $\int_M \omega \wedge d\eta = (-1)^{k+1} \int_M d\omega \wedge \eta$. So $a = (-1)^{k+1}$. □

38 Applications to Vector Analysis

1.

Proof. Let $M = \{x \in \mathbb{R}^3 : c \le \|x\| \le d\}$ oriented with the natural orientation. By the divergence theorem,

$$\int_M (\text{div}G)dV = \int_{\partial M} \langle G, N\rangle dV,$$

where N is the unit normal vector field to ∂M that points outwards from M. For the coordinate patch for M:

$$\begin{cases} x_1 = r\sin\theta\cos\phi \\ x_2 = r\sin\theta\sin\phi \quad (c \le r \le d, 0 \le \theta < \pi, 0 \le \phi < 2\pi) \\ x_3 = r\cos\theta, \end{cases}$$

we have

$$\det\frac{\partial(x_1,x_2,x_3)}{(r,\theta,\phi)} = \det\begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix} = r^2\sin\theta.$$

So $\int_M (\text{div}G)dV = \int \frac{1}{r}\left|\det\frac{\partial(x_1,x_2,x_3)}{(r,\theta,\phi)}\right| = 0$. Meanwhile $\int_{\partial M}\langle G, N\rangle dV = \int_{S^2(d)}\langle G, N_r\rangle dV - \int_{S^2(c)}\langle G, N_r\rangle dV$. So we conclude $\int_{S^2(d)}\langle G, N_r\rangle dV = \int_{S^2(c)}\langle G, N_r\rangle dV$. □

2. (a)

Proof. We let $M_3 = B^n(\varepsilon)$. Then for ε small enough, M_3 is contained by both $M_1 - \partial M_1$ and $M_2 - \partial M_2$. Applying the divergence theorem, we have $(i = 1, 2)$

$$0 = \int_{M_i - \text{Int}M_3} (\text{div}G)dV = \int_{\partial M_i}\langle G, N_i\rangle dV - \int_{\partial M_3}\langle G, N_3\rangle dV,$$

where N_3 is the unit outward normal vector field to ∂M_3. This shows that regardless $i = 1$ or $i = 2$, $\int_{\partial M_i}\langle G, N_i\rangle dV$ is a constant $\int_{\partial M_3}\langle G, N_3\rangle dV$. □

(b)

Proof. We have shown that if the origin is contained in $M - \partial M$, the integral $\int_{\partial M}\langle G, N\rangle dV$ is a constant. If the origin is not contained in $M - \partial M$, by the compactness of M, we conclude the origin is in the exterior of M. Applying the divergence theorem implies $\int_{\partial M}\langle G, N\rangle dV = 0$. So this integral has only two possible values. □

3.

Proof. Four possible values. Apply the divergence theorem (like in Exercise 3) and carry out the computation in the following four cases: 1) both p and q are contained by $M - \partial M$; 2) p is contained by $M - \partial M$ but q is not; 3) q is contained by $M - \partial M$ but p is not; 4) neither p nor q is contained by $M - \partial M$. □

4.

Proof. Follow the hint and apply Lemma 38.5. □

39 The Poincaré Lemma

2. (a)

Proof. Let $\omega \in \Omega^k(B)$ with $d\omega = 0$. Then $g^*\omega \in \Omega^k(A)$ and $d(g^*\omega) = g^*(d\omega) = 0$. Since A is homologically trivial in dimension k, there exists $\omega_1 \in \Omega^k(A)$ such that $d\omega_1 = g^*\omega$. Then $\omega_2 = (g^{-1})^*(\omega_1) \in \Omega^k(B)$ and $d\omega_2 = d(g^{-1})^*(\omega_1) = (g^{-1})^*(d\omega_1) = (g^{-1})^*g^*\omega = (g \circ g^{-1})^*\omega = \omega$. Since ω is arbitrary, we conclude B is homologically trivial in dimension k. □

(b)

Proof. Let $A = [\frac{1}{2}, 1] \times [0, \pi]$ and $B = \{(x, y) : \frac{1}{2} \le \sqrt{x^2 + y^2} \le 1, x, y \ge 0\}$. Define $g : A \to B$ as $g(r, \theta) = (r\cos\theta, r\sin\theta)$. By the Poincaré lemma, A is homologically trivial in every dimension. By part (a) of this exercise problem, B is homologically trivial in every dimension. But B is clearly not star-convex. □

3.

Proof. Let $p \in A$ and define $X = \{x \in A : x \text{ can be joined by a broken-line path in } A\}$. Since $\mathbb{R}^n$ is locally convex, it is easy to see X is an open subset of A.

(Sufficiency) Assume A is connected. Then $X = A$. For any closed 0-form f, $\forall x \in A$, denote by γ a broken-line path that joins x and p. We have by virtue of Newton-Leibnitz formula $0 = \int_\gamma df = f(x) - f(p)$. So f is a constant, i.e. an exact 0-form, on A. Hence A is homologically trivial in dimension 0.

(Necessity) Assume A is not connected. Then A can be decomposed into the joint union of at least two open subsets, say, A_1 and A_2. Define

$$f = \begin{cases} 1, & \text{on } A_1 \\ 0, & \text{on } A_2. \end{cases}$$

Then f is a closed 0-form, but not exact. So A is not homologically trivial in dimension 0. □

4.

Proof. Let $\eta = \sum_{[I]} f_I dx_I + \sum_{[J]} g_J dx_J \wedge dt$, where I denotes an ascending $(k+1)$-tuple and J denotes an ascending k-tuple, both from the set $\{1, \cdots, n\}$. Then $P\eta = \sum_{[J]} g_J dx_J$ and

$$(P\eta)(x)((x; v_1), \cdots, (x; v_k)) = \sum_{[J]} (-1)^k (\mathcal{L}g_J) \det[v_1 \cdots v_k]_J.$$

On the other hand,

$$w_i = D\alpha_t v_i = \begin{bmatrix} I_{n\times n} \\ 0 \end{bmatrix} v_i = \begin{bmatrix} v_i \\ 0 \end{bmatrix}.$$

So

$$\begin{aligned}
& \eta(y)((y; w_1), \cdots, (y; w_k), (y; e_{n+1})) \\
= \; & \sum_{[I]} f_I dx_I \left((y; \begin{bmatrix} v_1 \\ 0 \end{bmatrix}), \cdots, (y; \begin{bmatrix} v_k \\ 0 \end{bmatrix}), (y; \begin{bmatrix} 0_{n\times 1} \\ 1 \end{bmatrix}) \right) \\
& + \sum_{[J]} g_I dx_J \wedge dt \left((y; \begin{bmatrix} v_1 \\ 0 \end{bmatrix}), \cdots, (y; \begin{bmatrix} v_k \\ 0 \end{bmatrix}), (y; \begin{bmatrix} 0_{n\times 1} \\ 1 \end{bmatrix}) \right) \\
= \; & 0 + \sum_{[J]} g_J \det[v_1 \cdots v_k]_J \\
= \; & \sum_{[J]} g_J \det[v_1 \cdots v_k]_J.
\end{aligned}$$

Therefore

$$\begin{aligned}
& (-1)^k \int_{t=0}^{t=1} \eta(y)((y;w_1),\cdots,(y;w_k),(y;e_{n+1})) \\
= \quad & (-1)^k \sum_{[J]} \int_{t=0}^{t=1} g_J \det[v_1\cdots v_k]_J \\
= \quad & \sum_{[J]} (-1)^k (\mathcal{L} g_J) \det[v_1\cdots v_k]_J \\
= \quad & (P\eta)(x)((x;v_1),\cdots,(x;v_k)).
\end{aligned}$$

□

40 The deRham Groups of Punctured Euclidean Space

1. (a)

Proof. This is already proved on page 334 of the book, esp. in the last paragraph. □

(b)

Proof. To see $\widetilde{T}$ is well-defined, suppose $v+W=v'+W$. Then $v-v'\in W$ and $T(v)-T(v')=T(v-v')\in W'$ by the linearity of T and the fact that T carries W into W'. Therefore $T(v)+W'=T(v')+W'$, which shows $\widetilde{T}$ is well-defined. The linearity of $\widetilde{T}$ follows easily from that of T. □

2.

Proof. $\forall v\in V$, we can uniquely write v as $v=\sum_{i=1}^n c_i a_i$ for some coefficients $c_1,\cdots,c_n$. By the fact that a_1, $\cdots$, $a_k\in W$, we conclude $v+W=\sum_{i=k+1}^n c_i(a_i+W)$. So the cosets $a_{k+1}+W$, $\cdots$, a_n+W spans V/W. To see $a_{k+1}+W$, $\cdots$, a_n+W are linearly independent, let us assume $\sum_{i=k+1}^n c_i(a_i+W)=0$ for some coefficients $c_{k+1},\cdots,c_n$. Then $\sum_{i=k+1}^n c_i a_i\in W$ and there exist $d_1,\cdots,d_k$ such that $\sum_{i=k+1}^n c_i a_i=\sum_{j=1}^k d_j a_j$. By the linear independence of $a_1,\cdots,a_n$, we conclude $c_{k+1}=\cdots=c_n=0$, i.e. the cosets $a_{k+1}+W$, $\cdots$, a_n+W are linearly independent. □

4. (a)

Proof. $\dim H^i(U)=\dim H^i(V)=0$, for all i. □

(b)

Proof. $\dim H^i(U)=\dim H^i(V)=0$, for all i. □

(c)

Proof. $\dim H^0(U)=\dim H^0(V)=0$. □

5.

Proof. Step 1. We prove the theorem for $n=1$. Without loss of generality, we assume $p<q$. Let $A=\mathbb{R}^1-p-q$; write $A=A_0\cup A_1\cup A_2$, where $A_0=(-\infty,p)$, $A_1=(p,q)$, and $A_2=(q,\infty)$. If ω is a closed k-form in A, with $k>0$, then $\omega|A_0$, $\omega|A_1$ and $\omega|A_2$ are closed. Since A_0,A_1,A_2 are all star-convex, there are $k-1$ forms η_0, η_1 and η_2 on A_0, A_1 and A_2, respectively, such that $d\eta_i=\omega|A_i$ for $i=0,1,2$. Define $\eta=\eta_i$ on A_i, $i=0,1,2$. Then η is well-defined and of class C^∞, and $d\eta=\omega$.

Now let f_0 be the 0-form in A defined by setting $f_0(x)=0$ for $x\in A_1\cup A_2$ and $f_0(x)=1$ for $x\in A_0$; let f_1 be the 0-form in A defined by setting $f_1(x)=0$ for $x\in A_0\cup A_2$ and $f_1(x)=1$ for $x\in A_1$. Then f_0 and f_1 are closed forms, and they are not exact. We show the cosets $\{f_0\}$ and $\{f_1\}$ form a basis for $H^0(A)$.

Given a closed 0-form f in A, the forms $f|A_0$, $f|A_1$, and $f|A_2$ are closed and hence exact. Then there are constants c_0, c_1, and c_2 such that $f|A_i = c_i$, $i = 0, 1, 2$. It follows that

$$f(x) = (c_0 - c_2)f_0(x) + (c_1 - c_2)f_1(x) + c_2$$

for $x \in A$. Then $\{f\} = (c_0 - c_2)\{f_0\} + (c_1 - c_2)\{f_1\}$, as desired.

Step 2. Similar to the proof of Theorem 40.4, step 2, we can show the following: if B is open in $\mathbb{R}^n$, then $B \times \mathbb{R}$ is open in $\mathbb{R}^{n+1}$, and for all k, $\dim H^k(B) = \dim H^k(B \times \mathbb{R})$.

Step 3. Let $n \geq 1$. We assume the theorem true for n and prove it for $n+1$. We first prove the following

Lemma 40.1. *$\mathbb{R}^{n+1} - S \times \mathbb{H}^1$ and $\mathbb{R}^{n+1} - S \times \mathbb{L}^1$ are homologically trivial.*

Proof. Let $U_1 = \mathbb{R}^{n+1} - \{p\} \times \mathbb{H}^1$, $V_1 = \mathbb{R}^{n+1} - \{q\} \times \mathbb{H}^1$, $A_1 = U_1 \cap V_1 = \mathbb{R}^{n+1} - S \times \mathbb{H}^1$, and $X_1 = U_1 \cup V_1 = \mathbb{R}^{n+1}$. Since U_1 and V_1 are star-convex, U_1 and V_1 are homologically trivial in all dimensions. By Theorem 40.3, for $k \geq 0$, $H^k(A_1) = H^{k+1}(X_1) = H^{k+1}(\mathbb{R}^{n+1}) = 0$. So $\mathbb{R}^{n+1} - S \times \mathbb{H}^1$ is homologically trivial in all dimensions. Similarly, $\mathbb{R}^{n+1} - S \times \mathbb{L}^1$ is homologically trivial in all dimensions. □

Now, we define $U = \mathbb{R}^{n+1} - S \times \mathbb{H}^1$, $V = \mathbb{R}^{n+1} - S \times \mathbb{L}^1$, and $A = U \cap V = \mathbb{R}^{n+1} - S \times \mathbb{R}^1$. Then $X := \mathbb{R}^{n+1} - p - q = U \cup V$. We have shown U and V are homologically trivial. It follows from Theorem 40.3 that $H^0(X)$ is trivial, and that

$$\dim H^{k+1}(X) = \dim H^k(A) \text{ for } k \geq 0.$$

Now Step 2 tells us that $H^k(A)$ has the same dimension as the deRham group of $\mathbb{R}^n$ deleting two points, and the induction hypothesis implies that the latter has dimension 0 if $k \neq n-1$, and dimension 2 if $k = n-1$. The theorem follows. □

6.

Proof. The theorem of Exercise 5 can be restated in terms of forms as follows: Let $A = \mathbb{R}^n - p - q$ with $n \geq 1$.
(a) If $k \neq n-1$, then every closed k-form on A is exact on A.
(b) There are two closed $(n-1)$ forms, η_1 and η_2, such that η_1, η_2, and $\eta_1 - \eta_2$ are not exact. And if η is any closed $(n-1)$ form on A, then there exist unique scalars c_1 and c_2 such that $\eta - c_1\eta_1 - c_2\eta_2$ is exact. □

41 Differentiable Manifolds and Riemannian Manifolds

References

[1] J. Munkres. *Analysis on manifolds*, Westview Press, 1997. (document)

[2] P. Lax. *Linear algebra and its applications*, 2nd Edition, Wiley-Interscience, 2007.

1