Concise Complex Analysis, Revised Edition
Solution of Exercise Problems

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Abstract

This is a solution manual of selected exercise problems from Gong and Gong [4]. This version solves the exercise problems in Chapter 1-3, except the following: Chapter 1 problem 37-42; Chapter 2 problem 47, 49; Chapter 3 problem 15 (xi). If you find any typos/errors, please email me at zypublic@hotmail.com.

Contents

1 Calculus

► **1.** Verify the Newton-Leibniz formula, the Green' theorem, the Stokes's theorem and the Gauss's theorem by (1.9).

Proof. See, for example, Munkres [6], §38. □

► **2.** (1) Find the modulus and the principle values of the arguments for the following complex numbers: (i) $2i$; (ii) $1-i$; (iii) $3+4i$; (iv) $-5+12i$.

Solution. $|2i|=2$, $\arg(2i)=\frac{\pi}{2}$. $|1-i|=\sqrt{2}$, $\arg(1-i)=\frac{7}{4}\pi$. $|3+4i|=5$, $\arg(3+4i)=\arctan\frac{4}{3}\approx 0.9273$ (Matlab command: atan(4/3)). $|-5+12i|=13$, $\arg(-5+12i)=\arccos\left(-\frac{5}{13}\right)\approx 1.9656$ (Matlab command: acos(−5/13)). □

(2) Express the following complext numbers in the form of $x+iy$ where x and y are real numbers. (i) $(1+3i)^3$; (ii) $\frac{10}{4-3i}$; (iii) $\frac{2-3i}{4+i}$; (iv) $(1+i)^n+(1-i)^n$, where n is a positive integer.

Solution. $(1+3i)^3=-26-18i$. $\frac{10}{4-3i}=\frac{8}{5}+\frac{6}{5}i$. $\frac{2-3i}{4+i}=\frac{5}{17}-\frac{14}{17}i$. $(1+i)^n+(1-i)^n=2^{\frac{n}{2}+1}\cos\frac{n}{4}\pi$. □

(3) Find the absolute values (modulus) of the following complex numbers: (i) $-3i(2-i)(3+2i)(1+i)$; (ii) $\frac{(4-3i)(2-i)}{(1+i)(1+3i)}$.

Solution. $|-3i(2-i)(3+2i)(1+i)|=3\sqrt{130}$. $\left|\frac{(4-3i)(2-i)}{(1+i)(1+3i)}\right|=\frac{5\cdot\sqrt{5}}{\sqrt{2}\cdot\sqrt{10}}=\frac{5}{2}$. □

► **3.** Show that

$$\frac{\pi}{4}=4\arctan\frac{1}{5}-\arctan\frac{1}{239}$$

by calculating $(5-i)^4(1+i)$.

Proof. Let $\theta=\arctan\frac{1}{5}$, $\alpha=\arctan\frac{1}{239}$. Then $5-i=\sqrt{26}e^{-i\theta}$, $1+i=\sqrt{2}e^{i\frac{\pi}{4}}$ and $(5-i)^4(1+i)=676\sqrt{2}e^{i(\frac{\pi}{4}-4\theta)}$. Meanwhile $(5-i)^4(1+i)=956-4i=676\sqrt{2}e^{-\alpha i}$. So we must have $\frac{\pi}{4}-4\theta=-\alpha+2k\pi$, $k\in\mathbb{Z}$, i.e. $\frac{\pi}{4}=4\arctan\frac{1}{5}-\arctan\frac{1}{239}+2k\pi$, $k\in\mathbb{Z}$. Since $4\arctan\frac{1}{5}-\arctan\frac{1}{239}+2\pi>0-\frac{\pi}{2}+2\pi=\frac{7}{4}\pi>\frac{\pi}{4}$ and $4\arctan\frac{1}{5}-\arctan\frac{1}{239}-2\pi<4\cdot\frac{\pi}{2}-2\pi=0<\frac{\pi}{4}$, we must have $k=0$. Therefore $\frac{\pi}{4}=4\arctan\frac{1}{5}-\arctan\frac{1}{239}$. □

► **4.** Let $z=x+iy$, where x and y are real numbers. Find the real part and the imaginary part of each of the following complex numbers: (1) $\frac{1}{z}$; (2) z^2; (3) $\frac{1+z}{1-z}$; (4) $\frac{z}{z^2+1}$.

Solution. If $z=x+yi$, $\frac{1}{z}=\frac{x}{x^2+y^2}+\frac{y}{x^2+y^2}i$. $z^2=x^2-y^2+2xyi$. $\frac{1+z}{1-z}=\frac{1-x^2-y^2}{(1-x)^2+y^2}+\frac{2y}{(1-x)^2+y^2}i$. $\frac{z}{z^2+1}=\frac{x^3+xy^2+x+i(-y^3-x^2y+y)}{(x^2-y^2+1)^2+4x^2y^2}$. □

► **5.** Solve the quadratic equation

$$z^2+(\alpha+i\beta)z+\gamma+i\delta=0,$$

where α, β, γ and δ are real numbers.

Solution. $a=1$, $b=\alpha+i\beta$, $c=\gamma+i\delta$. So $\Delta=b^2-4ac=(\alpha^2-\beta^2-4\gamma)+i(2\alpha\beta-4\delta)$ and

$$z=\frac{-(\alpha+i\beta)\pm\sqrt{\Delta}}{2}.$$

□

▶ **6.** Assume $|z| = r > 0$. Show that:

$$\text{Re}z = \frac{1}{2}\left(z + \frac{r^2}{z}\right), \text{Im}z = \frac{1}{2i}\left(z - \frac{r^2}{z}\right).$$

Proof. Denote $\arg z$ by θ, then $z + \frac{r^2}{z} = re^{i\theta} + \frac{r^2}{re^{i\theta}} = r(e^{i\theta} + e^{-i\theta}) = 2r\cos\theta = 2\text{Re}z$, and $z - \frac{r^2}{z} = re^{i\theta} - \frac{r^2}{re^{i\theta}} = r(e^{i\theta} - e^{-i\theta}) = 2ir\sin\theta = 2i\text{Im}z$. □

▶ **7.** Show that:
(1) If $|a| < 1$, $|b| < 1$, then

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1;$$

(2) If $|a| = 1$ or $|b| = 1$, then

$$\left|\frac{a-b}{1-\bar{a}b}\right| = 1.$$

Discussing the case $|a| = 1$ and $|b| = 1$.

Proof. If $a = r_1e^{i\theta_1}$ and $b = r_2e^{i\theta_2}$, then

$$\left|\frac{a-b}{1-\bar{a}b}\right| = \left|\frac{r_1 - r_2e^{i(\theta_2-\theta_1)}}{1 - r_1r_2e^{i(\theta_2-\theta_1)}}\right|.$$

Denote $\theta_2 - \theta_1$ by θ, we can reduce the problem to comparing $|r_1 - r_2e^{i\theta}|^2$ and $|1 - r_1r_2e^{i\theta}|^2$. Note

$$|r_1 - r_2e^{i\theta}|^2 = (r_1 - r_2\cos\theta)^2 + r_2^2\sin^2\theta = r_1^2 - 2r_1r_2\cos\theta + r_2^2$$

and

$$|1 - r_1r_2e^{i\theta}|^2 = (1 - r_1r_2\cos\theta)^2 + r_1^2r_2^2\sin^2\theta = 1 - 2r_1r_2\cos\theta + r_1^2r_2^2.$$

So $|1 - r_1r_2e^{i\theta}|^2 - |r_1 - r_2e^{i\theta}|^2 = (r_1^2 - 1)(r_2^2 - 1)$. This observation shows $\left|\frac{a-b}{1-\bar{a}b}\right| = 1$ if and only if at least one of a and b has modulus 1; $\left|\frac{a-b}{1-\bar{a}b}\right| < 1$ if $|a| < 1$ and $|b| < 1$. □

▶ **8.** Prove the Lagrange equation in complex form

$$\left|\sum_{i=1}^{n} a_ib_i\right|^2 = \sum_{i=1}^{n}|a_i|^2\sum_{i=1}^{n}|b_i|^2 - \sum_{1\le i<j\le n}|a_i\bar{b}_j - a_j\bar{b}_i|^2.$$

Derive the Cauchy inequality

$$\left|\sum_{i=1}^{n} a_ib_i\right|^2 \le \sum_{i=1}^{n}|a_i|^2\sum_{i=1}^{n}|b_i|^2$$

from the Lagrange equation. The equality holds if and only if a_k and $\bar{b}_k$ are proportional, where $k = 1, \cdots, n$.

Proof. We prove by induction. The equation clearly holds for $n = 1$. Assume it holds for all n with $n \leq N$. Then

$$
\begin{aligned}
\left|\sum_{i=1}^{N+1} a_i b_i\right|^2 &= \left|\sum_{i=1}^{N} a_i b_i + a_{N+1}b_{N+1}\right|^2 = \left(\sum_{i=1}^{N} a_i b_i + a_{N+1}b_{N+1}\right)\left(\sum_{i=1}^{N} \bar{a}_i \bar{b}_i + \bar{a}_{N+1}\bar{b}_{N+1}\right) \\
&= \left|\sum_{i=1}^{N} a_i b_i\right|^2 + |a_{N+1}|^2|b_{N+1}|^2 + \bar{a}_{N+1}\bar{b}_{N+1}\sum_{i=1}^{N} a_i b_i + a_{N+1}b_{N+1}\sum_{i=1}^{N} \bar{a}_i \bar{b}_i \\
&= \sum_{i=1}^{N}|a_i|^2\sum_{i=1}^{N}|b_i|^2 - \sum_{i\leq i<j\leq N}|a_i\bar{b}_j - a_j\bar{b}_i|^2 + |a_{N+1}|^2|b_{N+1}|^2 + |b_{N+1}|^2\sum_{1\leq i<j=N+1}|a_i|^2 \\
&\quad +|a_{N+1}|^2\sum_{1\leq i<j=N+1}|b_i|^2 \\
&\quad -\sum_{1\leq i<j=N+1}\left[|a_i|^2|b_{N+1}|^2 + |a_{N+1}|^2|b_i|^2 - a_i\bar{a}_{N+1}b_i\bar{b}_{N+1} - \bar{a}_i a_{N+1}\bar{b}_i b_{N+1}\right] \\
&= \sum_{i=1}^{N+1}|a_i|^2\sum_{i=1}^{N+1}|b_i|^2 - \sum_{1\leq i<j\leq N}|a_i\bar{b}_j - a_j\bar{b}_i|^2 - \sum_{1\leq i<j=N+1}|a_i\bar{b}_j - a_j\bar{b}_i|^2 \\
&= \sum_{i=1}^{N+1}|a_i|^2\sum_{i=1}^{N+1}|b_i|^2 - \sum_{1\leq i<j\leq N+1}|a_i\bar{b}_j - a_j\bar{b}_i|^2.
\end{aligned}
$$

By method of mathematical induction, $|\sum_{i=1}^{n} a_i b_i|^2 = \sum_{i=1}^{n}|a_i|^2\sum_{i=1}^{n}|b_i|^2 - \sum_{1\leq i<j\leq n}|a_i\bar{b}_j - a_j\bar{b}_i|^2$ holds for all $n \in \mathbb{N}$. Cauchy's inequality follows directly from this equation. □

► **9.** Show that a_1, a_2, a_3 are the vertices of a equilateral triangle if and only if

$$a_1^2 + a_2^2 + a_3^2 = a_1a_2 + a_2a_3 + a_3a_1.$$

Proof. We first consider the special case where $a_1 = 0$ and $a_2 = 1$. Then for $a_3 = re^{i\theta}$, a_1, a_2, a_3 are vertices of an equilateral triangle if and only if $r = 1$ and $\cos\theta = \frac{1}{2}$. Meanwhile the equation $1 + re^{2i\theta} = re^{i\theta}$ is equivalent to

$$\begin{cases} 1 + r\cos 2\theta = r\cos\theta \\ r\sin 2\theta = r\sin\theta, \end{cases}$$

which implies $r = 1$ and $\cos\theta = \frac{1}{2}$. So the equality is proven for the special case $a_1 = 0$, $a_2 = 1$.

For general case, note $a_1^2 + a_2^2 + a_3^2 = a_1a_2 + a_2a_3 + a_3a_1$ if and only if $(a_2 - a_1)^2 + (a_3 - a_1)^2 = (a_2 - a_1)(a_3 - a_1)$, which is further equivalent to $1 + \left(\frac{a_3-a_1}{a_2-a_1}\right)^2 = \frac{a_3-a_1}{a_2-a_1}$. This shows the general case can be reduced to the special case of $a_1 = 0$ and $a_2 = 1$. Essentially, these reductions correspond to the composition of coordinate translation and coordinate rotation. □

► **10.** Show that the complex numbers α, β, γ are collinear if and only if

$$\begin{vmatrix} \alpha & \bar{\alpha} & 1 \\ \beta & \bar{\beta} & 1 \\ \gamma & \bar{\gamma} & 1 \end{vmatrix} = 0.$$

Proof.

$$D := \begin{vmatrix} \alpha & \bar{\alpha} & 1 \\ \beta & \bar{\beta} & 1 \\ \gamma & \bar{\gamma} & 1 \end{vmatrix} = \begin{vmatrix} \alpha-\gamma & \bar{\alpha}-\bar{\gamma} & 1 \\ \beta-\gamma & \bar{\beta}-\bar{\gamma} & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \alpha-\gamma & \bar{\alpha}-\bar{\gamma} \\ \beta-\gamma & \bar{\beta}-\bar{\gamma} \end{vmatrix}.$$

Suppose $\alpha-\gamma=r_1e^{i\theta_1}$ and $\beta-\gamma=r_2e^{i\theta_2}$. Then

$$\begin{vmatrix}\alpha-\gamma & \bar{\alpha}-\bar{\gamma}\\ \beta-\gamma & \bar{\beta}-\bar{\gamma}\end{vmatrix}=\begin{vmatrix}r_1e^{i\theta_1} & r_1e^{-i\theta_1}\\ r_2e^{i\theta_2} & r_2e^{-i\theta_2}\end{vmatrix}=r_1r_2e^{i(\theta_1-\theta_2)}-r_1r_2e^{-i(\theta_1-\theta_2)}.$$

So $D=0$ if and only if $\theta_1-\theta_2=k\pi$, which is equivalent to α, β, γ being colinear. Note the above reduction corresponds to the coordination transformation that places γ at origin. □

▶ **11.** Find the corresponding points of $1-i$, $4+3i$ on the Riemann sphere S^2.

Solution. We apply formula (1.11) (correction: the formula for x_2 should be $x_2=\frac{z-\bar{z}}{(1+|z|^2)i}$). If $z=1-i$, then $x_1=\frac{2}{3}$, $x_2=-\frac{2}{3}$, and $x_3=\frac{2}{3}$. If $z=4+3i$, then $x_1=\frac{4}{13}$, $x_2=\frac{3}{13}$, and $x_3=\frac{12}{13}$. □

▶ **12.** Tow points z_1, z_2 on the $\mathbb{C}$-plane correspond to the two end points of a diagonal of the Riemann sphere S^2 if and only if $z_1z_2=-1$.

Proof. Denote by $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)$ the points on the Riemann sphere S^2 corresponding to z_1 and z_2, respectively. Then x and y are two ends points of a diagonal of S^2 if and only if $x+y=0$. By formula (1.11), $x+y=0$ is equivalent to

$$\begin{cases}\frac{z_1+\bar{z}_1}{1+|z_1|^2}+\frac{z_2+\bar{z}_2}{1+|z_2|^2}=0\\ \frac{z_1-\bar{z}_1}{1+|z_1|^2}+\frac{z_2-\bar{z}_2}{1+|z_2|^2}=0\\ \frac{|z_1|^2-1}{|z_1|^2+1}+\frac{|z_2|^2-1}{|z_2|^2+1}=0.\end{cases}$$

Solving the third equation gives $|z_1z_2|=1$. So we can assume $z_1=re^{i\alpha}$ and $z_2=\frac{1}{r}e^{i\beta}$ $(r>0)$. Then $1+|z_1|^2=1+r^2$ and $1+|z_2|^2=1+\frac{1}{r^2}=\frac{1}{r^2}(1+|z_1|^2)$. So the other two equations become

$$\begin{cases}2r\cos\alpha+2r\cos\beta=0\\ 2r\sin\alpha+2r\sin\beta=0,\end{cases}$$

which is equivalent to

$$\begin{cases}\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=0\\ \sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=0.\end{cases}$$

So we must have $\frac{1}{2}(\alpha-\beta)=k\pi+\frac{\pi}{2}$ $(k\in\mathbb{Z})$, i.e. $\alpha-\beta=2k\pi+\pi$. So $z_1\bar{z}_2=e^{i(\alpha-\beta)}=e^{i(2k\pi+\pi)}=-1$. □

▶ **13.** Show that the equation of a circle is

$$A|z|^2+Bz+\bar{B}\bar{z}+C=0$$

where A, C are real numbers and $|B|^2>AC$. Also show that the image of this circle on the Riemann sphere S^2 is a great circle if and only if $A+C=0$.

Proof. Suppose $z\in\mathbb{C}$ is on a circle. Then there exists $z_0\in\mathbb{C}$ and $R>0$, such that $|z-z_0|=R$. Suppose $z=x+iy$ and $z_0=x_0+iy_0$. Then $|z-z_0|=R$ implies $(x-x_0)^2+(y-y_0)^2=R^2$. After simplification, this can be written as $|z|^2+\bar{z}_0z+z_0\bar{z}+|z_0|^2-R^2=0$. Let $A=1$, $B=\bar{z}_0$ and $C=|z_0|^2-R^2$. Then $A,C\in\mathbb{R}$ and $|B|^2>AC$.

Conversely, if $A|z|^2+Bz+\bar{B}\bar{z}+C=0$ is the equation satisfied by z, we have two cases to consider: (i) $A\neq 0$; (ii) $A=0$.

In case (i), $A|z|^2+Bz+\bar{B}\bar{z}+C=0$ can be written as $|z|^2+\frac{B}{A}z+\frac{\bar{B}}{A}\bar{z}+\frac{|B|^2}{A^2}=\frac{|B|^2}{A^2}-\frac{C}{A}>0$. Define $z_0=\frac{\bar{B}}{A}$ and $R=\sqrt{\frac{|B|^2}{A^2}-\frac{C}{A}}$. The equation can be written as $|z|^2+\bar{z}_0z+z_0\bar{z}+|z_0|^2=R^2$, which is equivalent to $|z-z_0|=R$. So z is on a circle.

In case (ii), the equation $Bz + \bar{B}\bar{z} + C = 0$ stands for a straight line in $\mathbb{C}$. Indeed, suppose $B = a + ib$ and $z = x + iy$, then $Bz + \bar{B}\bar{z} + C = 0$ becomes $2ax - 2by + C = 0$. If we regard straight lines as circles, $A|z|^2 + Bz + \bar{B}\bar{+}C = 0$ represents a circle in case (ii) as well.

To see the necessary and sufficient condition for the image of this circle on the Riemann sphere S^2 to be a great circle, we suppose a generic point z on this circle corresponds to a point (x_1, x_2, x_3) on the Riemann sphere S^2. Then by formula (1.10) $z = \frac{x_1+ix_2}{1-x_3}$, we can write the equation $A|z|^2 + Bz + \bar{B}\bar{z} + C = 0$ as $(B+\bar{B})x_1 + i(\bar{B}-B)x_2 + (A-C)x_3 = -(A+C)$. (x_1, x_2, x_3) falls on a great circle if and only if (x_1, x_2, x_3) lies on the intersection of S^2 and a plane that contains $(0, 0, 0)$, which is equivalent to $A + C = 0$ by the equation $(B + \bar{B})x_1 + i(\bar{B} - B)x_2 + (A - C)x_3 = -(A + C)$. □

► **14.** Let $d(z_1, z_2)$ be the spherical distance of z_1 and z_2 in $\mathbb{C}$, that is, the spherical distance between the corresponding points Z_1 and Z_2 on the Riemann sphere S^2. Prove

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}},$$

and

$$d(z_1, \infty) = \frac{2}{\sqrt{1 + |z_1|^2}}.$$

Proof. Suppose $Z_1 = (x_1, x_2, x_3)$ and $Z_2 = (y_1, y_2, y_3)$. Then by formula (1.11),

$$\begin{aligned}
d(z_1, z_2)^2 &= (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 \\
&= 2 - 2(x_1y_1 + x_2y_2 + x_3y_3) \\
&= 2 - 2\left[\frac{z_1 + \bar{z}_1}{1 + |z_1|^2} \cdot \frac{z_2 + \bar{z}_2}{1 + |z_2|^2} + \frac{z_1 - \bar{z}_1}{i(1 + |z_1|^2)} \cdot \frac{z_2 - \bar{z}_2}{i(1 + |z_2|^2)} + \frac{|z_1|^2 - 1}{|z_1|^2 + 1} \cdot \frac{|z_2|^2 - 1}{|z_2|^2 + 1}\right] \\
&= 4\frac{|z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - \bar{z}_1z_2}{(1 + |z_1|^2)(1 + |z_2|^2)} \\
&= 4\frac{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}{(1 + |z_1|^2)(1 + |z_2|^2)}.
\end{aligned}$$

So $d(z_1, z_2) = \frac{2|z_1-z_2|}{\sqrt{(1+|z_1|^2)(1+|z_2|^2)}}$. Consequently,

$$d(z_1, \infty) = 2 \lim_{z_2 \to \infty} \frac{\left|\frac{z_1}{z_2} - 1\right|}{\sqrt{(1 + |z_1|^2)\left(\frac{1}{|z_2|^2} + 1\right)}} = \frac{2}{\sqrt{1 + |z_1|^2}}.$$

□

► **15.** Explain the geometric meaning of the following relations:
(i) $\left|\frac{z-z_1}{z-z_2}\right| \leq 1$, $z_1 \neq z_2$;
(ii) $\mathrm{Re}\frac{z-z_1}{z-z_2} = 0$, $z_1 \neq z_2$;
(iii) $0 < \arg\frac{z+i}{z-i} < \frac{\pi}{4}$;
(iv) $|z + c| + |z - c| \leq 2a$, $a > 0$, $|c| < a$.

Solution. (i) The distance between z and z_1 is no greater than the distance between z and z_2.
(ii) $z - z_1$ and $z - z_2$ are perpendicular to each other.
(iii) The angle between $z + i$ and $z - i$ is less than $\frac{\pi}{4}$.
(iv) z is on an ellipse with c and $-c$ as the foci. □

► **16.** Prove Heine-Borel theorem and Bolzano-Weierstrass theorem on the complex plane.

Proof. Regard $\mathbb{C}$ as $\mathbb{R}^2$ and apply Heine-Borel theorem and Bolzano-Weierstrass theorem for $\mathbb{R}^n$ ($n \in \mathbb{N}$). □

▶ **17.** Show that the sequence $\{z_n\}$ converges to z_0 if and only if the sequences $\{\text{Re} z_n\}$, $\{\text{Im} z_n\}$ converges to $\text{Re} z_0$, $\text{Im} z_0$ respectively.

Proof. Note $\max\{|\text{Re} z_n - \text{Re} z_0|, |\text{Im} z_n - \text{Im} z_0|\} \le |z - z_0| \le |\text{Re} z_n - \text{Re} z_0| + |\text{Im} z_n - \text{Im} z_0|$. □

▶ **18.** Show that

(1) If $\lim_{n\to\infty} z_n = a$, $\lim_{n\to\infty} z'_n = b$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} z_k z'_{n-k} = ab.$$

Proof. We suppose $|a|, |b| < \infty$. Then $(z_n)_n$ and $(z'_n)_n$ are bounded sequences. Denote by M a common bound for both sequences. $\forall \varepsilon > 0$, $\exists N$, so that for any $n > N$, $\max\{|z_n - a|, |z'_n - b|\} < \frac{\varepsilon}{M}$. Then for any n satisfying $n > N\left(1 + \frac{M^2}{2}\right)$ (define $z_0 = z'_0 = 0$)

$$\begin{aligned}
\left|\frac{1}{n}\sum_{k=1}^{n} z_k z'_{n-k} - ab\right| &< \left|\frac{1}{n}\sum_{k=1}^{N} z_k z'_{n-k}\right| + \left|\frac{1}{n}\sum_{k=N+1}^{n} z_k z'_{n-k} - ab\right| \\
&< \frac{NM^2}{n} + \left|\frac{1}{n}\sum_{k=N+1}^{n} (z_k - a) z'_{n-k}\right| + \left|\frac{a}{n}\sum_{k=N+1}^{n} z'_{n-k} - ab\right| \\
&< \varepsilon + \varepsilon + |a| \left|\frac{1}{n}\sum_{k=0}^{n-N-1} z'_k - b\right| \\
&= 2\varepsilon + |a| \left|\frac{n-N-1}{n} \cdot \frac{1}{n-N-1}\sum_{k=0}^{n-N-1} z'_k - b\right|.
\end{aligned}$$

Taking upper limits on both sides, we get

$$\overline{\lim_{n\to\infty}} \left|\frac{1}{n}\sum_{k=1}^{n} z_k z'_{n-k} - ab\right| \le 2\varepsilon + 0 = 2\varepsilon.$$

Since ε is arbitrary, we conclude $\overline{\lim}_{n\to\infty} \left|\frac{1}{n}\sum_{k=1}^{n} z_k z'_{n-k} - ab\right| = 0$, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} z_k z'_{n-k} = ab.$$

□

(2) If $\lim_{n\to\infty} z_n = A$, then

$$\lim_{n\to\infty} \frac{1}{n}(z_1 + z_2 + \cdots + z_n) = A$$

(using the result of (1)).

Proof. Apply the result of (1) with $z'_n \equiv 1$. □

▶ **19.** Discuss the differentiability of the following functions.

(i) $f(z) = |z|$; (ii) $f(z) = \bar{z}$; (iii) $f(z) = \text{Re} z$.

Solution. (i) $f(z) = \sqrt{z\bar{z}}$ is a function of both z and $\bar{z}$. So f is not differentiable.

(ii) $f(z) = \bar{z}$ is a function of $\bar{z}$, so it is not differentiable.

(iii) $f(z) = \text{Re} z = \frac{1}{2}(z + \bar{z})$ is a function of both z and $\bar{z}$. So f is not differentiable. □

▶ **20.** Suppose $g(w)$ and $f(z)$ are holomorphic functions. Show that $g(f(z))$ is also holomorphic.

Proof. By chain rule, $\frac{\partial}{\partial \bar{z}} g(f(z)) = g'(f(z)) \frac{\partial f(z)}{\partial \bar{z}} = 0$. So $g(f(z))$ is also holomorphic. □

▶ **21.** On a region D, show that

(1) If a function $f(z)$ is holomorphic, and $f'(z)$ is identically zero, then $f(z)$ is a constant function.

Proof. Let $u(x,y) = \mathrm{Re} f(z)$ and $v(x,y) = \mathrm{Im} f(z)$. Then by C-R equations $f'(z) = u_x(x,y) + iv_x(x,y) = v_y(x,y) - iu_y(x,y)$. So $f'(z) \equiv 0$ if and only if $v_x = v_y = u_x = u_y \equiv 0$. By results for functions of real variables, we conclude u and v are constants on D. so f is a constant function on D. □

(2) If $f(z)$ is holomorphic and $f'(z)$ satisfies one of the following conditions: (i) $\mathrm{Re} f(z)$ is a constant function; (ii) $\mathrm{Im} f(z)$ is a constant function; (iii) $|f(z)|$ is a constant function; (iv) $\arg f(z)$ is a constant function, then $f(z)$ is a constant function.

Proof. (i) and (ii) are direct corollaries of C-R equations.

For (iii), we assume without loss of generality that $f(z) \not\equiv 0$ on D. We note $|f(z)|^2 = u^2(z) + v^2(z)$. So by the C-R equations, $0 = \frac{\partial}{\partial x}|f(z)|^2 = 2uu_x + 2vv_x = 2uu_x - 2vu_y$ and $0 = \frac{\partial}{\partial y}|f(z)|^2 = 2uu_y + 2vv_y = 2uu_y + 2vu_x$, i.e. $\begin{bmatrix} u & -v \\ v & u \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = 0$. Since $f \not\equiv 0$, $\begin{bmatrix} u & -v \\ v & u \end{bmatrix}$ is invertible on $D_1 := \{z \in D : f(z) \neq 0\}$. So $u_x = u_y \equiv 0$ on D_1. By the C-R equations $v_x = v_y \equiv 0$ on D_1. So f is a constant function on each simply connected component of D_1. Since $D_2 := \{z \in D : f(z) = 0\}$ has no accumulation points in D (see Theorem 2.13), f must be identical to the same constant throughout D.

For (iv), we assume $\arg f(z) \equiv \theta$. Then $g(z) := e^{-i\theta} f(z)$ is also holomorphic and f is a constant function if and only if g is a constant function. So without loss of generality, we can assume $\arg f(z) \equiv 0$. Then $\mathrm{Im} f(z) \equiv 0$. By C-R equations, $\mathrm{Re} f(z)$ is a constant too. Combined, we can conclude f is identically equal to a constant on D. □

▶ **22.** If $f(z) = u + iv$ is holomorphic and $f'(z) \neq 0$, then the curves $u(x,y) = c_1$ and $v(x,y) = c_2$ are orthogonal, where c_1, c_2 are constants.

Proof. The tangent vector $\mathbf{a}(x,y)$ of the curve $u(x,y) = c_1$ at point (x,y) is perpendicular to (u_x, u_y); the tangent vector $\mathbf{b}(x,y)$ of the curve $v(x,y) = c_2$ at point (x,y) is perpendicular to (v_x, v_y). Since $(v_y, -v_x) \perp (v_x, v_y)$ and $(u_x, u_y) = (v_y, -v_x)$ by C-R equations, we must have $\mathbf{a}(x,y) \perp \mathbf{b}(x,y)$, which means the curves $u(x,y) = c_1$ and $v(x,y) = c_2$ are orthogonal. □

▶ **23.** Show that

(1) If

$$f(z) = u(r,\theta) + iv(r,\theta),\ z = r(\cos\theta + i\sin\theta),$$

then the C-R equation become

$$u_r = \frac{1}{r} v_\theta,\ v_r = -\frac{1}{r} u_\theta,$$

and

$$f'(z) = \frac{r}{z}(u_r + iv_r).$$

Proof. Since $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, we have

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} := A(\theta, r) \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}.$$

It's easy to see $A^{-1}(\theta,r)=\frac{1}{r}\begin{bmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta\end{bmatrix}$. Writing the Cauchy-Riemann equations in matrix form, we get

$$\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{bmatrix} u = \begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{bmatrix} v.$$

Therefore, under the polar coordinate, the Cauchy-Riemann equations become

$$\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial \theta}\end{bmatrix} u = A(\theta,r)\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{bmatrix} u = A(\theta,r)\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{bmatrix} v = A(\theta,r)\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}A^{-1}(\theta,r)\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial \theta}\end{bmatrix} v = \begin{bmatrix} 0 & \frac{1}{r}\\ -r & 0\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial \theta}\end{bmatrix} v.$$

Fix $z=re^{i\theta}$, then

$$f'(z)=\lim_{\Delta r\to 0}\frac{u(r+\Delta r,\theta)+iv(r+\Delta r,\theta)-[u(r,\theta)+iv(r,\theta)]}{\Delta r\cdot e^{i\theta}}=\frac{\partial u}{\partial r}e^{-i\theta}+i\frac{\partial v}{\partial v}e^{-i\theta}=\frac{r}{z}\left[\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right].$$

□

(2) If $f(z)=R(\cos\phi+i\sin\phi)$, then the C-R equations are

$$\frac{\partial R}{\partial r}=-\frac{R}{r}\frac{\partial\phi}{\partial\theta},\ \frac{\partial R}{\partial\theta}=-Rr\frac{\partial\phi}{\partial r}.$$

Proof. If $f(z)=R(\cos\phi+i\sin\phi)$, then $u=R\cos\phi$ and $v=R\sin\phi$. So

$$u_r=\frac{\partial R}{\partial r}\cos\phi-R\sin\phi\frac{\partial\phi}{\partial r},\ v_r=\frac{\partial R}{\partial r}\sin\phi+R\cos\phi\frac{\partial\phi}{\partial r},$$
$$u_\theta=\frac{\partial R}{\partial\theta}\cos\phi-R\sin\phi\frac{\partial\phi}{\partial\theta},\ v_\theta=\frac{\partial R}{\partial\theta}\sin\phi+R\cos\phi\frac{\partial\phi}{\partial\theta}.$$

Writing the above equalities in matrix form, we have

$$\begin{bmatrix}u_r\\ v_r\end{bmatrix}=A\begin{bmatrix}\frac{\partial R}{\partial r}\\ \frac{\partial\phi}{\partial r}\end{bmatrix},\ \begin{bmatrix}u_\theta\\ v_\theta\end{bmatrix}=A\begin{bmatrix}\frac{\partial R}{\partial\theta}\\ \frac{\partial\phi}{\partial\theta}\end{bmatrix},\ \text{where } A=\begin{bmatrix}\cos\phi & -R\sin\phi\\ \sin\phi & R\cos\phi\end{bmatrix}.$$

Therefore the C-R equations become

$$A\begin{bmatrix}\frac{\partial R}{\partial r}\\ \frac{\partial\phi}{\partial r}\end{bmatrix}=\begin{bmatrix}u_r\\ v_r\end{bmatrix}=\begin{bmatrix}0 & \frac{1}{r}\\ -\frac{1}{r} & 0\end{bmatrix}\begin{bmatrix}u_\theta\\ v_\theta\end{bmatrix}=\begin{bmatrix}0 & \frac{1}{r}\\ -\frac{1}{r} & 0\end{bmatrix}A\begin{bmatrix}\frac{\partial R}{\partial\theta}\\ \frac{\partial\phi}{\partial\theta}\end{bmatrix}.$$

Note $A^{-1}=\frac{1}{R}\begin{bmatrix}R\cos\phi & R\sin\phi\\ \sin\phi & R\cos\phi\end{bmatrix}$. So $A^{-1}\begin{bmatrix}0 & \frac{1}{r}\\ -\frac{1}{r} & 0\end{bmatrix}A=\begin{bmatrix}0 & \frac{R}{r}\\ -\frac{1}{Rr} & 0\end{bmatrix}$. Therefore

$$\begin{bmatrix}\frac{\partial R}{\partial r}\\ \frac{\partial\phi}{\partial r}\end{bmatrix}=\begin{bmatrix}0 & \frac{R}{r}\\ -\frac{1}{Rr} & 0\end{bmatrix}\begin{bmatrix}\frac{\partial R}{\partial\theta}\\ \frac{\partial\phi}{\partial\theta}\end{bmatrix},\ \text{i.e.}\ \begin{cases}\frac{\partial R}{\partial r}=\frac{R}{r}\frac{\partial\phi}{\partial\theta}\\ \frac{\partial R}{\partial\theta}=-Rr\frac{\partial\phi}{\partial r}\end{cases}.$$

□

► **24.** Suppose the function $f(z)=u(x,y)+iv(x,y)$ $(z=x+iy)$ is holomorphic on a region D. Show that the Jacobi determinant of u and v with respect to x and y is

$$J=\begin{vmatrix}u_x & u_y\\ v_x & v_y\end{vmatrix}=|f'(z)|^2,$$

and give the geometric meaning of J.

Proof. By the C-R equations, $J=\begin{vmatrix}u_x & u_y\\ v_x & v_y\end{vmatrix}=u_xv_y-v_xu_y=u_x\cdot u_x-v_x\cdot(-v_x)=u_x^2+v_x^2=|f'(z)|^2$. J is the area of $f(D)$, see, for example, Munkres [6], §21 The volume of a parallelepiped. □

▶ **25.** Compute:

(1) The integral

$$\int_\gamma x dz$$

where $z = x + iy$ and γ is a directed segment from 0 to $1 + i$.

Solution. We can parameterize γ with $\gamma(t) = t + ti$ $(0 \leq t \leq 1)$. So

$$\int_\gamma x dz = \int_0^1 t(dt + idt) = \frac{1}{2} + \frac{1}{2}i.$$

□

(2) The integral

$$\int_\gamma |z - 1||dz|$$

where $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$.

Solution.

$$\begin{aligned}\int_\gamma |z-1||dz| &= \int_0^{2\pi} |e^{it} - 1|dt = \int_0^{2\pi} \sqrt{2 - 2\cos t}dt = \sqrt{2}\int_0^{2\pi} \left|\sin\left(\frac{\pi}{4} - \frac{t}{2}\right) - \cos\left(\frac{\pi}{4} - \frac{t}{2}\right)\right| dt \\ &= \sqrt{2}\int_{-\frac{3}{4}\pi}^{\frac{\pi}{4}} |\sin\theta - \cos\theta|\, d\theta = \sqrt{2}\int_{-\frac{3}{4}\pi}^{\frac{\pi}{4}} (\cos\theta - \sin\theta)\, d\theta 4.\end{aligned}$$

□

(3) The integral

$$\int_\gamma \frac{1}{z-a} dz$$

where $\gamma(t) = a + Re^{it}$, $0 \leq t \leq 2\pi$ and a is a complex constant.

Solution.

$$\int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{iRe^{it}}{Re^{it}} dt = 2\pi i.$$

□

▶ **26.** (1) Find the real parts and the imaginary parts of $\cos(x+iy)$, $\sin(x+iy)$ where x, y are real numbers.

(2) Show that

$$\sin iz = i\sinh z,\ \cos iz = \cosh z,\ (\sin z)' = \cos z, \cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2.$$

Solution. We first recall $\sinh z = \frac{e^z - e^{-z}}{2}$ and $\cosh z = \frac{e^z + e^{-z}}{2}$. Then straightforward calculations show

$$\cos(x+iy) = \cosh y \cos x - i \sinh y \sin x,\ \sin(x+iy) = \cosh y \sin x + i \sinh y \cos x.$$

By this result, we have

$$\sin iz = \frac{e^{i(iz)} - e^{-i(iz)}}{2i} = \frac{e^{-z} - e^z}{2i} = \sinh z,\ \cos iz = \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \frac{e^{-z} + e^z}{2} = \cosh z,$$

$$(\sin z)' = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)' = \frac{e^{iz} + e^{-iz}}{2} = \cos z.$$

and

$$
\begin{aligned}
& \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \\
= & \frac{e^{iz_1}+e^{-iz_1}}{2}\frac{e^{iz_2}+e^{-iz_2}}{2} - \frac{e^{iz_1}-e^{-iz_1}}{2i}\frac{e^{iz_2}-e^{-iz_2}}{2i} \\
= & \frac{e^{i(z_1+z_2)}+e^{-i(z_1+z_2)}+e^{i(z_1-z_2)}+e^{-i(z_1-z_2)}}{4} - \frac{e^{i(z_1+z_2)}+e^{-i(z_1+z_2)}-e^{i(z_1-z_2)}-e^{-i(z_1-z_2)}}{-4} \\
= & \frac{2e^{i(z_1+z_2)}+2e^{-i(z_1+z_2)}}{4} \\
= & \cos(z_1+z_2).
\end{aligned}
$$

□

▶ **27.** Evaluate

$$\sin i,\ \cos(2+i),\ \tan(1+i),\ 2^i,\ i^i,\ (-1)^{2i},\ \log(2-3i),\ \arccos\frac{1}{4}(3+i).$$

Solution. $\sin i = \sinh 1$.
$\cos(2+i) = \cosh 1 \cos 2 - i \sinh 1 \sin 2$.
$\tan(1+i) = \frac{\sin 1 \cos 1 + i \sinh 1 \cosh 1 \cos 2}{\cosh^2 1 \cos^2 1 + \sinh^2 1 \sin^2 1}$.
$2^i = e^{-2k\pi}[\cos(\log 2) + i\sin(\log 2)]$, $k \in \mathbb{Z}$.
$i^i = e^{-\frac{\pi}{2}+2k\pi}$, $k \in \mathbb{Z}$.
$(-1)^{2i} = e^{-2(2k+1)\pi}$, $k \in \mathbb{Z}$.
$\log(2-3i) = \log\sqrt{13} + i[2k\pi - \arctan\frac{3}{2}]$, $k \in \mathbb{Z}$.
$\arccos\frac{1}{4}(3+i) = \frac{3+i\pm\sqrt{10}e^{i(\frac{\pi}{2}-\frac{1}{2}\arctan\frac{3}{4})}}{4}$.

□

▶ **28.** (1) Find the values of e^z for $z = \pi i/2$, $-(2/3)\pi i$.
(2) Find the values of z, if z satisfies $e^z = i$.

Solution. $e^{\frac{\pi}{2}i} = i$, $e^{-\frac{2\pi}{3}i} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$. $\log i = (2k\pi + \frac{\pi}{2})i$, $k \in \mathbb{Z}$. □

▶ **29.** Find the real part and the imaginary part of z^z, where $z = x+iy$.

Solution. $z^z = e^{z\log z} = e^{(x+iy)\log(x+iy)} = e^{\frac{x}{2}\log(x^2+y^2)-y(2k\pi+\arctan\frac{y}{x})} \cdot e^{i[\frac{y}{2}\log(x^2+y^2)+x(2k\pi+\arctan\frac{y}{x})]}$, $k \in \mathbb{Z}$. □

▶ **30.** Show that the roots of $z^n = a$ are the vertices of a regular polygon.

Proof. The root of $z^n = a$ are $|a|^{\frac{1}{n}} e^{i\frac{2k\pi+\arg a}{n}}$, $k = 0, 1, \cdots, n-1$. □

▶ **31.** Show that the Cauchy criterion for convergence and the Weierstrass M-test hold for complex field.

Proof. If $u_n(z) = \mathrm{Re} f_n(z)$ and $v_n(z) = \mathrm{Im} f_n(z)$, then by $\max\{|u_n(z)-u_m(z)|, |v_n(z)-v_m(z)|\} \le |f_n(z) - f_m(z)| \le |u_n(z)-u_m(z)| + |v_n(z)-v_m(z)|$, the Cauchy criterion for convergence for complex field is reduced to that for real numbers.

Note $|\sum_{n=k}^m f_n(z)| \le M\sum_{n=k}^m a_n$. So $\sum_{n=1}^\infty f_n(z)$ converges uniformly if $\sum_{n=1}^\infty a_n$ converges. □

▶ **32.** Find the radius of convergence of $\sum_{n=1}^\infty a_n z^n$ if
(i) $a_n = n^{1/n}$;

Solution. Since $\lim_{n\to\infty} n^{\frac{1}{n^2}} = e^{\lim_{n\to\infty}\frac{\ln n}{n^2}} = 1$, Hadamard's formula implies $R = 1$. □

(ii) $a_n = n^{\ln n}$;

Solution. Since $\lim_{n\to\infty} n^{\frac{\ln n}{n}} = e^{\lim_{n\to\infty}[\frac{\ln}{n}\cdot \ln n]}1$, Hadamard's formula implies $R = 1$. □

(iii) $a_n = n!/n^n$;

Solution. By Stirling's formula $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, we have

$$\lim_{n\to\infty}\left[\frac{n!}{n^n}\right]^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln\frac{n!}{n^n}} = \lim_{n\to\infty} e^{\frac{1}{n}\left[\ln(n!)-\ln\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right]} \cdot \lim_{n\to\infty} e^{\frac{1}{n}\ln\sqrt{2\pi n}\left(\frac{n}{e}\right)^n-\ln n} = e^{-1}.$$

So Hadamard's formula implies $R = e$. □

(iv) $a^n = n^n$.

Solution. Since $\lim_{n\to\infty} n = \infty$, Hadamard's formula implies $R = 0$. □

▶ **33.** Prove Theorem 1.15.

Proof. Let $u(z) = \mathrm{Re} f(z)$ and $v(z) = \mathrm{Im} f(z)$. Then f is uniformly convergent on a set A if and only if u and v are uniformly convergent on A. Since $dz = dx + idy$, we can reduce the theorem to that of real-valued functions of real variables. □

▶ **34.** Let $f(z)$ be holomorphic on $\mathbb{C}$ and $f(0) = 1$. Show that:
(1) If $f'(z) = f(z)$ for all $z \in \mathbb{C}$, then $f(z) = e^z$.

Proof. $[f(z)e^{-z}]' = f'(z)e^{-z} - f(z)e^{-z} = 0$. So $f(z) = Ce^z$ for some constant C. $f(0) = 1$ dictates $C = 1$. □

(2) If for every $z \in \mathbb{C}$ and every $\omega \in \mathbb{C}$, $f(z+\omega) = f(z)f(\omega)$ and $f'(0) = 1$, then $f(z) = e^z$.

Proof. $f'(z) = \lim_{\omega\to 0}\frac{f(z+\omega)-f(z)}{\omega} = \lim_{\omega\to 0} f(z)\frac{f(\omega)-1}{\omega} = f(z)f'(0) = f(z)$. By part (1), we conclude $f(z) = e^z$. □

▶ **35.** Let f be a holomorphic function on $\mathbb{C}\backslash(-\infty, 0]$. Show that:
(1) If $f(z) = e^{-f(z)}$ for every $z \in \mathbb{C}\backslash(-\infty, 0]$, then $f(z) = \log z$.

Proof. $[e^{f(z)}]' = f'(z)e^{f(z)} = 1$. So $e^{f(z)} = 1$. □

(2) If $f(z\omega) = f(z) + f(\omega)$ for every $z \in \mathbb{C}\backslash(-\infty, 0]$, $\omega \in (0, +\infty)$ and $f'(1) = 1$, then $f(z) = \log z$.

Proof. It's clear $f(1) = 0$. So

$$f'(z) = \lim_{\omega\to 0}\frac{f(z+z\omega)-f(z)}{z\omega} = \lim_{\omega\to 0}\frac{f(z(1+\omega))-f(z)}{z\omega} = \frac{1}{z}\lim_{\omega\to 0}\frac{f(1+\omega)}{\omega} = \frac{1}{z}f'(1) = \frac{1}{z}.$$

Therefore

$$\left[\frac{1}{z}e^{f(z)}\right]' = -\frac{1}{z^2}e^{f(z)} + \frac{1}{z}e^{f(z)}f'(z) = 0.$$

This shows $e^{f(z)} = z$. □

▶ **36.** If

$$\varphi(z) = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

then φ is univalent on the following four regions:
(1) upper half plane $\{z \in \mathbb{C} : \mathrm{Im} z > 0\}$.
(2) lower half plane $\{z \in \mathbb{C} : \mathrm{Im} z < 0\}$.
(3) unit disc $D(0,1)$ except the center point $\{0\}$, $D(0,1)\backslash\{0\}$.
(4) outside of the unit disc $\{z \in \mathbb{C} : |z| > 1\}$.

Proof. $2[\varphi(z_1) - \varphi(z_2)] = (z_1 - z_2)\left(1 - \frac{1}{z_1 z_2}\right)$. So $\varphi(z)$ is univalent in any region that makes $z_1 z_2 \neq 1$ whenever $z_1 \neq z_2$. Since $z_1 z_2 = 1$ implies $|z_1| = \frac{1}{|z_2|}$ and $\arg z_1 = -\arg z_2$, we conclude φ is univalent in any of the four regions in the problem. □

► **43.** If $f(z)$ is a univalent holomorphic function on a region U, then the area of $f(U)$ is

$$\int\int_U |f'(z)|^2 dxdy$$

where $z = x + iy$.

Proof. This is straightforward from Problem 24. □

► **44.** Show that if $\{a_n\}$ and $\{b_n\}$ satisfy the conditions:
(1) the sequence $\{S_n\}$ is bounded, where

$$S_n = \sum_{k=1}^{n} a_k,$$

(2) $\lim_{n\to\infty} b_n = 0$.
(3) $\sum_{n=1}^{\infty} |b_n - b_{n+1}| < \infty$,
then the series

$$\sum_{n=1}^{\infty} a_n b_n$$

is convergent. Moreover, verify that this is a generalization of the Dirichlet criterion and the Abel criterion for series in the real number field.

Proof. We first use Abel's Lemma on partial sum: for $n > m$,

$$\sum_{k=m+1}^{n} a_k b_k = \sum_{k=m+1}^{n} (S_k - S_{k-1}) b_k = S_b b_{n-1} + \sum_{k=m+1}^{n} S_k (b_k - b_{k+1}) - S_m b_{m+1}.$$

Denote by K a bound of the sequence $(S_n)_{n=1}^{\infty}$, then $|S_n b_{n+1}| \leq K|b_{n+1}|$, $|S_m b_{m+1}| \leq K|b_{m+1}|$, and $|\sum_{k=m+1}^{n} S_k(b_k - b_{k+1})| \leq K \sum_{k=m+1}^{n} |b_k - b_{k+1}|$. Since $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=1}^{\infty} |b_n - b_{n+1}| < \infty$, we conclude $\sum_{k=m+1}^{n} a_k b_k$ can be arbitrarily small if m is sufficiently large. So under conditions (1)-(3), $\sum_{n=1}^{\infty} a_k b_k$ is convergent.

In the Dirichlet criterion, we require
(1) $(S_n)_{n=1}^{\infty}$ is bounded;
(2) $\lim_{n\to\infty} b_n = 0$;
(3') $(b_n)_{n=1}^{\infty}$ is monotone.

Clearly, (2)+(3') imply (3), so the Dirichlet criterion is a special case of the result in current problem. In the Abel criterion, we require
(1") $(S_n)_{n=1}^{\infty}$ is convergent;
(2") $(b_n)_{n=1}^{\infty}$ is bounded;
(3") $(b_n)_{n=1}^{\infty}$ is monotone.

Define $b'_n = b_n - b$, where $b = \lim_{n\to\infty} b_n$ is a finite number. Then $(S_n)_{n=1}^{\infty}$ satisfies (1) and $(b'_n)_{n=1}^{\infty}$ satisfies (2) and (3'). By the Dirichlet criterion, $\sum_{n=1}^{\infty} a_n b'_n$ converges. Note $\sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N} a_n b'_n + b S_N$. By condition (1"), we conclude $\sum_{n=1}^{\infty} a_n b_n$ converges. □

Remark 1. *The result in this problem is the so-called Abel-Dedekind-Dirichlet Theorem.*

► **45.** Let $\sum_{n=1}^{\infty} a_n$ be a complex series, and $\overline{\lim}_{n\to\infty} |a_n|^{1/n} = q$. Prove:
(1) If $q < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Proof. For any $\varepsilon \in (0, 1-q)$, there exists $N \in \mathbb{N}$, such that for any $n \geq N$, $|a_n|^{\frac{1}{n}} < q + \varepsilon < 1$. By comparing $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} (q+\varepsilon)^n$, we conclude $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. □

(2) If $q > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof. By definition of upper limit and the fact $q > 1$, we can find $\varepsilon > 0$ and infinitely many n, such that $|a_n|^{\frac{1}{n}} > q - \varepsilon > 1$. Therefore $\overline{\lim}_{n\to\infty} |a_n| = \infty$ and $\sum_{n=1}^{\infty} a_n$ is divergent. □

▶ **46.** Show that if $a_n \in \mathbb{C}\backslash\{0\}$, $n = 1, 2, \cdots$, and

$$\overline{\lim}_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = q,$$

then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $q < 1$. Discuss the convergence and divergence of the series if $q > 1$.

Proof. If $q < 1$, for any $\varepsilon \in (0, 1-q)$, we can find $N \in \mathbb{N}$, such that for any $n \geq N$, $\frac{|a_{n+1}|}{|a_n|} < q + \varepsilon < 1$. So $|a_{N+k}| \leq |a_N|(q+\varepsilon)^k$, $\forall k \geq 0$. Since $\sum_{k=1}^{\infty} (q+\varepsilon)^k$ is convergent, we conclude $\sum_{n=1}^{\infty} a_n$ is also convergent.

If $q > 1$, we have two cases to consider. In the first case, the upper limit is assumed to be a limit. Then we can find $\varepsilon > 0$ such that $q - \varepsilon > 1$ and for n sufficiently large, $|a_{n+1}| > |a_n|(q-\varepsilon) > |a_n|$. This implies $\lim_{n\to\infty} a_n \neq 0$, so the series is divergent. In the second case, the upper limit is assumed not to be a limit. Then we can manufacture counter examples where the series is convergent. Indeed, consider ($k \geq 0$)

$$a_n = \begin{cases} \frac{1}{(k+1)^2}, & \text{if } n = 2k+1 \\ \frac{2}{(k+1)^2}, & \text{if } n = 2k+2. \end{cases}$$

Then $\sum_{n=1}^{\infty} a_n$ is convergent and $\overline{\lim}_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 2$. □

▶ **47.** Let $a_n \in \mathbb{C}\backslash\{0\}$ ($n = 1, 2, \cdots$) and

$$\overline{\lim}_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1.$$

Show that if

$$\overline{\lim}_{n\to\infty} n\left(\left|\frac{a_{n+1}}{a_n}\right| - 1\right) < -1,$$

then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (*Reabe criterion*).

Proof. We choose $\varepsilon > 0$, such that $\overline{\lim}_{n\to\infty} n\left(|\frac{a_{n+1}}{a_n}| - 1\right) < -1 - 2\varepsilon$. By the definition of upper limit, there exists $N_1 \in \mathbb{N}$, such that for any $n \geq N_1$, $n\left(|\frac{a_{n+1}}{a_n}| - 1\right) < -1 - 2\varepsilon$, i.e. $\frac{|a_{n+1}|}{|a_n|} < 1 - \frac{1+2\varepsilon}{n}$. Define $b_n = \frac{1}{n^{1+\varepsilon}}$. Then

$$\frac{b_{n+1}}{b_n} = \frac{n^{\varepsilon}}{(n+1)^{1+\varepsilon}} = \left(1 - \frac{1}{n+1}\right)^{1+\varepsilon} = 1 - \frac{1+\varepsilon}{n+1} + O\left(\frac{1}{(n+1)^2}\right).$$

So there exists $N_2 \in \mathbb{N}$, such that for $n \geq N_2$, $\frac{b_{n+1}}{b_n} > 1 - \frac{1+2\varepsilon}{n+1}$. Define $N = \max\{N_1, N_2\}$, then for any $n \geq N$, $\frac{a_{n+1}}{a_n} < \frac{b_{n+1}}{b_n}$. In particular, we have

$$\left|\frac{a_{N+1}}{a_N}\right| < \frac{b_{N+1}}{b_N}, \ \left|\frac{a_{N+2}}{a_{N+1}}\right| < \frac{b_{N+2}}{b_{N+1}}, \ \cdots, \ \left|\frac{a_{N+k}}{a_{N+k-1}}\right| < \frac{b_{N+k}}{b_{N+k-1}}.$$

Multiply these inequalities, we get

$$|a_{N+k}| \leq \frac{|a_N|}{b_N} b_{N+k}.$$

Since $\sum_{n=1}^{\infty} b_n$ is convergent, we conclude $\sum_{n=1}^{\infty} a_n$ must also converge. [1] □

[1] It seems the condition $\overline{\lim}_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ is redundant.

► **48.** Let $\{a_n\}$ be a series of positive numbers that converges to zero monotonically. Prove:
(1) If R is the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$, then $R \geq 1$.

Proof. Since $(a_n)_{n=1}^{\infty}$ monotonically decreases to 0, for n sufficiently large, $\ln a_n < 0$ and $\overline{\lim}_{n\to\infty} \frac{\ln a_n}{n} \leq 0$. So $\overline{\lim}_{n\to\infty} a_n^{\frac{1}{n}} = e^{\overline{\lim}_{n\to\infty} \frac{\ln a_n}{n}} \leq e^0 = 1$. By Hadamard's formula, $R \geq 1$. □

(2) The series $\sum_{n=0}^{\infty} a_n z^n$ is convergent on $\partial D(0,R)\backslash\{R\}$, where $\partial D(0,R)$ is the boundary of the disc $D(0,R)$.

Proof. The claims seems problematic. For a counter example, assume $R > 1$ and $a_n = \frac{1}{R^n}$. If $z = Re^{i\theta}$ with $\theta \neq 0$, we have

$$\sum_{n=0}^{N} a_n z^n = \sum_{n=0}^{N} \frac{1}{R^n} \cdot R^n e^{in\theta} = \sum_{n=0}^{N} \cos n\theta + i \sum_{n=0}^{N} \sin n\theta = \frac{\sin\frac{(N+1)\theta}{2} \cos\frac{N\theta}{2}}{\sin\frac{\theta}{2}} + i \frac{\sin\frac{(N+1)\theta}{2} \sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}}.$$

So $\sum_{n=0}^{\infty} a_n z^n$ is not convergent. □

► **49.** Show that the power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is uniformly convergent on its disc of convergence D if and only if it is uniformly convergent on $\overline{D}$. Here $z_0 \in \mathbb{C}$ is a fixed point and $\overline{D}$ is the closure of D.

Proof. By definition of uniform convergence, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that for any $m \geq N$, $\sum_{n=m}^{\infty} |a_n||z - z_0|^n < \varepsilon$, $\forall z \in D$. In particular, for any $z^* \in \partial D$, by letting $z \to z^*$, we get $\sum_{n=m}^{\infty} |a_n||z^* - z_0|^n \leq \varepsilon$. That is, for this given ε, we can pick up N, such that for any $m \geq N$, $\sum_{n=m}^{\infty} |a_n||z - z_0|^n \leq \varepsilon$, $\forall z \in \bar{D}$. This is exactly the definition of uniform convergence on $\bar{D}$. □

► **50.** Suppose that the radius of convergence of the series

$$\sum_{n=0}^{\infty} a_n z^n = f(z)$$

is 1, and $z_0 \in \partial D(0,1)$. Prove that if $\lim_{n\to\infty} n a_n = 0$ and $\lim_{r\to 1} f(rz_0)$ exists, then $\sum_{n=0}^{\infty} a_n z_0^n$ converges to $\lim_{n\to\infty} f(rz_0)$.

Proof. We choose a sequence $(r_n)_{n=1}^{\infty}$ such that $r_n \in (0,1)$ and $\lim_{n\to\infty} r_n = 1$. Define $\beta_n = \sum_{k=0}^{n} a_k z_0^k - f(r_n z_0)$. We show $\lim_{n\to\infty} \beta_n = 0$. Indeed, we note

$$\begin{aligned} |\beta_n| &= \left| \sum_{k=0}^{n} a_k z_0^k - \sum_{k=0}^{\infty} a_k (r_n z_0)^k \right| = \left| \sum_{k=0}^{n} a_k (1 - r_n^k) z_0^k - \sum_{k=n+1}^{\infty} a_k (r_n z_0)^k \right| \\ &\leq \sum_{k=0}^{n} |a_k| (1 - r_n)(1 + r_n + \cdots + r_n^{k-1}) + \sum_{k=n+1}^{\infty} \frac{k}{n} |a_k| r_n^k \\ &\leq \sum_{k=0}^{n} |a_k| (1 - r_n) k + \frac{\varepsilon_{n+1}}{n} \frac{1}{1 - r_n}, \end{aligned}$$

where $\varepsilon_n = \sum_{k\geq n} k|a_k|$. We choose r_n in such a way that $\lim_{n\to\infty} n(1 - r_n) = 1$, e.g. $r_n = 1 - \frac{1}{n}$. Then $\lim_{n\to\infty} \frac{\varepsilon_{n+1}}{n(1-r_n)} = 0$ since $\lim_{n\to\infty} n a_n = 0$. By Problem 18 (2), we have

$$\sum_{k=0}^{n} |a_k| (1 - r_n) k = (1 - r_n) n \cdot \frac{\sum_{k=0}^{n} |a_k| k}{n} \to 0, \text{ as } n \to \infty.$$

Combined, we conclude $\lim_{n\to\infty} \beta_n = 0$. □

Remark 2. *The result in this problem is a special case of the so-called Landau's Theorem, see, for example, Boos [1], §5.1 Boundary behavior of power series.*

2 Cauchy Integral Theorem and Cauchy Integral Formula

► **1.** Calculate the following by using the Cauchy Integral Formula.

(i) $\int_{|z+i|=3} \sin z \frac{dz}{z+i}$;

(ii) $\int_{|z|=2} \frac{e^z}{z-1} dz$;

(iii) $\int_{|z|=4} \frac{\cos z}{z^2-\pi^2} dz$;

(iv) $\int_{|z|=2} \frac{dz}{(z-1)^n(z-3)}$, $n = 1, 2, \cdots$;

(v) $\int_{|z|=\frac{3}{2}} \frac{dz}{(z^2+1)(z^2+4)}$;

(vi) $\int_{|z|=2} \frac{dz}{z^5-1}$;

(vii) $\int_{|z|=R} \frac{dz}{(z-a)^n(z-b)}$, where a, b are not on the circle $|z| = R$ and n is a positive integer.

(viii) $\int_{|z|=3} \frac{dz}{(z^3-1)(z-2)^2}$.

Solution. (i) The integral is equal to $2\pi i \cdot \sin i = \pi(e^{-1} - e)$.

(ii) The integral is equal to $2e\pi i$.

(iii) For sufficiently small $\varepsilon > 0$, the integral is equal to

$$\int_{|z-\pi|=\varepsilon} \frac{\cos z}{(z+\pi)(z-\pi)} dz + \int_{|z+\pi|=\varepsilon} \frac{\cos z}{(z+\pi)(z-\pi)} dz = 2\pi i \left[\frac{\cos \pi}{2\pi} + \frac{\cos(-\pi)}{-2\pi}\right] = 0.$$

(iv) The integral is equal to $2\pi i \left(\frac{1}{z-3}\right)^{(n-1)}\Big|_{z=1} = 2\pi i \cdot \frac{-(n-1)!}{2^n} = -\frac{(n-1)!}{2^{n-1}}\pi i$.

(v) For sufficiently small $\varepsilon > 0$he integral is equal to

$$\int_{|z+i|=\varepsilon} \frac{1}{(z-i)(z^2+4)} \cdot \frac{dz}{(z+i)} + \int_{|z-i|=\varepsilon} \frac{1}{(z+i)(z^2+4)} \cdot \frac{dz}{(z-i)} = 2\pi i \cdot \left[\frac{1}{(-2i)\cdot 3} + \frac{1}{2i \cdot 3}\right] = 0.$$

(vi) By the Fundamental Theorem of Algebra, the equation $z^5 - 1 = 0$ has five solutions z_1, z_2, z_3, z_4, z_5. For any $i \in \{1, 2, 3, 4, 5\}$, we have

$$(z_i - z_1)\cdots(z_i - z_{i-1})(z_i - z_{i+1})\cdots(z_i - z_5) = \lim_{z\to z_i} \frac{z^5-1}{z-z_i} = \lim_{z\to z_i} \frac{z^5 - z_i^5}{z-z_i} = 5z_i^4 = \frac{5}{z_i}.$$

So for sufficiently small $\varepsilon > 0$, the integral is equal to

$$\begin{aligned} & \sum_{i=1}^{5} \int_{|z-z_i|=\varepsilon} \frac{dz}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)} \\ = \ & 2\pi i \sum_{i=1}^{5} \frac{1}{(z_i - z_1)\cdots(z_i - z_{i-1})(z_i - z_{i+1})\cdots(z_i - z_5)} \\ = \ & 2\pi i \sum_{i=1}^{6} \frac{z_i}{5} \\ = \ & 0, \end{aligned}$$

where the last equality comes from the fact that the expansion of $(z-z_1)\cdots(z-z_5)$ is z^5-1 and $-(z_1+\cdots+z_5)$ is the coefficient of z^4.

(vii) If $|a|, |b| > R$, the integral is equal to 0. If $|a| > R > |b|$, the integral is equal to $\frac{2\pi i}{(b-a)^n}$. If $|b| > R > |a|$, the integral is equal to $2\pi i \left(\frac{1}{z-b}\right)^{(n-1)}\Big|_{z=a} = 2\pi i \cdot (-1)^{n-1}(a-b)^{-n}$. If $R > |a|, |b|$, the integral is equal to $2\pi i \left[\frac{1}{(b-a)^n} + (-1)^{n-1}\frac{1}{(a-b)^n}\right] = 0$.

(viii) Denote by z_1, z_2, z_3 the three roots of the equation $z^3 - 1 = 0$. Then the integral is equal to

$$\begin{aligned}
& 2\pi i \left[\frac{1}{(z_1-z_2)(z_1-z_3)(z_1-2)^2} + \frac{1}{(z_2-z_1)(z_2-z_3)(z_2-2)^2} + \frac{1}{(z_3-z_1)(z_3-z_2)(z_3-2)^2} \right. \\
& \left. + \left(\frac{1}{z^3-1}\right)^{(1)} \Bigg|_{z=2} \right] \\
= \; & 2\pi i \left[\frac{z_1}{3(z_1-2)^2} + \frac{z_2}{3(z_2-2)^2} + \frac{z_3}{3(z_3-2)^2} - \frac{12}{49} \right].
\end{aligned}$$

□

► **2.** Prove that

$$\left(\frac{z^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_C \frac{z^n e^{z\zeta}}{n!\zeta^n} \cdot \frac{d\zeta}{\zeta},$$

where C is a simple closed curve around the origin.

Proof.

$$\frac{1}{2\pi i}\int_C \frac{e^{z\zeta}}{\zeta^n}\cdot\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int_C \sum_{k=0}^{\infty}\frac{z^k}{k!}\frac{1}{\zeta^{n-k}}\cdot\frac{d\zeta}{\zeta} = \sum_{k=0}^{\infty}\frac{z^k}{k!}\cdot\frac{1}{2\pi i}\int_C \frac{1}{\zeta^{n-k}}\cdot\frac{d\zeta}{\zeta} = \frac{z^n}{n!},$$

where the last equality is due to Cauchy's integral formula. We need to justify the exchange of integration and infinite series summation. Indeed, for $m > n$, we have

$$\begin{aligned}
& \left| \sum_{k=0}^{m}\frac{z^k}{k!}\cdot\frac{1}{2\pi i}\int_C \frac{1}{\zeta^{n-k}}\cdot\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int_C \sum_{k=0}^{\infty}\frac{z^k}{k!}\frac{1}{\zeta^{n-k}}\cdot\frac{d\zeta}{\zeta} \right| \\
= \; & \left| \frac{1}{2\pi i}\int_C \sum_{k=m+1}^{\infty}\frac{z^k}{k!}\frac{1}{\zeta^{n-k}}\cdot\frac{d\zeta}{\zeta} \right| \\
\le \; & \frac{1}{2\pi}\int_C \sum_{k=m+1}^{\infty}\frac{|z\zeta|^k}{k!}\frac{|d\zeta|}{|\zeta|^{n+1}} \\
\le \; & \frac{1}{2\pi}\int_C \sum_{k=m+1}^{\infty}\frac{M^k}{k!}\frac{|d\zeta|}{|\zeta|^{n+1}},
\end{aligned}$$

where $M = |z|\max_{\zeta\in C}|\zeta|$. Since $\sum_{k=m+1}^{\infty}\frac{M^k}{k!}$ is the remainder of e^M, for m sufficiently large, $\sum_{k=m+1}^{\infty}\frac{M^k}{k!}$ can be smaller than any given positive number ε. So for m sufficiently large,

$$\left| \sum_{k=0}^{m}\frac{z^k}{k!}\cdot\frac{1}{2\pi i}\int_C \frac{1}{\zeta^{n-k}}\cdot\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int_C \sum_{k=0}^{\infty}\frac{z^k}{k!}\frac{1}{\zeta^{n-k}}\cdot\frac{d\zeta}{\zeta} \right| \le \varepsilon\frac{1}{2\pi}\int_C \frac{|d\zeta|}{|\zeta|^{n+1}}.$$

This shows $\sum_{k=0}^{\infty}\frac{z^k}{k!}\cdot\frac{1}{2\pi i}\int_C\frac{1}{\zeta^{n-k}}\cdot\frac{d\zeta}{\zeta}$ converges to $\frac{1}{2\pi i}\int_C\sum_{k=0}^{\infty}\frac{z^k}{k!}\frac{1}{\zeta^{n-k}}\cdot\frac{d\zeta}{\zeta}$. □

► **3.** Let f and g be holomorphic in the unit disc $|z| < 1$ and continuous on $|z| \le 1$. Show that

$$\frac{1}{2\pi i}\int_{|\zeta|=1}\left(\frac{f(\zeta)}{\zeta - z} + \frac{zg(\zeta)}{z\zeta - 1}\right)d\zeta = \begin{cases} f(z), & |z| < 1, \\ g\left(\frac{1}{z}\right), & |z| > 1. \end{cases}$$

Proof. We note $\frac{g(\zeta)}{z\zeta-1} = \frac{g(\zeta)}{\zeta-\frac{1}{z}}$. If $|z| < 1$, we apply Cauchy's integral formula to $\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta-z}d\zeta$ and apply Cauchy's integral theorem to $\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{g(\zeta)}{\zeta-\frac{1}{z}}d\zeta$ (regarding $\frac{1}{z}$ as a parameter and $\frac{g(\zeta)}{\zeta-\frac{1}{z}}$ an analytic function of ζ). If $|z| > 1$, we apply Cauchy's integral theorem to $\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta-z}d\zeta$ (regarding z as a parameter and $\frac{f(\zeta)}{\zeta-z}$ an analytic function of ζ) and apply Cauchy's integral formula to $\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{g(\zeta)}{\zeta-\frac{1}{z}}d\zeta$. □

▶ **4.** Let

$$f(z) = \int_{|\zeta|=3} \frac{3\zeta^2 + 7\zeta + 1}{\zeta - z} d\zeta.$$

Find $f'(1+i)$.

Solution. For $z \in \{z : |z| < 3\}$, by Cauchy's integral formula, $f(z) = 2\pi i(3z^3 + 7z + 1)$. So $f'(1+i) = 2\pi i[6(1+i) + 7] = -12\pi + 26\pi i$. □

▶ **5.** Show that

$$\int_0^{2\pi} \cos^{2n} \theta d\theta = 2\pi \cdot \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}$$

by calculating

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}.$$

Proof. We note

$$\int_{|z|=1} (z + \frac{1}{z})^{2n} \frac{dz}{z} = \int_0^{2\pi} (e^{i\theta} + e^{-i\theta})^{2n} i d\theta = 2^{2n} i \int_0^{2\pi} \cos^{2n} \theta d\theta.$$

On the other hand,

$$(z + \frac{1}{z})^{2n} \cdot \frac{1}{z} = \sum_{k=0}^{2n} C_{2n}^k z^k (z^{-1})^{2n-k} \cdot z^{-1} = \sum_{k=0}^{2n} \frac{C_{2n}^k}{z^{-2n+2k-1}}.$$

Therefore

$$\int_0^{2\pi} \cos^{2n} \theta d\theta = \frac{1}{2^{2n} i} \int_{|z|=1} (z + \frac{1}{z})^{2n} \frac{dz}{z} = \frac{1}{2^{2n} i} \int_{|z|=1} \frac{C_{2n}^n dz}{z} = \frac{2\pi C_{2n}^n}{2^{2n}} = 2\pi \cdot \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}.$$

□

▶ **6.** Let

$$p_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n}[(z^2 - 1)^n].$$

This is referred to as *Legendre polynomial*. Show that

$$p_n(z) = \frac{1}{2\pi i} \int_\gamma \frac{(\zeta^2 - 1)^n}{2^n (\zeta - z)^{n+1}} d\zeta,$$

where γ is a simple closed curve containing z.

Proof. This is straightforward from Cauchy's integral formula. □

▶ **7.** Let $f(z)$ be a holomorphic function on $\mathbb{C}$ and $|f(z)| \le Me^{|z|}$. Show that $|f(0)| \le M$ and

$$\frac{|f^n(0)|}{n!} \le M \left(\frac{e}{n}\right)^n, \; n = 1, 2, \cdots.$$

Proof. For any $R > 0$, we have $f^{(n)}(0) = \frac{n!}{2\pi i} \int_{|z|=R} \frac{f(\xi)}{\xi^{n+1}} d\xi$. So

$$\left| \frac{f^{(n)}(0)}{n!} \right| = \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{|f(Re^{i\theta})|}{R^n} d\theta \right| \le \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{Me^{|Re^{i\theta}|}}{R^n} d\theta \right| = M \frac{e^R}{R^n}.$$

Let $R = n$, we get the desired inequality. □

Remark 3. *Note $h(R) = \ln \frac{e^R}{R^n} = R - n \ln R$ takes minimal value at n. This is the reason why we choose $R = n$, to get the best possible estimate. Another perspective is to assume $f(z) = e^z$. Then $\left|\frac{f^{(n)}(0)}{n!}\right| = \frac{1}{n!} \sim \frac{1}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = \frac{1}{\sqrt{2\pi n}}\left(\frac{e}{n}\right)^n$ by Stirling's formula. So the bound $M\frac{e^R}{R^n}$ is not "far" from the best one.*

► **8.** (*Cauchy integral formula for the outside region of a simple closed curve*) Let γ be a rectifiable simple closed curve, D_1 be the inside of γ and D_2 be the outside of γ. Suppose a function $f(z)$ is holomorphic in D_2 and continuous on $D_2 \cup \gamma$. Show that

(1) If $\lim_{z\to\infty} f(z) = A$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta = \begin{cases} -f(z) + A, & z \in D_2, \\ A, & z \in D_1; \end{cases}$$

Proof. For any $z \in D_2$, we can find $R > 0$ sufficiently large, so that z is in the region enclosed by γ and $\{\zeta : |\zeta - z| = R\}$. By Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_{-\gamma} \frac{f(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)}{\zeta - z}d\zeta.$$

So

$$\begin{aligned} \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta &= -f(z) + \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)}{\zeta - z}d\zeta \\ &= -f(z) + \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta) - f(\infty)}{\zeta - z}d\zeta + \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\infty)}{\zeta - z}d\zeta. \end{aligned}$$

We note

$$\left|\frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta) - f(\infty)}{\zeta - z}d\zeta\right| \le \sup_{\zeta \in D(z,R)} |f(\zeta) - A| \to 0, \text{ as } R \to \infty,$$

and Cauchy's integral formula implies

$$\frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\infty)}{\zeta - z}d\zeta = f(\infty) \cdot 1 = A.$$

So by letting $R \to \infty$ in the above equality, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta = -f(z) + A.$$

If $z \in D_1$, $h(\zeta) = \frac{f(\zeta)}{\zeta - z}$ is a holomorphic function in D_2, with z a parameter. So for R sufficiently large, Cauchy's integral theorem implies

$$0 = \frac{1}{2\pi i}\int_{-\gamma} \frac{f(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)}{\zeta - z}d\zeta.$$

An argument similar to (1) can show $\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta = A$. □

(2) If the origin is in D_1, then

$$\frac{1}{2\pi i}\int_\gamma \frac{zf(\zeta)}{z\zeta - \zeta^2}d\zeta = \begin{cases} f(z), & z \in D_2, \\ 0, & z \in D_1. \end{cases}$$

Proof. We note $\frac{z}{z\zeta-\zeta^2} = \frac{1}{\zeta} - \frac{1}{\zeta-z}$. For R sufficiently large, Cauchy's integral theorem implies

$$0 = \frac{1}{2\pi i}\int_{-\gamma}\frac{f(\zeta)}{\zeta}d\zeta + \frac{1}{2\pi i}\int_{|\zeta|=R}\frac{f(\zeta)}{\zeta}d\zeta.$$

So $\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta}d\zeta = \frac{1}{2\pi i}\int_{|\zeta|=R}\frac{f(\zeta)}{\zeta}d\zeta$. Similarly,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-z}d\zeta = \begin{cases} \frac{1}{2\pi i}\int_{|\zeta|=R}\frac{f(\zeta)}{\zeta-z}d\zeta - f(z) & z\in D_2 \\ \frac{1}{2\pi i}\int_{|\zeta|=R}\frac{f(\zeta)}{\zeta-z}d\zeta & z\in D_1. \end{cases}$$

Therefore

$$\frac{1}{2\pi i}\int_\gamma \frac{zf(\zeta)}{z\zeta-\zeta^2}d\zeta = \frac{1}{2\pi i}\int_\gamma\left[\frac{f(\zeta)}{\zeta} - \frac{f(\zeta)}{\zeta-z}\right]d\zeta = \begin{cases} f(z) & z\in D_2 \\ 0 & z\in D_1. \end{cases}$$

□

▶ **9.** Let $f(z)$ be holomorphic in a bounded region D and continuous on $\overline{D}$ with $f(z)\neq 0$. Show that if $|f(z)| = M$ on ∂D, then $f(z) = Me^{i\alpha}$, where α is a real number.

Proof. Since $f(z)\neq 0$ in D, $1/f(z)$ is holomorphic in D. Applying Maximum Modulus Principle to $1/f(z)$, we conclude $|f(z)|$ achieves its minimum on ∂D. Applying Maximum Modulus Principle to $f(z)$, we conclude $|f(z)|$ achieves its maximum on ∂D. Since $|f(z)|\equiv M$ on ∂D, $|f(z)|$ must be a constant on $\bar{D}$. By Chapter 1, exercise problem 21 (iii), we conclude $f(z)$ is a constant function. □

▶ **10.** Suppose $f(z)$ is holomorphic and bounded in $\mathbb{C}$, a, b are complex numbers. Find

$$\lim_{R\to\infty}\int_{|z|=R}\frac{f(z)}{(z-a)(z-b)}dz.$$

This leads to another proof of Liouville Theorem.

Proof. For R large enough, $|a|, |b| < R$. So we can find $\varepsilon > 0$ such that (assume $a\neq b$)

$$\int_{|z|=R}\frac{f(z)}{(z-a)(z-b)}dz = \int_{|z-a|=\varepsilon}\frac{f(z)}{(z-a)(z-b)}dz + \int_{|z-b|=\varepsilon}\frac{f(z)}{(z-a)(z-b)}dz = \frac{2\pi i}{a-b}[f(a)-f(b)].$$

Meanwhile, denote by M a bound of f, we have

$$\begin{aligned}\left|\int_{|z|=R}\frac{f(z)}{(z-a)(z-b)}dz\right| &\leq M\int_{|z|=R}\frac{|dz|}{|(z-a)(z-b)|} \\ &\leq M\int_{|z|=R}\frac{|dz|}{(R-|a|)(R-|b|)} \\ &= \frac{2\pi MR}{(R-|a|)(R-|b|)}\to 0, \text{ as } R\to\infty.\end{aligned}$$

Combined, we conclude $0 = \frac{2\pi i}{a-b}[f(a)-f(b)]$, $\forall a, b\in\mathbb{C}$. This shows f is a constant function. □

▶ **11.** If $f(z)$ is holomorphic in the unit disc $|z| < 1$ and

$$|f(z)| \leq \frac{1}{1-|z|}\ (|z|<1),$$

then

$$|f^{(n)}(0)| \leq \frac{n!}{r^n(1-r)}$$

for $0 < r < 1$. Especially, if $r = 1 - 1/(n+1)$, then

$$|f^{(n)}(0)| < e(n+1)!,\ n = 1, 2, \cdots.$$

Proof. By Cauchy's integral formula,

$$|f^{(n)}(0)| \leq \frac{n!}{2\pi}\left|\int_{|z|=r}\frac{f(\xi)}{\xi^{n+1}}d\xi\right| \leq \frac{n!}{2\pi}\int_0^{2\pi}\frac{\frac{1}{1-r}}{r^n}d\theta = \frac{n!}{r^n(1-r)}.$$

□

► **12.** (*Integral of Cauchy type*) If the function $\varphi(\zeta)$ is continuous on a rectifiable curve γ, show that the function

$$\Phi(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z}$$

is holomorphic in any region D which does not contain any point of γ. Moreover,

$$\Phi^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{\varphi(\zeta)d\zeta}{(\zeta-z)^{n+1}},\ n = 1, 2, \cdots.$$

Proof. We note

$$\lim_{z\to z_0}\frac{\Phi(z)-\Phi(z_0)}{z-z_0} = \frac{1}{2\pi i}\lim_{z\to z_0}\int_\gamma \frac{\varphi(\zeta)}{(\zeta-z)(\zeta-z_0)}d\zeta = \frac{1}{2\pi i}\int_\gamma\frac{\varphi(\zeta)}{(\zeta-z_0)^2}d\zeta,$$

where the exchange of limit and integration can be seen as an application of Lebesgue's Dominated Convergence Theorem (We can find $\rho > 0$ so that $\forall z$ in a neighborhood of z_0, $\mathrm{dist}(\gamma, z) \geq \rho$. Then $\frac{1}{(\zeta-z)(\zeta-z_0)}) \leq \frac{1}{\rho^2}$). This shows Φ is holomorphic in any region D which does not contain any point of γ.

The formula for n-th derivative of Φ can be proven similarly and by induction. For details, we refer the reader to 方企勤 [3], Chapter 4, §3, Lemma 2. □

Remark 4. *The result still holds if φ is piecewise continuous on γ. See, for example, 方企勤 [3], Chapter 7, proof of Theorem 7.*

► **13.** Suppose that a non-constant function $f(z)$ is holomorphic on a bounded region D and continuous on $\overline{D}$. Let $m = \inf_{\tau\in\partial D}|f(z)|$, $M = \sup_{z\in\partial D}|f(z)|$ and $f(z) \neq 0$. Then $m < |f(z)| < M$ for any point $z \in D$.

Proof. Since $f(z) \neq 0$ in D, $1/f(z)$ is holomorphic in D. Applying Maximum Modulus Principle to $1/f(z)$, we conclude $|f(z)|$ does not achieve its minimum in D unless it's a constant function. Applying Maximum Modulus Principle to $f(z)$, we conclude $f(z)$ does not achieve its maximum in D unless it's a constant function. Combined, we have $m < |f(z)| < M$ for any point $z \in D$. □

► **14.** If $p_n(z)$ is a polynomial with degree n and $|p_n(z)| \leq M$ for $|z| < 1$, then $|p_n(z)| \leq M|z|^n$ for $1 \leq |z| < \infty$.

Proof. Define $f(z) = \frac{p_n(z)}{z^n}$ and $g(z) = f(\frac{1}{z})$. Then $g(z)$ is holomorphic and bounded on $D(0,1)\setminus\{0\}$, since $\lim_{z\to 0} g(z) = \lim_{z\to\infty}\frac{p_n(z)}{z^n}$ is a finite number and $g(z)$ is clearly continuous in a neighborhood of $\partial D(0,1)$. By Theorem 2.11, Riemann's Theorem, $g(z)$ can be analytically continued to $D(0,1)$. By a corollary of Theorem 2.18, Maximum Modulus Principle, the maximum value of $|g(z)|$ can only be reached on $\partial D(0,1)$. So $|g(z)| \leq \max_{|z|=1}|g(z)| = \max_{|z|=1}|f(z)| = \max_{|z|=1}|p_n(z)| \leq M$, $\forall z \in D(0,1)$, i.e. $|f(z)| = \frac{|p_n(z)|}{|z^n|} \leq M$ for $1 \leq |z| < \infty$. □

► **15.** Suppose $f(z)$ is a holomorphic function on the disc $|z| < R$, $|f(z)| \leq M$ and $f(0) = 0$. Show that

$$|f(z)| \leq \frac{M}{R}|z|,\ |f'(0)| \leq \frac{M}{R},$$

where the equalities hold if and only if $f(z) = (M/R)e^{i\alpha}z$ with $\alpha \in \mathbb{R}$.

Proof. Define $g(z) = \frac{f(Rz)}{M}$ ($z \in D(0,1)$) and apply Theorem 2.19, Schwarz Lemma, to $g(z)$, we have $|g(z)| \leq |z|$ ($\forall z \in D(0,1)$) and $|g'(0)| \leq 1$. So for any $z \in D(0,R)$, $f(z) = f(R \cdot \frac{z}{R}) = Mg(\frac{z}{R}) \leq M\frac{|z|}{R}$ and $|f'(0)| = |g'(0)| \cdot \frac{M}{R} \leq \frac{M}{R}$. □

► **16.** Prove Corollary 2.1 by applying Theorem 2.7.

Proof. Define $\rho = \text{dist}(\mathbb{C} \setminus V, K)$. Then $\rho \in (0,\infty)$. By Theorem 2.7, for any $z \in K$, we have

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi}\int_{\partial V} \frac{|f(\xi)|}{|\xi - z|^{n+1}}|d\xi| \leq \frac{n!}{2\pi}\int_{\partial V} \frac{|d\xi|}{\rho^{n+1}} \cdot \sup_{z\in V}|f(z)|.$$

Define $c_n = \frac{n!}{2\pi}\int_{\partial V}\frac{|d\xi|}{\rho^{n+1}}$ and take supremum on the left side of the above inequality, we have

$$\sup_{z\in K}|f^{(n)}(z)| \leq c_n \sup_{z\in V}|f(z)|.$$

□

► **17.** Prove that if $\{f_n(z)\}$ is a sequence of continuous functions on a region D that converges uniformly to $f(z)$, then

$$\int_\gamma f(z)dz = \lim_{n\to\infty}\int_\gamma f_n(z)dz$$

for any curve γ in D. (This assertion was used in the proof of Theorem 2.15.)

Proof. We have to assume γ has finite length, that is, $L(\gamma) := \int_\gamma |dz| < \infty$. Then by the definition of uniform convergence, for any $\varepsilon > 0$, there exists n_0 such that $\forall n \geq n_0$, $\sup_{z\in D}|f(z) - f_n(z)| \leq \frac{\varepsilon}{L(\gamma)}$. So for any $n \geq n_0$,

$$\left|\int_\gamma f(z)dz - \int_\gamma f_n(z)dz\right| \leq \int_\gamma |f(z) - f_n(z)||dz| \leq \int_\gamma \frac{\varepsilon}{L(\gamma)}|dz| = \varepsilon.$$

Therefore $\lim_{n\to\infty}\int_\gamma f_n(z)dz = \int_\gamma f(z)dz$. □

► **18.** Let $f(z)$ be a holomorphic function in the disc $|z| < 1$ with $\text{Re}f(z) > 0$ and $f(0) = \alpha > 0$. Show that

$$|f(z)| \leq \frac{2A|z|}{1-|z|}.$$

Proof. We note the linear fractional transformation $\omega(z) = \frac{z-\alpha}{z+\alpha}$ maps $\{z : \text{Re}z > 0\}$ to $D(0,1)$. So $h(z) = \omega(f(z))$ maps $D(0,1)$ to itself (see, for example, 方企勤 [3], Chapter 3, §7) and $h(0) = 0$. By Schwarz Lemma, $h(z)| = |z|$ and $|h'(0)| \leq 1$. This is equivalent to

$$\left|\frac{f(z)-\alpha}{f(z)+\alpha}\right| \leq |z|, \text{ and } |f'(0)| \leq 2\alpha.$$

□

► **19.** Let $f(z)$ be a holomorphic function in the disc $|z| < 1$ with $f(0) = 0$ and $\text{Re}f(z) \leq A$ ($A > 0$). Show that

$$|f(z)| \leq \frac{2A|z|}{1-|z|}.$$

Proof. The linear fractional transformation $\omega(z) = \frac{z}{z-2A}$ maps $\{z : \text{Re}z < A\}$ to $D(0,1)$. So $h(z) = \omega(f(z)) = \frac{f(z)}{f(z)-2A}$ maps $D(0,1)$ to itself and $h(0) = 0$. By Schwarz Lemma, $|h(z)| \leq |z|$. So $\forall z \in D(0,1)$

$$|f(z)| \leq |z| \cdot |f(z) - 2A| \leq |z||f(z)| + 2A|z|, \text{ i.e. } |f(z)| \leq \frac{2A|f(z)|}{1-|z|}.$$

□

► **20.** Find the Taylor expansions and their radius of convergence of the following functions at $z = 0$.

(i) $\frac{e^z+e^{-z}+2\cos z}{4}$;

Solution. We note

$$\frac{e^z + e^{-z} + 2\cos z}{4} \quad = \quad \frac{1}{4}[e^z + e^{-z} + e^{iz} + e^{-iz}] = \frac{1}{4}\sum_{k=0}^{\infty} \frac{1}{k!}[z^k + (-z)^k + (iz)^k + (-iz)^k] = \sum_{m=0}^{\infty} \frac{z^{4m}}{(4m)!}.$$

The radius of convergence is ∞. □

(ii) $\frac{z^2+4z^4+z^6}{(1-z^2)^3}$;

Solution. By induction, it is easy to show $\left[\frac{1}{(1-\zeta)^3}\right]^{(n)} = \frac{(n+2)!}{2!}(1-\zeta)^{-(n+3)}$. So the Taylor expansion of $\frac{1}{(1-\zeta)^3}$ is $\frac{1}{(1-\zeta)^3} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2}\zeta^n$, with radius of convergence equal to 1. This implies

$$\frac{\zeta + 4\zeta^2 + \zeta^3}{(1-\zeta)^3} = \zeta + 7\zeta^4 + \sum_{n=3}^{\infty}(3n^2 - 3n + 1)\zeta^n.$$

Replace ζ with z^2, we can get the Taylor expansion at $z = 0$: $z^2 + 7z^8 + \sum_{n=3}^{\infty}(3n^2 - 3n + 1)z^{2n}$, and the radius of convergence is equal to 1 (this can be verified by calculating $\lim_{n\to\infty}(3n^2 - 3n + 1)^{\frac{1}{n}} = 1$). □

(iii) $(1 - z^{-5})^{-4}$;

Solution. By induction, it is easy to show $\left[(1-\zeta)^{-4}\right]^{(n)} = \frac{(n+3)!}{3!}(1-\zeta)^{-(n+4)}$. So for $\zeta \in D(0,1)$, $(1-\zeta)^{-4} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{6}\zeta^n$. Replace ζ with z^5, we get for $z \in D(0,1)$

$$\frac{1}{(1-z^5)^4} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} z^{5n}.$$

Therefore

$$(1 - z^{-5})^{-4} = \frac{z^{20}}{(1-z^5)^4} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} z^{5n+20}.$$

By Hadamard's formula, the radius of convergence is 1. □

(iv) $\frac{z^6}{(z^2-1)(z+1)}$.

Solution. By induction, we can show $\left[(1+z)^{-2}\right]^{(n)} = (-1)^n(n+1)!(1+z)^{-(n+2)}$. So $\forall z \in D(0,1)$,

$$\frac{1}{(1+z)^2} = \sum_{n=0}^{\infty}(-1)^n(n+1)z^n.$$

It is also easy to verify that

$$\frac{1}{(z+1)^2(z-1)} = -\frac{1}{4(z+1)} + \frac{1}{4(z-1)} - \frac{1}{2(z+1)^2} = -\frac{1}{4}\sum_{n=0}^{\infty}(-z)^n - \frac{1}{4}\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+1)z^n.$$

Therefore

$$\frac{z^6}{(z+1)(z^2-1)} = -\frac{z^6}{4}\sum_{n=0}^{\infty}\left[(-1)^n + 1 + 2\cdot(-1)^n(n+1)\right] z^n = -\frac{1}{4}\sum_{n=0}^{\infty}\left[(-1)^n(2n+3) + 1\right] z^{n+6}.$$

By Hadamard's formula, the radius of convergence is 1. □

▶ **21.**

(1) If $f(z)$ is holomorphic on $|z| \leq r$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is its Taylor expansion at $z = 0$, then

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta.$$

Proof. We note

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=0}^{\infty} a_n r^n e^{in\theta} \right) \cdot \left(\sum_{m=0}^{\infty} \bar{a}_m r^m e^{-im\theta} \right) d\theta \\
&= \frac{1}{2\pi} \sum_{m,n=0}^{\infty} \int_{-\pi}^{\pi} a_n \bar{a}_m r^{n+m} e^{i(n-m)\theta} d\theta \\
&= \sum_{m,n=0}^{\infty} a_n \bar{a}_m r^{n+m} \delta_{nm} \\
&= \sum_{n=0}^{\infty} |a_n|^2 r^{2n},
\end{aligned}$$

where $\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$ and the exchange of integration and summation is justified by Theorem 1.14 (Abel Theorem) and Problem 17. □

(2) If $r = 1$ in (1), then

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} = \frac{1}{\pi} \int \int_D |f(z)|^2 dA,$$

where D is the unit disc $|z| \leq 1$ and dA is the area element.

Proof. In the result of part (1), multiply both sides by r and integrate with respect to r from 0 to 1, we have

$$\sum_{n=0}^{\infty} |a_n|^2 \int_0^1 r^{2n+1} dr = \sum_{n=0}^{\infty} \frac{|a_n|^2}{2n+2} = \frac{1}{2\pi} \int_0^1 \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 r dr d\theta = \frac{1}{2\pi} \int \int_D |f(z)|^2 dA.$$

So $\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} = \frac{1}{\pi} \int \int_D |f(z)|^2 dA$. □

▶ **22.** Verify the equations (2.32) and (2.33).

Proof. Suppose u is harmonic on $D(0, R)$ and continuous $\bar{D}(0, R)$, then $\forall z \in \bar{D}(0, 1)$, $v(z) := u(zR)$ is harmonic on $D(0, 1)$ and continuous on $\bar{D}(0, 1)$. By formula (2.31), for any $z \in D(0, R)$,

$$\begin{aligned}
u(z) &= v(\frac{z}{R}) = \frac{1}{2\pi} \int_0^{2\pi} v(e^{i\tau}) \frac{1 - \left|\frac{z}{R}\right|^2}{\left|1 - \frac{\bar{z}}{R} e^{i\tau}\right|^2} d\tau = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\tau}) \frac{R^2 - |z|^2}{|R - \bar{z} e^{i\tau}|^2} d\tau \\
&= \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\tau}) \frac{R^2 - |z|^2}{\left|R - z e^{-i\tau}\right|^2} d\tau = \frac{1}{2\pi} \int_0^{2\pi} u(\zeta) \frac{R^2 - |z|^2}{|\zeta - z|^2} d\tau.
\end{aligned}$$

This is formula (2.32). To verify formula (2.33), note

$$\text{Re} \frac{Re^{i\tau} + z}{Re^{i\tau} - z} = \text{Re} \frac{(Re^{i\tau} + z)(Re^{-i\tau} - \bar{z})}{|Re^{i\tau} - z|^2} = \text{Re} \frac{R^2 - |z|^2 + R(ze^{-i\tau} - \bar{z} e^{i\tau})}{|Re^{i\tau} - z|^2} = \frac{R^2 - |z|^2}{|Re^{i\tau} - z|^2}.$$

Therefore,

$$\begin{aligned}
u(z) &= \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\tau})\frac{R^2-|z|^2}{|Re^{i\tau}-z|^2}d\tau = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\tau})\mathrm{Re}\frac{Re^{i\tau}+z}{Re^{i\tau}-z}d\tau \\
&= \mathrm{Re}\left[\frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\tau})\frac{Re^{i\tau}+z}{Re^{i\tau}-z}d\tau\right] \\
&= \mathrm{Re}\left[\frac{1}{2\pi i}\int_{|\zeta|=R} u(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}\right].
\end{aligned}$$

This is formula (2.33). □

► **23.** Show that one of the zeros of the equation $z^4-6z+3=0$ is in the disc $|z|<1$ and the other three zeros are in $1<|z|<2$.

Proof. On $\partial D(0,1)$, $|(-6z)-(z^4-6z+3)| = |z^4+3| \le 4 < |-6z|$. By Rouché Theorem, z^4-6z+3 and $-6z$ have the same number of zeros in $D(0,1)$, which is one. On $\partial D(0,2)$, $|z^4-(z^4-6z+3)| = |6z-3| \le 15 < |z^4|$. So z^4-6z+3 and z^4 have the same number of zeros in $D(0,2)$, which is four. Combined, we conclude $z^4-6z+3=0$ has one root in $D(0,1)$ and three roots in the annulus $\{z : 1<|z|<2\}$. □

► **24.** Find the number of zeros of the equation $z^7-5z^4-z+2=0$ in the disc $|z|<1$.

Solution. On $\partial D(0,1)$, $|(-5z^4)-(z^7-5z^4-z+2)| = |z^7-z+2| \le 4 < |-5z^4|$. So z^7-5z^4-z+2 and $-5z^4$ have the same number of zeros in $D(0,1)$, which is four (counting multiplicity). □

► **25.** Show that there is exactly one zero of the equation $z^4+2z^3-2z+10=0$ in each quadrant.

Proof. Let $P(z)=z^4+2z^3-2z+10$. We note $P(z)=(z^2-1)(z+1)^2+11$. If $z\in\mathbb{R}$ and $|z|\ge 1$, $P(z)\ge 11$; if $z\in\mathbb{R}$ and $|z|\le 1$, $P(z)\ge -1\cdot(1+1)^2+11=7$. So $P(z)=0$ has no root on the real axis. If $z=iy$ with $y\in\mathbb{R}$, we have $P(iy)=y^4+10-2iy(y^2+1)\ne 0$. So $P(z)=0$ has no root on the imaginary axis. Consider the region D enclosed by the curve $\gamma=\gamma_1\cup\gamma_2\cup\gamma_3$, where ($R$ is a positive number) $\gamma_1=\{z: 0\le \mathrm{Re}z\le R, \mathrm{Im}z=0\}$, $\gamma_2=\{z: |z|=R, \arg z\in[0,\frac{\pi}{2}]\}$, and $\gamma_3=\{z: 0\le \mathrm{Im}z\le R, \mathrm{Re}z=0\}$.

On γ_1, $\Delta_{\gamma_1}\arg P(z)=0$. On γ_2, $P(z)=z^4\left(1+\frac{2z^3-2z+10}{z^4}\right)$. So $\Delta_{\gamma_2}\arg P(z)=4\cdot\frac{\pi}{2}+o(1)=2\pi+o(1)$ $(R\to\infty)$. On γ_3, $\Delta_{\gamma_3}\arg P(z)=\arg P(0)-\arg P(iR)=\arg 10-\arg(R^4+10-2iR(R^2+1))=0-\arg\left(1-2i\frac{R(R^2+1)}{R^4+10}\right)=o(1)$ $(R\to\infty)$. Combined, we have $\Delta_\gamma \arg P(z)=\sum_{k=1}^3\Delta_{\gamma_k}\arg P(z)=2\pi+o(1)$ $(R\to\infty)$.

By The Argument Principle, $P(z)=0$ has only one root in the first quadrant. The root conjugate to this root must lie in the fourth quadrant. The other two roots are conjugate to each other and are located in the second and third quadrant. So one must lie in the second quadrant and the other must lie in the third quadrant. □

Remark 5. *The above solution can be found in 方企勤 [3], Chapter 6, §3 Example 5.*

► **26.** Find the number of zeros of the equation $z^4-8z+10=0$ in the disc $|z|<1$ and in the annulus $1<|z|<3$.

Solution. On $\partial D(0,1)$, $|(-8z+10)-(z^4-8z+10)| = |z^4| = 1 < |10-8|z|| \le |-8z+10|$. So $z^4-8z+10$ and $-8z+10$ have the same number of zeros in $D(0,1)$, which is zero.

On $\partial D(0,3)$, $|z^4-(z^4-8z+10)| = |8z-10| < 34 < |z^4|$. So $z^4-8z+10$ and z^4 have the same number of zeros in $D(0,3)$, which is four (counting multiplicity). So $z^4-8z+10=0$ has no root in $D(0,1)$ and four roots in the annulus $\{z : 1<|z|<3\}$. □

► **27.** Show that if $a>e$, then there are n zeros in the disc $|z|<1$ for the equation $e^z=az^n$.

Proof. If $a > e$, on $\partial D(0,1)$, $|az^n - (az^n - e^z)| = |e^z| \le e^{|z|} = e < |az^n|$. By Rouché Theorem, $e^z - az^n$ and az^n have the same number of zeros in $D(0,1)$, which is n. ☐

▶ **28.** If $f(z)$ is holomorphic in the disc $|z| < 1$ and continuous on $|z| \le 1$ and $|f(z)| < 1$, then there is a unique fixed point of f in the disc $|z| < 1$.

Proof. On $\partial D(0,1)$, $|z - (z - f(z))| = |f(z)| < 1 = |z|$. So $z - f(z)$ and z have the same number of zeros in $D(0,1)$, which is one. So there is a unique fixed point of f in $D(0,1)$. ☐

Remark 6. *This is a special case of Brower's Fixed Point Theorem.*

▶ **29.**

(1) Find a holomorphic function whose real part is

$$e^x(x\cos y - y\sin y),$$

where $x + iy = z$.

Solution. $\mathrm{Re}(ze^z) = e^x(x\cos y - y\sin y)$. ☐

(2) Find the most general harmonic function of the form

$$ax^3 + bx^2y + cxy^2 + dy^3$$

where a, b, c, d are real numbers.

Solution. Let $u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$. Then $\Delta u = (6a+2c)x + (2b+6d)y$. So u is harmonic if and only if $3a+c = 0$ and $b+3d = 0$. So the most general harmonic function of the form $ax^3+bx^2y+cxy^2+dy^3$ is $ax^3 - 3dx^2y - 3axy^2 + dy^3$. ☐

▶ **30.** By applying the mean-value property of harmonic functions, show that if $-1 < r < 1$, then

$$\int_0^\pi \ln(1 - 2r\cos\theta + r^2)d\theta = 0.$$

Proof. Let $u(r,\theta) = \ln(1 - 2r\cos\theta + r^2)$. Then using the fact $\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$, we can verify $\Delta u = 0$. So by the mean-value property of harmonic functions, we have $\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)d\theta = u(0,0) = 0$. Since $\cos\theta$ is an even function of θ, it's not hard to show $\int_0^{2\pi} \ln(1 - 2r\cos\theta + r^2)d\theta = 2\int_0^\pi \ln(1 - 2r\cos\theta + r^2)d\theta$. So $\int_0^\pi \ln(1 - 2r\cos\theta + r^2)d\theta = 0$. ☐

▶ **31.** Prove that if a harmonic function $u(z)$ is bounded on the entire complex plane $\mathbb{C}$, then $u(z)$ is equal to a constant identically.

Proof. Let $M = \sup_{z\in\mathbb{C}} |u(z)|$. Then $\forall z \in \mathbb{C}$ and $R > |z|$, the holomorphic function defined in formula (2.34) satisfies

$$|f(z)| \le \frac{1}{2\pi}\int_0^{2\pi} |u(Re^{i\theta})| \left|\frac{Re^{i\theta} + z}{Re^{i\theta} - z}\right| d\theta \le \frac{M}{2\pi}\int_0^{2\pi} \frac{R + |z|}{R - |z|} d\theta \to M, \text{ as } R \to \infty.$$

So $f(z)$ is bounded on the entire complex plane, and by Liouville's Theorem, f is a constant function. Therefore $u = \mathrm{Re}(f)$ is a constant function. ☐

▶ **32.** Find a harmonic function on $|z| < 1$ such that its value is 1 on an arc of $|z| = 1$ and zero on the rest of $|z| = 1$.

Solution. If the arc is $\partial D(0,1)$, the desired harmonic function $u \equiv 1$. So without loss of generality, we assume the arc $\gamma = \{z \in \partial D(0,1) : \theta_1 \le \arg z \le \theta_2\}$ with $0 < \theta_2 - \theta_1 < 2\pi$.

For n sufficiently large, we consider $\gamma_n = \{z \in \partial D(0,1) : \theta_1 - \frac{1}{n} \le \arg z \le \theta_2 + \frac{1}{n}\}$. By the Partition of Unity Theorem for $\mathbb{R}^1$, we can find $\varphi_n \in C(\partial D(0,1))$ such that $0 \le \varphi_n \le 1$, $\varphi_n \equiv 1$ on γ, and $\varphi_n \equiv 0$ on $\partial D(0,1) \setminus \gamma_n$. Define $u_n(z) = \int_0^{2\pi} P(\zeta, z)\varphi_n(\zeta)d\theta$ where $P(\cdot,\cdot)$ is the Poisson kernel and $\zeta = Re^{i\theta}$. Then u_n is the unique solution of the Dirichlet problem with boundary value φ_n. So φ_n is harmonic in $D(0,1)$.

It is easy to see that if $(\varphi_n)_n$ uniformly converges to the indicator function $1_\gamma(z)$, $(u_n)_n$ uniformly converges to $u(z) := \int_0^{2\pi} P(\zeta, z)1_\gamma(\zeta)d\theta$ on $D(0,\rho)$ ($\rho \in (0,1)$). Since each u_n satisfies the local mean value property, u must also satisfies the local mean value property in $D(0,1)$ and is continuous. By Theorem 2.22, u is harmonic on $D(0,1)$. This suggests the harmonic function we are looking for is exactly u.

What is left to prove is that for any $z_0 \in \partial D(0,1)$, $\lim_{|z|=1, z\to z_0} u(z) = 1_\gamma(z_0)$. Indeed, in general, we have the following classical result: Suppose function φ is piecewise continuous on $\partial D(0,1)$, then the function $u(z) := \int_0^{2\pi} P(\zeta, z)\varphi(\zeta)d\theta$ is harmonic in $D(0,1)$ and for any continuity point z_0 of φ, we have $\lim_{z\to z_0, |z|<1} u(z) = \varphi(z_0)$.

For the proof of continuity on the boundary, we refer to 方企勤 [3], Chapter 7, Theorem 7. □

► **33.** Suppose U is a region, $f_i(z)$ $(i = 1, 2, \cdots)$ are holomorphic on U and continuous on $\overline{U}$. Show that if

$$\sum_{n=1}^{\infty} f_n(z)$$

converges uniformly on the boundary of U, then it converges uniformly on $\overline{U}$.

Proof. $\forall \varepsilon > 0$, there exists N, such that for any $n \ge N$, $p \in \mathbb{N}$, we have

$$\sup_{\zeta\in\partial U} \left| \sum_{k=n+1}^{n+p} f_k(\zeta) \right| < \varepsilon.$$

By Maximum Modulus Principle,

$$\sup_{z\in\bar{U}} \left| \sum_{k=n+1}^{n+p} f_k(z) \right| \le \sup_{\zeta\in\partial U} \left| \sum_{k=n+1}^{n+p} f_k(\zeta) \right| < \varepsilon.$$

So Cauchy criterion implies $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on $\bar{U}$. □

Remark 7. *This result is the so-called Weierstrass' Second Theorem.*

► **34.** Suppose $f(z)$ is holomorphic on $D(0,R)$ and continuous on $\overline{D(0,R)}$. Let $M = \max_{|z|=R} |f(z)|$. Show that if $z_0 \in D(0,R)\backslash\{0\}$ is a zero of $f(z)$, then

$$R|f(0)| \le (M + |f(0)|)|z_0|.$$

Proof. Define $g(z) = \frac{f(z)-f(z_0)}{z-z_0} = \frac{f(z)}{z-z_0}$, $\forall z \in \bar{D}(0,R) \setminus \{z_0\}$. Then $g(z)$ can be analytically continued to $D(0,R)$ (Theorem 2.11, Riemann Theorem) and is therefore continuous on $\bar{D}(0,R)$. By Cauchy's integral formula, we have

$$\frac{f(0)}{-z_0} = g(0) = \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{g(\zeta)}{\zeta}d\zeta = \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)}{\zeta(\zeta - z_0)}d\zeta.$$

Therefore

$$\left|\frac{f(0)}{z_0}\right| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(Re^{i\theta})}{|Re^{i\theta} - z_0|}d\theta \le \frac{M}{R - |z_0|},$$

which is equivalent to $R|f(0)| \le (M + |f(0)|)|z_0|$. □

Remark 8. *The point of this problem is to give an estimate of a holomorphic function's modulus at $z = 0$ in terms of its zeros.*

▶ **35.** Let $|z_1| > 1$, $|z_2| > 1$, $\cdots$, $|z_n| > 1$. Show that there exists a point z_0 such that $\prod_{k=1}^n |z_0 - z_k| > 1$.

Proof. This problem seems suspicious. For example, when $z_0 = 0$, of course $\prod_{k=1}^n |z_0 - z_k| = \prod_{k=1}^n |z_k| > 1$. When z_0 is sufficiently large, it is also clear $\prod_{k=1}^n |z_0 - z_k| > 1$. So the point z_0 can be both inside and outside $D(0, 1)$. I'm not sure what proof is needed for such a trivial claim. □

▶ **36.** Suppose that $f(z)$ is holomorphic on $D(0, R)$. Show that

$$M(r) = \max_{|z|=r} |f(z)|$$

is an increasing function on $[0, R)$.

Proof. By Maximum Modulus Principle, $M(r) = \max_{|z|\leq r} |f(z)|$. So $M(r)$ is an increasing function on $[0, R)$. □

▶ **37.** Use the Maximum Modulus Principle to prove the Fundamental Theorem of Algebra.

Proof. Suppose there is an n-th degree polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ such that $n \geq 1$, $a_n \neq 0$, and $p(z) \neq 0$ for any $z \in \mathbb{C}$. Then $q(z) = 1/p(z)$ is holomorphic over the whole complex plane. For any $R > 0$, we have by Maximum Modulus Principle

$$\max_{|z|\leq R} |q(z)| \leq \max_{|z|=R} |q(z)|.$$

Since on $\{z : |z| = R\}$,

$$|p(z)| \geq |a_n|R^n - |a_{n-1}|R^{n-1} - \cdots - |a_1|R - |a_0| = R^n \left(|a_n| - \frac{|a_{n-1}|}{R} - \cdots - \frac{|a_1|}{R^{n-1}} - \frac{|a_0|}{R^n} \right) \to \infty \text{ as } R \to \infty,$$

we must have $\lim_{R\to\infty} \max_{|z|=R} |q(z)| = 0$. This implies $\sup_{z\in\mathbb{C}} |q(z)| = 0$, i.e. $p(z) \equiv \infty$. This is a contradiction and our assumption must be incorrect. □

▶ **38.** If $f(z)$ is a non-constant holomorphic function on a region U and has no zeros in U, then $|f(z)|$ can not attain its minimal value in U.

Proof. Apply the Maximum Modulus Principle to the function $g(z) = 1/f(z)$, which is holomorphic on U. □

▶ **39.** (*Hadamard's Three Circles Theorem*) Suppose that $0 < r_1 < r_2 < \infty$, $U = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$, $f(z)$ is holomorphic on U and continuous on $\overline{U}$. Define $M(r) = \max_{|z|=r} |f(z)|$. Show that $\log M(r)$ is a convex function of $\log r$ on $[r_1, r_2]$. In other words, the inequality

$$\log M(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2)$$

holds for $r \in [r_1, r_2]$.

Proof. Note the Laplace operator $\Delta = 4\frac{\partial}{\partial}\frac{\partial}{\partial}$, so $\log |z|$ is a harmonic function on $\mathbb{C} \setminus \{0\}$ and $\log |f(z)|$ is a harmonic function outside the zeros of $f(z)$. Define $K = \{z \in U : f(z) = 0\}$ and $V_\varepsilon = \cup_{z\in K} D(z, \varepsilon)$. Then for any $\alpha \in \mathbb{R}$, $\alpha \log |z| + \log |f(z)|$ is harmonic in $U \setminus \bar{V}_\varepsilon$ and continuous on $\bar{U} \setminus V_\varepsilon$. By Maximum Modulus Principle for harmonic functions, $\max_{z\in\bar{U}\setminus V_\varepsilon}(\alpha \log |z| + \log |f(z)|) = \max_{z\in\partial(\bar{U}\setminus V_\varepsilon)}(\alpha \log |z| + \log |f(z)|)$.

When z approaches to zeros of $f(z)$, $\log|f(z)| \to -\infty$, so by letting $\varepsilon \to 0$, we can further deduce that $\max_{z\in U}(\alpha\log|z| + \log|f(z)|) = \max_{z\in\partial U}(\alpha\log|z| + \log|f(z)|)$. Therefore

$$\alpha\log|z| + \log|f(z)| \le \max\{\alpha\log r_1 + \log M(r_1), \alpha\log r_2 + \log M(r_2)\},\ \forall z \in U,$$

which is the same as

$$\alpha\log r + \log M(r) \le \max\{\alpha\log r_1 + \log M(r_1), \alpha\log r_2 + \log M(r_2)\},\ r \in [r_1, r_2].$$

Now let α be such that the two values inside the parentheses on the right are equal, that is

$$\alpha = \frac{\log M(r_2) - \log M(r_1)}{\log r_1 - \log r_2}.$$

Then from the previous inequality, we get

$$\log M(r) \le \alpha\log r_1 + \log M(r_1) - \alpha\log r,$$

which upon substituting value for α gives

$$\log M(r) \le (1-s)\log M(r_1) + s\log M(r_2),$$

where $s = \frac{\log r_1 - \log r}{\log r_2 - \log r_1}$. □

► **40.** If $f(z)$ is holomorphic on $D(0,1)$ and $f(0) = 0$, show that $\sum_{n=1}^{\infty} f(z^n)$ converges absolutely on $D(0,1)$ and converges uniformly on every compact subset of $D(0,1)$.

Proof. Fix $r \in (0,1)$. For any $\rho \in (0,r)$ and any $z \in D(0,\rho)$, we have $z^n \in D(0,\rho)$ $(\forall n \in \mathbb{N})$. By Cachy's integral formula, $\forall n, p \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{k=n+1}^{n+p} |f(z^k)| &= \sum_{k=n+1}^{n+p} |f(z^k) - f(0)| = \sum_{k=n+1}^{n+p} \frac{1}{2\pi}\left|\int_{\partial D(0,r)} \left[\frac{f(\zeta)}{\zeta - z^k} - \frac{f(\zeta)}{\zeta}\right] d\zeta\right| \\
&\le \sum_{k=n+1}^{n+p} \frac{1}{2\pi}\left|\int_{\partial D(0,r)} \frac{f(\zeta)}{\zeta(\zeta - z^k)} d\zeta \cdot z^k\right| \le \sum_{k=n+1}^{n+p} \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(re^{i\theta})|d\theta}{r - \rho^k} \cdot \rho^k \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(re^{i\theta})|d\theta}{r - \rho} \sum_{k=n+1}^{n+p} \rho^k \to 0
\end{aligned}$$

as $n \to \infty$. So $\sum_{n=1}^{\infty} f(z^n)$ converges absolutely and uniformly on $\bar{D}(0,\rho)$. □

Remark 9. *In general, there is no Mean Value Theorem for holomorphic functions. For example,* $f(z) = \exp\left(i\frac{2z-(z_1+z_2)}{z_2-z_1}\pi\right)$ *is analytic but* $f(z_2) - f(z_1) \ne f'(z)(z_2 - z_1)$, $\forall z \in \mathbb{C}$ *(for more details, see Qazi [7]). But as the proof of Theorem 2.7 shows, the formula*

$$f(z) - f(z_0) = \frac{z - z_0}{2\pi i}\int_{\partial U} \frac{f(\zeta)d\zeta}{(\zeta - z)(\zeta - z_0)}$$

more or less fills the void. In particular, it shows that holomorphic functions are locally Lipschitz.

► **41.** Let $f(z)$ be a holomorphic function on $D(0,R)$ and $f(0) = 0$. If $f(D(0,R)) \subset D(0,M)$, then

(i) the inequalities

$$|f(z)| \le \frac{M}{R}|z|,\ |f'(0)| \le \frac{M}{R}$$

hold for $z \in \partial D(0,R)\backslash\{0\}$.

(ii) the equalities in (1) hold if and only if

$$f(z) = \frac{M}{R}e^{i\theta}z,$$

where θ is a real number.

Proof. This problem is the same as Problem 15. ☐

▶ **42.** Let $f(z)$ be a holomorphic function on $D(0,1)$ and $f(0)=0$. If there exists a constant $A>0$ such that $\mathrm{Re}f(z)\le A$ for $z\in D(0,1)$, then

$$|f(z)|\le \frac{2A|z|}{1-|z|}$$

for $z\in D(0,1)$.

Proof. This problem is the same as Problem 19. ☐

▶ **43.** Let $f(z)$ be a holomorphic function on $D(0,1)$ and $f(0)=1$. If $\mathrm{Re}f(z)\ge 0$ for every $z\in D(0,1)$, using Schwarz Lemma, show that

(1) the inequality

$$\frac{1-|z|}{1+|z|}\le \mathrm{Re}f(z)\le |f(z)|\le \frac{1+|z|}{1-|z|}$$

holds for $z\in D(0,1)$;

Proof. By Problem 18, $\left|\frac{f(z)-1}{f(z)+1}\right|\le |z|$. So $|f(z)|-1\le |f(z)-1|\le |z||f(z)+1|\le |z||f(z)|+|z|$. Since $|z|<1$, the above inequality implies $|f(z)|\le \frac{1+|z|}{1-|z|}$.

From the same inequality $\left|\frac{f(z)-1}{f(z)+1}\right|\le |z|$, we have

$$(\mathrm{Re}f(z)-1)^2+(\mathrm{Im}f(z))^2\le |z|^2[(\mathrm{Re}f(z)+1)^2+(\mathrm{Im}f(z))^2]\le |z|^2(\mathrm{Re}f(z)+1)^2+(\mathrm{Im}f(z))^2.$$

So $|\mathrm{Re}f(z)-1|\le |z|(\mathrm{Re}f(z)+1)$. If $\mathrm{Re}f(z)>1$, we can deduce from this inequality $\frac{1-|z|}{1+|z|}\le 1<\mathrm{Re}f(z)$. If $\mathrm{Re}f(z)\le 1$, we can deduce from this inequality $1-\mathrm{Re}f(z)\le |z|\mathrm{Re}f(z)+|z|$, i.e. $\frac{1-|z|}{1+|z|}\le \mathrm{Re}f(z)$. ☐

(2) the equality in (1) holds for $z\ne 0$ if and only if

$$f(z)=\frac{1+e^{i\theta}z}{1-e^{i\theta}z},$$

where θ is an arbitrary real number.

Proof. We note $f(z)=\frac{1+e^{i\theta}z}{1-e^{i\theta}z}$ if and only if $\frac{f(z)-1}{f(z)+1}=e^{i\theta}$. So the claim is straightforward from Schwarz Lemma. ☐

▶ **44.** Suppose $f(z)$ is holomorphic on $D(0,1)$. Show that there exists a sequence $\{z_n\}$ in $D(0,1)$ that converges to a point $z_0\in\partial D(0,1)$ such that $\lim_{n\to\infty}f(z_n)$ exists.

Proof. Because $D(0,1)$ is pre-compact, by Theorem 1.11 (Bolzano-Weierstrass Theorem), it suffices to find a sequence $(z_n)_{n=1}^{\infty}\subset D(0,1)$, such that $\lim_{n\to\infty}z_n=z_0\in\partial D(0,1)$ and $(f(z_n))_{n=1}^{\infty}$ is bounded. Assume this does not hold. Then $\forall z_0\in\partial D(0,1)$ and $\forall n\in\mathbb{N}$, there exists $\delta(z_0)>0$ such that $\forall z\in D(z_0,\delta(z_0))\cap D(0,1)$, $|f(z)|\ge n$. The family of these open sets $(D(z,\delta(z))_{z\in\partial D(0,1)}$ is an open covering of the compact set $\partial D(0,1)$. By Theorem 1.10 (Heine-Borel Theorem), there is a finite sub-covering $(D(z_k,\delta(z_k))_{k=1}^{p}$ of $\partial D(0,1)$. Consequently, for each $n\in\mathbb{N}$, we can find $\varepsilon_n>0$ such that $\forall z\in\{z:1-\varepsilon_n\le |z|<1\}$, $|f(z)|\ge n$. Without loss of generality, we can assume $(\varepsilon_n)_{n=1}^{\infty}$ monotonically decreases to 0.

Since $f(z)$ uniformly approaches to ∞ as $z\to\partial D(0,1)$, by Theorem 2.13, f has finitely many zeros in $D(0,1)$. So f can be written as $f(z)=\prod_{i=1}^{m}(z-z_i)^{k_i}\cdot h(z)$, where h is a holomorphic function on $\partial D(0,1)$ with no zeros in $D(0,1)$. So h satisfies the Minimum Modulus Principle: $\forall\rho>0$, $|h(z)|\ge\min_{|\zeta|=\rho}|h(\zeta)|$ for any $z\in D(0,\rho)$. Therefore, for any $z\in D(0,1-\varepsilon_n)$,

$$|h(z)|\ge \min_{|\zeta|=1-\varepsilon_n}|h(\zeta)|=\min_{|\zeta|=1-\varepsilon_n}\frac{|f(\zeta)|}{|\prod_{i=1}^{m}(\zeta-z_i)^{k_i}|}\ge \frac{n}{\max_{|\zeta|=1-\varepsilon_n}\prod_{i=1}^{m}|\zeta-z_i|^{k_i}}\ge\frac{n}{2^{\sum_{i=1}^{m}k_i}}.$$

Fix $z\in D(0,1-\varepsilon_n)$ and let $n\to\infty$, we get $h(z)=\infty$. Contradiction. ☐

▶ **45.** Let $f(z)$ be a holomorphic function on $D(0,1)$ and $f(D(0,1)) \subset D(0,1)$. If $z_1, z_2, \cdots, z_n$ are all of the different zeros of $f(z)$ in $D(0,1)$ and their multiplicities are $k_1, k_2, \cdots, k_n$ respectively, then

$$|f(z)| \le \prod_{j=1}^{n} \left| \frac{z - z_j}{1 - \overline{z_j} z} \right|^{k_j}$$

for all $z \in D(0,1)$. Especially,

$$|f(0)| \le \prod_{j=1}^{n} |z_j|^{k_j}.$$

Proof. By assumption, we can write $f(z)$ as $f(z) = \prod_{j=1}^{n}(z - z_j)^{k_j} \cdot h(z)$, where $h(z)$ is a holomorphic function in $D(0,1)$ with no zeros in $D(0,1)$. Define $G(z) = h(z) \cdot \prod_{j=1}^{n}(1 - \bar{z}_j z)^{k_j}$. Then G is holomorphic on $D(0,1)$. Apply Maximum Modulus Principle to $G(z)$ on $\{z : |z| \le 1 - \varepsilon\}$ ($\varepsilon > 0$), we get

$$|G(z)| \le \max_{|z|=1-\varepsilon} |h(z)| \prod_{j=1}^{n} |1 - \bar{z}_j z|^{k_j} = \max_{|z|=1-\varepsilon} \frac{|f(z)|}{\prod_{j=1}^{n} \left| \frac{z-z_j}{1-\bar{z}_j z} \right|^{k_j}} \le \max_{|z|=1-\varepsilon} \frac{1}{\prod_{j=1}^{n} \left| \frac{z-z_j}{1-\bar{z}_j z} \right|^{k_j}}.$$

Since each $\frac{z-z_j}{1-\bar{z}_j z}$ maps $\partial D(0,1)$ to $\partial D(0,1)$, by letting $\varepsilon \to 0$, we get $G(z) \le 1$, i.e. $|f(z)| \le \prod_{j=1}^{n} \left| \frac{z-z_j}{1-\bar{z}_j z} \right|^{k_j}$. □

▶ **46.** If $f(z)$ is holomorphic on $D(0,1)$ and $f(D(0,1)) \subset D(0,1)$, then

$$M|f'(0)| \le M^2 - |f(0)|^2.$$

Proof. Let $a = f(0)$ and define $\varphi_a(\zeta) = \frac{-\zeta + a}{1 - \bar{a}\zeta}$. Then $h(z) = \varphi_a(f(z))$ maps $D(0,1)$ to $D(0,1)$ with $h(0) = \varphi_a(a) = 0$. So Schwarz Lemma implies $|h'(0)| \le 1$. We note $h'(z) = \varphi_a(f(z)) \cdot f'(z)$. Since

$$\varphi_a'(\zeta) = \frac{-(1 - \bar{a}\zeta) - (-\zeta + a)(-\bar{a})}{(1 - \bar{a}\zeta)^2} = \frac{-1 + |a|^2}{(1 - \bar{a}\zeta)^2},$$

we have $h'(0) = \varphi_a'(a) f'(0) = \frac{f'(0)}{|a|^2 - 1}$. So we have

$$\frac{|f'(0)|}{1 - |a|^2} \le 1, \text{ i.e. } |f'(0)| \le 1 - |a|^2.$$

□

▶ **48.** Find the group of holomorphic automorphisms of the upper half plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$, $\mathrm{Aut}(\mathbb{C}^+)$.

Proof. The Möbius transformation $w(z) = \frac{z-i}{z+i}$ maps $\mathbb{C}^+$ to $D = D(0,1)$. So $f \in \mathrm{Aut}(\mathbb{C}^+)$ if and only if $w \circ f \circ w^{-1} \in \mathrm{Aut}(D)$. By Theorem 2.20, $\exists a \in D$ and $\tau \in \mathbb{R}$, such that $w \circ f \circ w^{-1}(\zeta) = \varphi_a \circ \rho_\tau(\zeta)$. Plain calculation shows

$$f(z) = w^{-1} \circ \varphi_a \circ \rho_\tau \circ w(z) = \frac{(1 + a)(z + i) - (1 + \bar{a})e^{i\tau}(z - i)}{(1 - a)(z + i) - (\bar{a} - 1)e^{i\tau}(z - i)}.$$

□

▶ **50.** If $f(z)$ is a bounded entire function and z_1, z_2 are arbitrary points in $D(0,r)$, then

$$\int_{|z|=r} \frac{f(z)}{(z - z_1)(z - z_2)} dz = 0,$$

and this implies the Liouville Theorem.

Proof. This problem is the same as Problem 10. □

3 Theory of Series of Weierstrass

Throughout this chapter, all the integration paths will take the following convention on orientation: all the arcs take counterclockwise as their orientation and all the segments take the natural orientation of $\mathbb{R}^1$ as their orientation.

A very readable presentation of the power of analytic functions is *Complex Proofs of Real Theorems* [5].

▶ **1.** Let a_1, a_2, $\cdots$ be a sequence of distinct points and $\lim_{n\to\infty}|a_n| = \infty$. If

$$\psi_n(z) = \sum_{j=1}^{\infty} \frac{c_{n,j}}{(z-a_n)^j},\ n = 1, 2, \cdots$$

are holomorphic on $\mathbb{C}$ except at points a_n $(n = 1, 2, \cdots)$, then there exists a holomorphic function $f(z)$ on $\mathbb{C}\backslash\{a_1, a_2, \cdots\}$ such that the principle parts of the Laurent expansion of $f(z)$ at a_n $(n = 1, 2, \cdots)$ are $\psi_n(z)$ $(n = 1, 2, \cdots)$ respectively.

Proof. This is just Mittag-Leffler Theorem (Theorem 3.8). □

▶ **2.** Prove Theorem 3.11 and Theorem 3.12.

Proof. To prove Theorem 3.1, choose *varepsilon* > 0 sufficiently small so that the circles $\{z : |z - z_k| = \varepsilon\}$ $(1 \le k \le n)$ don't intersect with each other or with ∂U. Then by Cauchy's integration theorem,

$$\int_{\partial U} f(z)dz - \sum_{k=1}^{n} \int_{|z-z_k|=\varepsilon} f(z)dz = 0,$$

which is $\int_{\partial U} f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k)$.

To prove Theorem 3.12, we choose $R > 0$ sufficiently large so that $\bar{U} \subset D(0, R)$. Then

$$\int_{\partial U} f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k) \text{ and } \int_{\partial D(0,R)} f(z)dz - \int_{\partial U} f(z)dz = 0.$$

Since $\int_{\partial D(0,R)} f(z)dz = -2\pi i \text{Res}(f, \infty)$, we conclude $\sum_{k=1}^{n} \text{Res}(f, z_k) + \text{Res}(f, \infty) = 0$. □

▶ **3.** Find the Laurent series of the following function son the indicated regions.

(i) $\frac{1}{z^3(z+i)}$, $0 < |z + i| < 1$;

Solution. Suppose $\frac{1}{z^3(z+i)}$ has Laurent series $\sum_{n=-\infty}^{\infty} c_n(z + i)^n$. Then by Theorem 3.2 and Cauchy integral theorem, for $\varepsilon \in (0, 1)$,

$$\begin{aligned} c_n &= \frac{1}{2\pi i}\int_{|\zeta+i|=\varepsilon} \frac{1}{(\zeta+i)^{n+1}} \cdot \frac{1}{\zeta^3(\zeta+i)} d\zeta = \frac{1}{(n+1)!} \frac{d^{n+1}}{d\zeta^{n+1}}(\zeta^{-3})\bigg|_{\zeta=-i} \\ &= \frac{1}{(n+1)!}(-1)^{n+1}\frac{(n+3)!}{2!}(-i)^{-(n+4)} = (-1)^{n+1}\frac{(n+2)(n+3)}{2} i^n. \end{aligned}$$

□

(ii) $\frac{z^2}{(z+1)(z+2)}$, $1 < |z| < 2$;

Solution. We note

$$\begin{aligned} \frac{z^2}{(z+1)(z+2)} &= \frac{(z+1)(z+2) - 3z - 2}{(z+1)(z+2)} = 1 - \frac{3(z+1) - 1}{(z+1)(z+2)} = 1 + \frac{1}{z+1} - \frac{4}{z+2} \\ &= 1 + \frac{1}{z}\cdot\frac{1}{1+\frac{1}{z}} - 2\cdot\frac{1}{1+\frac{z}{2}} = \sum_{k=0}^{\infty}\frac{(-1)^k}{z^{k+1}} - 1 - \sum_{k=1}^{\infty}\frac{(-1)^k}{2^{k-1}}z^k. \end{aligned}$$

□

(iii) $\log\left(\frac{z-a}{z-b}\right)$, $\max(|a|,|b|)<|z|<\infty$;

Solution. We note

$$\begin{aligned}\log\left(\frac{z-a}{z-b}\right) &= \log\left(\frac{1-\frac{a}{z}}{1-\frac{b}{z}}\right)=\log\left(1-\frac{a}{z}\right)-\log\left(1-\frac{b}{z}\right)\\ &= -\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a}{z}\right)^n+\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{b}{z}\right)^n=\sum_{n=1}^{\infty}\frac{1}{n}(b^n-a^n)z^{-n}.\end{aligned}$$

□

(iv) $z^2e^{\frac{1}{z}}$, $0<|z|<\infty$;

Solution. We note

$$z^2e^{\frac{1}{z}} = z^2\sum_{k=0}^{\infty}\frac{z^{-k}}{k!}=z^2+z+\frac{1}{2}+\sum_{k=1}^{\infty}\frac{z^{-k}}{(k+2)!}.$$

□

(v) $\sin\frac{z}{1+z}$, $0<|z+1|<\infty$.

Solution. We note $\sin z=\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!}$. So $\sin\frac{z}{z+1}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\frac{z^{2k+1}}{(z+1)^{2k+1}}$. Suppose $\sin\frac{z}{z+1}$ has Laurent series $\sum_{n=-\infty}^{\infty}c_n(z+1)^n$. Then by Theorem 3.2, for $\varepsilon>0$,

$$\begin{aligned}c_n &= \frac{1}{2\pi i}\int_{|\zeta+1|=\varepsilon}\frac{1}{(\zeta+1)^{n+1}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\frac{\zeta^{2k+1}}{(\zeta+1)^{2k+1}}d\zeta\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\cdot\frac{1}{2\pi i}\int_{|\zeta+1|=\varepsilon}\frac{\zeta^{2k+1}}{(\zeta+1)^{2k+n+2}}d\zeta\\ &= \sum_{k\geq\frac{-n-1}{2}}\frac{(-1)^k}{(2k+1)!}\frac{1}{(2k+n+1)!}\frac{d^{2k+n+1}}{d\zeta^{2k+n+1}}\left(\zeta^{2k+1}\right)\Big|_{\zeta=-1}.\end{aligned}$$

Since

$$\frac{d^{2k+n+1}}{d\zeta^{2k+n+1}}\left(\zeta^{2k+1}\right)\Big|_{\zeta=-1}=\begin{cases}0 & n\geq 1,\\ (2k+1)! & n=0,\\ \frac{(2k+1)!}{(-n)!}(-1)^{-n} & n\leq -1,\end{cases}$$

we have

$$\begin{aligned}c_n &= \begin{cases}0 & n\geq 1,\\ \sum_{k\geq 0}\frac{(-1)^k}{(2k+1)!} & n=0,\\ \frac{(-1)^{-n}}{(-n)!}\sum_{k\geq\frac{-n-1}{2}}\frac{(-1)^k}{(2k+n+1)!} & n\leq -1\end{cases}\\ &= \begin{cases}0 & n\geq 1,\\ \sin 1 & n=0,\\ \frac{1}{(-n)!}\cos 1 & n\leq -1,\ n\text{ is odd},\\ \frac{1}{(-n)!}\sin 1 & n\leq -1,\ n\text{ is even}.\end{cases}\end{aligned}$$

Therefore

$$\sin\frac{z}{z+1}=\sum_{k=0}^{\infty}\left[\frac{\sin 1}{(2k)!}(z+1)^{-2k}+\frac{\cos 1}{(2k+1)!}(z+1)^{-(2k+1)}\right].$$

□

► **4.** Find and classify the singularities of the following functions. Find the order if the singularity is a pole.

(i) $\frac{\sin z}{z}$;

Solution. $z = 0$ is a removable singularity. □

(ii) $\frac{1}{z^2-1}\cos\frac{\pi z}{z+1}$;

Solution. $z = 1$ is a pole of order 1. The Laurent series of the function in a neighborhood of $z = -1$ can be obtained by

$$\frac{1}{z^2-1}\cos\frac{\pi z}{z+1} \quad = \quad \frac{1}{(z-1)(z+1)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\pi z}{z+1}\right)^{2n}.$$

By discussion (3) of page 91, $z = -1$ is an essential singularity. □

(iii) $z(e^{\frac{1}{z}} - 1)$;

Solution. Since $e^{\frac{1}{z}} = \sum_{n=0}^{\infty} z^{-n}$, we have $z(e^{\frac{1}{z}} - 1) = \sum_{n=1}^{\infty} z^{-n+1}$. Therefore $z = 0$ is an essential singularity. □

(iv) $\sin\frac{1}{1-z}$;

Solution. $z = 1$ is an essential singularity. □

(v) $\frac{\exp\left\{\frac{1}{1-z}\right\}}{e^z-1}$;

Solution. $z = 1$ is an essential singularity. $z = 0$ is a pole of order 1. □

(vi) $\tan z$.

Solution. $z = k\pi + \frac{\pi}{2}$ $(k \in \mathbb{Z})$ are poles of order 1. □

► **5.** Prove:

(1) If a is an essential singularity of $f(z)$ and $f(z) \neq 0$, then a is also an essential singularity of $1/f(z)$.

Proof. The limit $\lim_{z\to a} f(z)$ exists (finite or ∞) if and only if $\lim_{z\to a}\frac{1}{f(z)}$ exists. By discussion (3) on page 93, a is an essential singularity of $f(z)$ if and only if a is an essential singularity of $\frac{1}{f(z)}$. □

(2) If a is an essential singularity of $f(z)$ and $P(\zeta)$ is a non-constant polynomial, then a is also an essential singularity of $P(f(z))$.

Proof. Clearly, a is an isolated singularity of $P(f(z))$. We note by Fundamental Theorem of Algebra, $P(\zeta)$ maps $\mathbb{C}$ to $\mathbb{C}$. Consequently, $P(\zeta)$ maps a dense subset of $\mathbb{C}$ to a dense subset of $\mathbb{C}$. By Weierstrass Theorem (Theorem 3.3), $f(z)$ maps a neighborhood of a to a dense subset of $\mathbb{C}$. Therefore, $P(f(z))$ maps a neighborhood of a to a dense subset of $\mathbb{C}$. So it is impossible for a to become a removable singularity or a pole of $P(f(\zeta))$. Hence a must be also an essential singularity of $P(f(z))$. □

► **6.** Prove:

(i) $\prod_{n=0}^{\infty}(1 + z^{2^n}) = \frac{1}{1-z}$.

Proof. We note any positive integer can be uniquely represented in the form $\sum_{k=0}^{\infty} a_k \cdot 2^k$, where each $a_k \in \{0, 1\}$ and only finitely many a_k's are non-zero. In the expansion of $\prod_{n=0}^{\infty}(1 + z^{2^n})$, each representation $z^{\sum_{k=0}^{\infty} a_k\cdot 2^k}$ appears once and only once. Therefore, after a rearrangement of the terms, we must have

$$\prod_{n=0}^{\infty}(1 + z^{2^n}) = 1 + z + z^2 + z^3 + z^4 + \cdots = \frac{1}{1-z}.$$

□

(ii) $\sinh \pi z = \pi z \prod_{n=1}^{\infty}\left(1+\frac{z^2}{n^2}\right)$;

Proof. $\sinh \pi z = \frac{1}{2}[e^{\pi z} - e^{-\pi z}]$ has zeros $a_n = ni$ $(n \in \mathbb{Z})$, where $a_0 = 0$ is a zero of order 1. Since for any $R > 0$, we have $\sum_{n=-\infty, n\neq 0}^{\infty}\left(\frac{R}{|a_n|}\right)^2 = \sum_{n=-\infty, n\neq 0}^{\infty}\frac{R^2}{n^2} < \infty$, by Weierstrass Factorization Theorem, we can find an entire function $h(z)$ such that

$$\sinh \pi z = ze^{h(z)} \prod_{n=-\infty, n\neq 0}^{n=\infty}\left\{\left(1-\frac{z}{ni}\right)e^{\frac{z}{ni}}\right\} = ze^{h(z)}\prod_{n=1}^{\infty}\left(1+\frac{z^2}{n^2}\right).$$

Since $\lim_{z\to 0}\frac{\sinh \pi z}{z} = \pi$, we conclude $e^{h(0)} = \pi$. Meanwhile we have

$$\begin{aligned}\pi \coth \pi z &= \frac{(\sinh \pi z)'}{\sinh \pi z} = \frac{1}{\sinh \pi z}\left[ze^{h(z)}\prod_{n=1}^{\infty}\left(1+\frac{z^2}{n^2}\right)\right]' \\ &= \frac{1}{z} + h'(z) + \sum_{n=1}^{\infty}\frac{\frac{2z}{n^2}}{1+\frac{z^2}{n^2}} = \frac{1}{z} + h'(z) + \sum_{n=1}^{\infty}\frac{2z}{n^2+z^2}.\end{aligned}$$

Let γ_n be the rectangular path $[n+(n+\frac{1}{2})i, -n+(n+\frac{1}{2})i, -n-(n+\frac{1}{2})i, n-(n+\frac{1}{2})i, n+(n+\frac{1}{2})i]$. Then for any given a, when n is large enough, we have

$$\int_{\gamma_n}\frac{\coth \pi z}{z^2-a^2}dz = I + II + III + IV,$$

where

$$I = \int_n^{-n}\frac{\frac{e^{2\pi[x+(n+\frac{1}{2})i]}+1}{e^{2\pi[x+(n+\frac{1}{2})i]}-1}}{\left[x+(n+\frac{1}{2})i\right]^2-a^2}dx = \int_n^{-n}\frac{\frac{-e^{2\pi x}+1}{-e^{2\pi x}-1}}{\left[x+(n+\frac{1}{2})i\right]^2-a^2}dx,$$

$$II = \int_{n+\frac{1}{2}}^{-(n+\frac{1}{2})}\frac{\frac{e^{2\pi(-n+yi)}+1}{e^{2\pi(-n+yi)}-1}}{(-n+yi)^2-a^2}idy,$$

$$III = \int_{-n}^{n}\frac{\frac{e^{2\pi[x-(n+\frac{1}{2})i]}+1}{e^{2\pi[x-(n+\frac{1}{2})i]}-1}}{[x-(n+\frac{1}{2})i]^2-a^2}dx = \int_{-n}^{n}\frac{\frac{-e^{2\pi x}+1}{-e^{2\pi x}-1}}{[x-(n+\frac{1}{2})i]^2-a^2}dx$$

and

$$IV = \int_{-(n+\frac{1}{2})}^{n+\frac{1}{2}}\frac{\frac{e^{2\pi(n+yi)}+1}{e^{2\pi(n+yi)}-1}}{(n+iy)^2-a^2}idy.$$

We note $\frac{e^x-1}{e^x+1} \in (-1,1)$ for any $x \in \mathbb{R}$. So we have

$$|I| \le \int_{-n}^{n}\frac{1}{x^2+(n+\frac{1}{2})^2-a^2}dx = \frac{2}{\sqrt{(n+\frac{1}{2})^2-a^2}}\arctan\frac{n}{\sqrt{(n+\frac{1}{2})^2-a^2}} \to 0,$$

as $n \to \infty$. Similarly we can conclude $\lim_{n\to\infty} III = 0$. We also note for $x \in \mathbb{R}$ large enough, $\frac{e^x+1}{e^x-1} < 2$. So for n large enough,

$$|II| \le \int_{-(n+\frac{1}{2})}^{n+\frac{1}{2}}\frac{e^{\frac{1+e^{-2\pi n}}{1-e^{-2\pi n}}}}{n^2+y^2-a^2}dy \le \frac{4}{\sqrt{n^2-a^2}}\arctan\frac{n+\frac{1}{2}}{\sqrt{n^2-a^2}} \to 0,$$

as $n \to \infty$. Similarly we can conclude $\lim_{n\to\infty} IV = 0$. Combined, we have shown

$$\lim_{n\to\infty}\int_{\gamma_n}\frac{\coth \pi z}{z^2-a^2}dz = 0.$$

Define $f(z) = \frac{\coth \pi z}{z^2 - a^2}$. By Residue Theorem, for $a \notin i\mathbb{Z}$, we have

$$\int_{\gamma_n} f(z)dz = 2\pi i \text{Res}(f; a) + 2\pi i \text{Res}(f; -a) + 2\pi i \sum_{k=-n}^{n} \text{Res}(f; a_n).$$

It's easy to see $\text{Res}(f; a) = \text{Res}(f; -a) = \frac{\coth \pi a}{2a}$. Since

$$\lim_{z \to a_n} (z - a_n) f(z) = \lim_{z \to a_n} \frac{e^{2\pi z} + 1}{z^2 - a^2} \frac{z - a_n}{e^{2\pi z} - 1} = \frac{2}{2\pi(a_n^2 - a^2)} \lim_{z \to a_n} \frac{2\pi(z - a_n)}{e^{2\pi z} - e^{2\pi a_n}} = -\frac{1}{\pi(n^2 + a^2)},$$

we must have $\text{Res}(f; a_n) = -\frac{1}{\pi(n^2+a^2)}$. Therefore

$$0 = \frac{1}{2\pi i} \lim_{n \to \infty} \int_{\gamma_n} f(z)dz = \lim_{n \to \infty} \left(\frac{\coth \pi a}{a} - \sum_{k=-n}^{n} \frac{1}{\pi(a^2 + k^2)} \right) = \frac{\coth \pi a}{a} - \sum_{k=-\infty}^{\infty} \frac{1}{\pi(a^2 + k^2)}$$

This implies for $a \notin i\mathbb{Z}$,

$$\pi \coth \pi a = \frac{1}{a} + \sum_{k=1}^{\infty} \frac{2a}{a^2 + n^2}.$$

Plugging this back into the formula $\pi \coth \pi z = \frac{1}{z} + h'(z) + \sum_{n=1}^{\infty} \frac{2z}{n^2+z^2}$, we conclude $h'(z) = 0$ for $z \notin i\mathbb{Z}$. By Theorem 2.13, $h'(z) \equiv 0$ for any $z \in \mathbb{C}$. So

$$\sinh \pi z = z e^{h(0)} \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2}\right) = \pi z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2}\right).$$

□

Remark 10. *The above solution is a variant of the proof for factorization formula of* $\sin \pi z$*. See Conway[2] VII, §6 for more details.*

(iii) $\cos \pi z = \prod_{n=0}^{\infty} \left(1 - \left(\frac{z}{n+\frac{1}{2}}\right)^2\right)$;

Proof. From problem (v), we have $\frac{\pi}{2} \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right) = 1$ and

$$\begin{aligned}
\cos \pi z &= \sin \pi \left(\frac{1}{2} - z\right) = \pi \left(\frac{1}{2} - z\right) \prod_{n=1}^{\infty} \left(1 - \frac{\left(\frac{1}{2} - z\right)^2}{n^2}\right) \\
&= \pi \left(\frac{1}{2} - z\right) \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{\frac{1}{2} - z}{n}\right) \left(1 - \frac{\frac{1}{2} - z}{n}\right) \right\} \\
&= \pi \left(\frac{1}{2} - z\right) \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{1}{2n}\right) \left(1 - \frac{z}{n + \frac{1}{2}}\right) \left(1 - \frac{1}{2n}\right) \left(1 + \frac{z}{n - \frac{1}{2}}\right) \right\} \\
&= \lim_{N \to \infty} \frac{\pi}{2} \left(1 - \frac{z}{\frac{1}{2}}\right) \prod_{n=1}^{N} \left(1 - \frac{1}{4n^2}\right) \cdot \prod_{n=1}^{N} \left(1 - \frac{z}{n + \frac{1}{2}}\right) \cdot \prod_{n=1}^{N} \left(1 + \frac{z}{n - \frac{1}{2}}\right) \\
&= \lim_{N \to \infty} \left\{ \frac{\pi}{2} \prod_{n=1}^{N} \left(1 - \frac{1}{4n^2}\right) \right\} \cdot \left(1 - \frac{z}{N + \frac{1}{2}}\right) \cdot \prod_{n=1}^{N} \left[1 - \frac{z^2}{(n - \frac{1}{2})^2}\right] \\
&= \prod_{n=0}^{\infty} \left[1 - \left(\frac{z}{n + \frac{1}{2}}\right)^2\right].
\end{aligned}$$

□

(iv) $e^z - 1 = ze^{\frac{z}{2}} \prod_{n=1}^{\infty} \left(1 + \frac{z}{4\pi^2 n^2}\right)$;

Proof. The function $e^{\pi z} - 1$ has zeros $a_n = 2ni$ ($n \in \mathbb{Z}$), where $a_0 = 0$ is a zero of order 1. Since for any $R > 0$, we have $\sum_{n=-\infty,n\neq 0}^{\infty} \left(\frac{R}{|a_n|}\right)^2 = \sum_{n=-\infty,n\neq 0}^{\infty} \frac{R^2}{4n^2} < \infty$, by Weierstrass Factorization Theorem, we can find an entire function $h(z)$ such that

$$e^{\pi z} - 1 = ze^{h(z)} \prod_{n=-\infty,n\neq 0}^{\infty} \left\{\left(1 - \frac{z}{2ni}\right) e^{\frac{z}{2ni}}\right\} = ze^{h(z)} \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4n^2}\right).$$

To determine $h(z)$, we note

$$\frac{\pi e^{\pi z}}{e^{\pi z} - 1} = \frac{(e^{\pi z} - 1)'}{e^{\pi z} - 1} = \frac{1}{z} + h'(z) + \sum_{n=1}^{\infty} \frac{\frac{2z}{4n^2}}{1 + \frac{z^2}{4n^2}} = \frac{1}{z} + h'(z) + \sum_{n=1}^{\infty} \frac{2z}{4n^2 + z^2}.$$

Let γ_n be the rectangular path $[2n+(2n+1)i, -2n+(2n+1)i, -2n-(2n+1)i, 2n-(2n+1)i, 2n+(2n+1)i]$. Define $f(z) = \frac{\pi e^{\pi z}}{(z^2-a^2)(e^{\pi z}-1)}$. Then for any given a, when n is large enough, we have

$$\int_{\gamma_n} f(z)dz = I + II + III + IV,$$

where

$$I = \int_{2n}^{-2n} f(x + (2n+1)i)dx = \int_{2n}^{-2n} \frac{\pi e^{e^{\pi x}}}{[(x + (2n+1)i)^2 - a^2](e^{\pi x} + 1)} dx,$$

$$II = \int_{2n+1}^{-(2n+1)} f(-2n + yi)idy = \int_{2n+1}^{-(2n+1)} \frac{\pi e^{\pi(-2n+yi)}}{[(-2n + yi)^2 - a^2](e^{\pi(-2n+yi)} - 1)} idy,$$

$$III = \int_{-2n}^{2n} f(x - (2n+1)i)dx = \int_{-2n}^{2n} \frac{\pi e^{e^{\pi x}}}{[(x - (2n+1)i)^2 - a^2](e^{\pi x} + 1)} dx,$$

and

$$IV = \int_{-(2n+1)}^{2n+1} f(2n + yi)idy = \int_{-(2n+1)}^{2n+1} \frac{\pi e^{\pi(2n+yi)}}{[(2n + yi)^2 - a^2](e^{\pi(2n+yi)} - 1)} idy.$$

It is easy to see

$$|I| \le \int_{-2n}^{2n} \frac{\pi dx}{x^2 + (2n+1)^2 - a^2} \le \frac{2\pi}{\sqrt{(2n+1)^2 - a^2}} \arctan \frac{2n}{\sqrt{(2n+1)^2 - a^2}} \to 0$$

as $n \to \infty$. Similarly, we can show $\lim_{n\to\infty} III = 0$. Also, we note

$$|II| \le \int_{-(2n+1)}^{2n+1} \frac{\pi dy}{(4n^2 + y^2 - a^2)(e^{2n\pi} - 1)} \le \frac{2\pi}{(e^{2n\pi} - 1)\sqrt{4n^2 - a^2}} \arctan \frac{2n+1}{\sqrt{4n^2 - a^2}} \to 0,$$

as $n \to \infty$. For n sufficiently large, $\frac{e^{2n\pi}}{e^{2n\pi}-1} \le 2$, so

$$|IV| \le \int_{-(2n+1)}^{2n+1} \frac{\pi e^{2\pi n} dy}{(4n^2 + y^2 - a^2)(e^{2\pi n} - 1)} \le \frac{4\pi}{\sqrt{4n^2 - a^2}} \arctan \frac{2n+1}{\sqrt{4n^2 - a^2}} \to 0,$$

as $n \to \infty$. Combined, we have shown

$$\lim_{n\to\infty} \int_{\gamma_n} f(z)dz = 0.$$

By Residue Theorem, for $a \notin 2\mathbb{Z}i$, we have

$$\int_{\gamma_n} f(z)dz = 2\pi i \text{Res}(f; a) + 2\pi i \text{Res}(f; -a) + 2\pi i \sum_{k=-n}^{n} \text{Res}(f; a_n).$$

It's easy to see $\text{Res}(f;a) = \frac{\pi e^{\pi a}}{2a(e^{\pi a}-1)}$ and $\text{Res}(f;-a) = \frac{\pi}{2a(e^{\pi a}-1)}$. Since

$$\lim_{z\to a_n}(z-a_n)f(z) = \lim_{z\to a_n}\frac{\pi e^{\pi z}}{(z^2-a^2)}\cdot\frac{z-a_n}{e^{\pi z}-1} = -\frac{1}{4n^2+a^2},$$

$\text{Res}(f;a_n) = -\frac{1}{4n^2+a^2}$. Therefore

$$0 = \frac{1}{2\pi i}\lim_{n\to\infty}\int_{\gamma_n} f(z)dz = \lim_{n\to\infty}\left(\frac{\pi}{2a}\frac{e^{\pi a}+1}{e^{\pi a}-1} - \sum_{k=-n}^{n}\frac{1}{4n^2+a^2}\right).$$

So $\frac{\pi}{2a}\frac{e^{\pi a}+1}{e^{\pi a}-1} = \frac{1}{a^2} + \sum_{n=1}^{\infty}\frac{2}{4n^2+a^2}$ for $a \notin 2\mathbb{Z}i$. Therefore for $z \notin 2\mathbb{Z}i$, we have

$$h'(z) = \frac{\pi e^{\pi z}}{e^{\pi z}-1} - \frac{1}{z} - \sum_{n=1}^{\infty}\frac{2z}{4n^2+z^2} = \frac{\pi e^{\pi z}}{e^{\pi z}-1} - \frac{\pi}{2}\frac{e^{\pi z}+1}{e^{\pi z}-1} = \frac{\pi}{2}.$$

So $h(z) = \frac{\pi z}{2} + h(0)$. It's easy to see $\lim_{z\to 0}\frac{e^{\pi z}-1}{z} = \pi$. So $e^{h(0)} = \pi$. Combined, we conclude $e^{h(z)} = \pi e^{\frac{\pi z}{2}}$. So

$$e^{\pi z} - 1 = \pi z e^{\frac{\pi z}{2}}\prod_{n=1}^{\infty}\left(1 + \frac{z^2}{4n^2}\right).$$

Equivalently, we have

$$e^z - 1 = z e^{\frac{z}{2}}\prod_{n=1}^{\infty}\left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

□

(v) $\sin\pi z = \pi z\prod_{n\neq 0}\left(1 - \frac{z}{n}\right)e^{\frac{z}{n}}$, hence $\sin\pi z = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$.

Proof. Using the formula $\sin\pi z = -i\sinh(iz)$ and the factorization formula for $\sinh\pi z$, we have

$$\sin\pi z = -i\pi(iz)\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) = \pi z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right).$$

□

► **7.** (*Blaschke Product*) Suppose a sequence of complex numbers $\{a_k\}$ satisfies $0 < |a_k| < 1$ $(k = 1, 2, \cdots)$ and $\sum_{k=1}^{\infty}(1 - |a_k|) < \infty$. Show that the infinite product

$$f(z) = \prod_{k=1}^{\infty}\frac{a_k - z}{1 - \overline{a_k}z}\cdot\frac{|a_n|}{a_n}$$

converges uniformly on the disc $\{z : |z| \leq r\}$ $(0 < r < 1)$, and $f(z)$ is holomorphic in $|z| < 1$ with $|f(z)| \leq 1$. Also show that the zeros of $f(z)$ are a_k $(k = 1, 2, \cdots)$ and there is no other zeros.

Proof. Fix $r \in (0,1)$. Since $\sum_{k=1}^{\infty}(1 - |a_k|) < \infty$, $\lim_{k\to\infty}|a_k| = 1$. So there exists $k_0 \in \mathbb{N}$, such that for any $k \geq k_0$, $\frac{1+r}{2} < |a_k| < 1$. So for any $k \geq k_0$,

$$\left|\frac{a_k - z}{1 - \bar{a}_k z}\cdot\frac{|a_k|}{a_k} - 1\right| \leq \frac{|a_k|(1 - |a_k|) + |z||a_k|(1 - |a_k|)}{|1 - \bar{a}_k z||a_k|} \leq \frac{(1+r)(1 - |a_k|)}{\frac{1+r}{2}(1-r)} = \frac{2}{1-r}(1 - |a_k|).$$

Since $\sum_{k=1}^{\infty}(1 - |a_k|) < \infty$, we conclude $\sum_{k=1}^{\infty}\left|\frac{a_k - z}{1 - \bar{a}_k z}\cdot\frac{|a_k|}{a_k} - 1\right|$ is absolutely and uniformly convergent on $\{z : |z| \leq r\}$. Therefore $\prod_{k=1}^{\infty}\frac{a_k - z}{1 - \bar{a}_k z}\cdot\frac{|a_k|}{a_k}$ is absolutely and uniformly convergent on $\{z : |z| \leq r\}$. By Weierstrass Theorem (Theorem 3.1), $f(z) = \prod_{k=1}^{\infty}\frac{a_k - z}{1 - \bar{a}_k z}\cdot\frac{|a_k|}{a_k}$ represents a non-zero holomorphic function on $\{z : |z| < 1\}$. Since the mapping $z \mapsto \frac{a_k - z}{1 - \bar{a}_k z}$ $(0 < |a_k| < 1)$ maps $D(0,1)$ to $D(0,1)$, we conclude $|f(z)| \leq 1$. By the theorem quoted at the beginning of the solution, it's clear that $(a_k)_{k=1}^{\infty}$ are the only zeros of $f(z)$. □

Remark 11. *For a clear presentation of convergence of infinite products, we refer to Conway [2], Chapter VII, §5. In particular, we quote the following theorem (Conway [2], Chapter VII, §5, Theorem 5.9)*

Let G be an open subset of the complex plane. Denote by $H(G)$ the collection of analytic functions on G, equipped with the topology determined by uniform convergence on compact subsets of G. Then $H(G)$ is a complete metric space. Furthermore, we have the following theorem (proof omitted).

Theorem 3.1. *Let G be a region in $\mathbb{C}$ and let $(f_n)_{n=1}^{\infty}$ be a sequence in $H(G)$ such that no f_n is identically zero. If $\sum[f_n(z)-1]$ converges absolutely and uniformly on compact subsets of G, then $\prod_{n=1}^{\infty} f_n(z)$ converges in $H(G)$ to an analytic function $f(z)$. If a is a zero of f then a is a zero of only a finite number of the functions f_n, and the multiplicity of the zero of f at a is the sum of the multiplicities of the zeros of the functions f_n at a.*

► **8.** (*Poisson-Jensen Formula*) Suppose $f(z)$ is a meromorphic function on the disc $\{z : |z| \leq R\}$ ($0 < R < \infty$), and is not equal to zero identically, the points $a_1, a_2, \cdots, a_s$ and $b_1, b_2, \cdots, b_t$ are zeros and poles of $f(z)$ on $|z| < R$ respectively. Suppose that if a is a zero with multiplicity n, then a appears n times in $a_1, a_2, \cdots, a_s$. Same assumption for the poles. Then we have

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\varphi})|\mathrm{Re}\frac{Re^{i\varphi}+2}{Re^{i\varphi}-z}d\varphi + \sum_{j=1}^{t}\log\left|\frac{R^2-\overline{b_j}z}{R(z-b_j)}\right| - \sum_{i=1}^{s}\log\left|\frac{R^2-\overline{a_i}z}{R(z-a_i)}\right|$$

for any points in the disc $|z| < 1$ which is different from a_i $(i = 1, 2, \cdots, s)$ and b_j $(j = 1, 2, \cdots, t)$.

This formula is the starting point of Nevanlinna Value Distribution theory.

Proof. Let $R_\varepsilon = R - \varepsilon$, $\varepsilon \in (0, R)$. Define

$$F_\varepsilon(z) = f(z) \cdot \frac{\prod_{j=1}^{t} \frac{R_\varepsilon(z-b_j)}{R_\varepsilon^2 - \bar{b}_j z}}{\prod_{i=1}^{s} \frac{R_\varepsilon(z-a_i)}{R_\varepsilon^2 - \bar{a}_i z}},\ \forall z \in D(0, R_\varepsilon).$$

It's easy to verify that each of $\frac{R_\varepsilon^2(z-b_j)}{R_\varepsilon^2-\bar{b}_j z}$ and $\frac{R_\varepsilon^2(z-a_i)}{R_\varepsilon^2-\bar{a}_i z}$ maps $D(0, R_\varepsilon)$ onto itself and takes the boundary to the boundary. Therefore $F_\varepsilon(z)$ is analytic on $D(0, R_\varepsilon)$, has no zeros in $D(0, R_\varepsilon)$, and $|F_\varepsilon(z)| = |f(z)|$ for $|z| = R_\varepsilon$. Since $\log|F_\varepsilon(z)|$ is a harmonic function, by formula (2.33) on page 71,

$$\log|F_\varepsilon(z)| = \frac{1}{2\pi}\int_0^{2\pi}\log|F_\varepsilon(R_\varepsilon e^{i\varphi})|\mathrm{Re}\frac{R_\varepsilon e^{i\varphi}+z}{R_\varepsilon e^{i\varphi}-z}d\varphi = \frac{1}{2\pi}\int_0^{2\pi}\log|f(R_\varepsilon e^{i\varphi})|\mathrm{Re}\frac{R_\varepsilon e^{i\varphi}+z}{R_\varepsilon e^{i\varphi}-z}d\varphi.$$

Plugging the formula of F_ε into the above equality, we get

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi}\log|f(R_\varepsilon e^{i\varphi})|\mathrm{Re}\frac{R_\varepsilon e^{i\varphi}+z}{R_\varepsilon e^{i\varphi}-z}d\varphi + \sum_{j=1}^{t}\log\left|\frac{R_\varepsilon^2-\bar{b}_j z}{R_\varepsilon(z-b_j)}\right| - \sum_{i=1}^{s}\log\left|\frac{R_\varepsilon^2-\bar{a}_i z}{R_\varepsilon(z-a_i)}\right|.$$

Letting $\varepsilon \to 0$ yields the desired formula. □

► **9.** Suppose a meromorphic function $f(z)$ only has two poles on the extended complex plane $\mathbb{C}^*$. The point $z = -1$ is a pole of multiplicity 1 and with principle part $1/(z+1)$; the point $z = 2$ is a pole of multiplicity 2 and with principle part $2/(z-2) + 3/(z-2)^2$, and $f(0) = 7/4$. Find the Laurent expansion of $f(z)$ in $1 < |z| < 2$.

Solution. By Theorem 3.4 (correction: "holomorphic function $f(z)$" in the theorem's statement should be "meromorphic function $f(z)$"), $f(z)$ has the form of $\frac{1}{z+1} + \frac{2}{z-2} + \frac{3}{(z-2)^2} + c$, where c is a constant. $f(0) = \frac{7}{4}$ implies $c = 1$. Since for any $\zeta \in D(0, 1)$,

$$\frac{1}{(1-\zeta)^2} = \left(\frac{1}{1-\zeta}\right)' = \left(\sum_{n=0}^{\infty}\zeta^n\right)' = \sum_{n=0}^{\infty}(n+1)\zeta^n,$$

the Laurent expansion of $f(z)$ in $1 < |z| < 2$ is

$$\begin{aligned} f(z) &= \frac{1}{z+1} + \frac{2}{z-2} + \frac{3}{(z-2)^2} + 1 \\ &= \frac{1}{z}\frac{1}{1+\frac{1}{z}} - \frac{1}{1-\frac{z}{2}} + \frac{3}{4}\cdot\frac{1}{(1-\frac{z}{2})^2} + 1 \\ &= \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n - \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \frac{3}{4}\sum_{n=0}^{\infty}(n+1)\left(\frac{z}{2}\right)^n + 1 \\ &= -\sum_{n=1}^{\infty}\frac{(-1)^n}{z^n} + \frac{3}{4} + \sum_{n=1}^{\infty}\frac{3n-1}{2^{n+2}}z^n. \end{aligned}$$

□

► **10.** Suppose a meromorphic function $f(z)$ has poles at $z = 1, 2, 3, \cdots$ with multiplicity 2, and the principle parts of Laurent expansions in the neighborhood of $z = n$ are $\psi_n(z) = n/(z-n)^2$ $(n = 1, 2, \cdots)$. Find a general form of $f(z)$.

Solution. We follow the construction outlined in the proof of Mittag-Leffler Theorem (Theorem 3.9). For each $n \in \mathbb{N}$, when $|z| < \frac{n}{2}$, $\psi_n(z)$ has Taylor expansion

$$\psi_n(z) = \frac{n}{(z-n)^2} = \frac{1}{n}\cdot\frac{1}{\left(1-\frac{z}{n}\right)^2} = \frac{1}{n}\sum_{k=0}^{\infty}(k+1)\left(\frac{z}{n}\right)^k.$$

Let λ_n be a positive integer to be determined later. We estimate the tail error obtained by retaining only the first λ_n terms of the Taylor expansion of $\psi_n(z)$:

$$\begin{aligned} \left|\psi_n(z) - \frac{1}{n}\sum_{k=0}^{\lambda_n}(k+1)\left(\frac{z}{n}\right)^k\right| &\le \frac{1}{n}\sum_{k=\lambda_n+1}^{\infty}\frac{(k+1)}{2^k} = \frac{1}{n}\sum_{k=\lambda_n}^{\infty}\frac{(k+2)}{2^{k+1}} = \frac{1}{n}\left(\sum_{k=\lambda_n}^{\infty}\frac{k}{2^{k+1}} + \frac{1}{2^{\lambda_n-1}}\right) \\ &\le \frac{1}{n\cdot 2^{\lambda_n-1}} + \frac{1}{n}\sum_{k=\lambda_n}^{\infty}\int_k^{k+1}\frac{x}{2^x}dx = \frac{1}{n\cdot 2^{\lambda_n-1}} + \frac{1}{n}\int_{\lambda_n}^{\infty}\frac{x}{\log\frac{1}{2}}d\left(\frac{1}{2}\right)^x \\ &= \frac{1}{n\cdot 2^{\lambda_n-1}} + \frac{1}{n\cdot 2^{\lambda_n}}\left[\frac{\lambda_n}{\log 2} + \frac{1}{(\log 2)^2}\right]. \end{aligned}$$

If we let $\lambda_n = n$, then $\varepsilon_n := \frac{1}{n\cdot 2^{\lambda_n-1}} + \frac{1}{n\cdot 2^{\lambda_n}}\left[\frac{\lambda_n}{\log 2} + \frac{1}{(\log 2)^2}\right] = \frac{1}{n\cdot 2^{n-1}} + \frac{1}{n\cdot 2^n}\left[\frac{n}{\log 2} + \frac{1}{(\log 2)^2}\right]$ satisfies $\sum_{n=1}^{\infty}\varepsilon_n < \infty$. By the proof of Mittag-Leffler Theorem, we can write $f(z)$ as

$$f(z) = U(z) + \sum_{n=1}^{\infty}\left[\frac{n}{(z-n)^2} - \frac{1}{n}\sum_{k=0}^{n}(k+1)\left(\frac{z}{n}\right)^k\right],$$

where $U(z)$ is an entire function. □

► **11.**

(i) Expand $f(z) = 1/(e^z - 1)$ to a partial fraction.

Solution. Define $f(z) = \frac{1}{(z^2-a^2)(e^{\pi z}-1)}$, where $a \in \mathbb{C} \setminus 2i\mathbb{Z}$. Let γ_n be the rectangular path $[2n + (2n+1)i, -2n + (2n+1)i, -2n - (2n+1)i, 2n - (2n+1)i, 2n + (2n+1)i]$. Then for any given a, when n is large enough, we have

$$\int_{\gamma_n} f(z)dz = I + II + III + IV,$$

where

$$I = \int_{2n}^{-2n} f(x+(2n+1)i)dx = \int_{-2n}^{2n} \frac{dx}{[(x+(2n+1)i)^2 - a^2](e^{\pi x}+1)},$$

$$II = \int_{2n+1}^{-(2n+1)} f(2n+yi)idy = \int_{2n+1}^{-(2n+1)} \frac{idy}{[(-2n+yi)^2 - a^2](e^{\pi(-2n+yi}-1)},$$

$$III = \int_{-2n}^{2n} f(x-(2n+1)i)dx = \int_{-2n}^{2n} \frac{-dx}{[(x-(2n+1)i)^2 - a^2](e^{\pi x}+1)},$$

and

$$IV = \int_{-(2n+1)}^{2n+1} f(2n+iy)idy = \int_{-(2n+1)}^{2n+1} \frac{idy}{[(2n+iy)^2 - a^2](e^{\pi(2n+iy)}-1)}.$$

It's easy to see

$$|I| \le \int_{-2n}^{2n} \frac{dx}{x^2+(2n+1)^2-a^2} \le \frac{2}{\sqrt{(2n+1)^2-a^2}} \arctan\frac{2n}{\sqrt{(2n+1)^2-a^2}} \to 0$$

as $n \to \infty$. Similarly, we can show $\lim_{n\to\infty} III = 0$. Also, we note

$$|II| \le \int_{-(2n+1)}^{2n+1} \frac{dy}{(4n^2+y^2-a^2)(1-e^{-2n\pi})} = \frac{2}{(1-e^{-2n\pi})\sqrt{4n^2-a^2}} \arctan\frac{2n+1}{\sqrt{4n^2-a^2}} \to 0$$

as $n \to \infty$, and

$$|IV| \le \int_{-(2n+1)}^{2n+1} \frac{dy}{(4n^2+y^2-a^2)(e^{2\pi n}-1)} = \frac{2}{(e^{2\pi n}-1)\sqrt{4n^2-a^2}} \arctan\frac{2n+1}{4n^2-a^2} \to 0$$

as $n \to \infty$. Combined, we have shown $\lim_{n\to\infty}\int_{\gamma_n} f(z)dz = 0$. By Residue Theorem, for $a \notin 2i\mathbb{Z}$, we have

$$\int_{\gamma_n} f(z)dz = 2\pi i \text{Res}(f;a) + 2\pi i \text{Res}(f;-a) + 2\pi i \sum_{k=-n}^{n} \text{Res}(f;2ni).$$

It's easy to see $\text{Res}(f;a) = \frac{1}{2a(e^{\pi a}-1)}$ and $\text{Res}(f;-a) = \frac{1}{2a(1-e^{-\pi a})}$. Since

$$\lim_{z\to 2ni}(z-2ni)f(z) = \lim_{z\to 2ni} \frac{\pi(z-2ni)}{\pi(z^2-a^2)(e^{\pi z}-1)} = -\frac{1}{\pi(4n^2+a^2)},$$

we have $\text{Res}(f;2ni) = -\frac{1}{\pi(4n^2+a^2)}$. Therefore

$$0 = \frac{1}{2\pi i}\lim_{n\to\infty}\int_{\gamma_n} f(z)dz = \lim_{n\to\infty}\left(\frac{1}{2a}\cdot\frac{e^{\pi a}+1}{e^{\pi a}-1} - \frac{1}{\pi}\sum_{k=-n}^{n}\frac{1}{4k^2+a^2}\right) = \frac{1}{2a}\cdot\frac{e^{\pi a}+1}{e^{\pi a}-1} - \frac{1}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{4k^2+a^2}.$$

Replacing πa with z, we get after simplification

$$\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty}\frac{2z}{4k^2\pi^2+z^2},\ \forall z \in \mathbb{C}\setminus 2\mathbb{Z}\pi i.$$

This is the partial fraction of $1/(e^z-1)$. □

(ii) Show that

$$\frac{1}{\sin^2 \pi z} = \frac{1}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(z-n)^2};$$

Proof. Define $f(z) = \frac{1}{(z-a)\sin^2(\pi z)}$, where $a \in \mathbb{C}\backslash\mathbb{Z}$. Let γ_n be the rectangular path $[\left(n+\frac{1}{2}\right)+ni, -\left(n+\frac{1}{2}\right)+ni, -\left(n+\frac{1}{2}\right)-ni, \left(n+\frac{1}{2}\right)-ni, \left(n+\frac{1}{2}\right)+ni]$. Then when n is large enough, we have

$$\int_{\gamma_n} f(z)dz = I + II + III + IV,$$

where

$$I = \int_{n+\frac{1}{2}}^{-(n+\frac{1}{2})} \frac{dx}{(x+ni-a)\sin^2[\pi(x+ni)]} = \int_{-(n+\frac{1}{2})}^{(n+\frac{1}{2})} \frac{4dx}{(x+ni-a)(e^{-2n\pi+2ix\pi}+e^{2n\pi-2ix\pi}-2)},$$

$$II = \int_{n}^{-n} \frac{idy}{[-(n+\frac{1}{2})+yi-a]\sin^2[-\pi(n+\frac{1}{2})+\pi yi]} = \int_{-n}^{n} \frac{-4idy}{[-(n+\frac{1}{2})-a+yi](e^{2\pi y}+e^{-2\pi y}+2)},$$

$$III = \int_{-(n+\frac{1}{2})}^{(n+\frac{1}{2})} \frac{dx}{(x-ni-a)\sin^2[\pi(x-ni)]} = \int_{-(n+\frac{1}{2})}^{(n+\frac{1}{2})} \frac{-4dx}{(x-a-ni)(e^{2n\pi+2i\pi x}+e^{-2n\pi-2i\pi x}-2)},$$

and

$$IV = \int_{-n}^{n} \frac{idy}{[(n+\frac{1}{2})+yi-a]\sin^2[\pi(n+\frac{1}{2})+\pi yi]} = \int_{-n}^{n} \frac{4idy}{[(n+\frac{1}{2})-a+yi](e^{-2y\pi}+e^{2y\pi}+2)}.$$

It's easy to see

$$|I| \leq \int_{-(n+\frac{1}{2})}^{(n+\frac{1}{2})} \frac{4dx}{\sqrt{(x-a)^2+n^2}(e^{2n\pi}-e^{-2n\pi}-2)} \leq \frac{4(2n+1)}{n(e^{2n\pi}-e^{-2n\pi}-2)} \to 0$$

as $n \to \infty$,

$$\begin{aligned} |II| &\leq \int_{-n}^{n} \frac{4dy}{\sqrt{(n+\frac{1}{2}+a)^2+y^2}(e^{2\pi y}+e^{-2\pi y}+2)} = \int_{0}^{n} \frac{8dy}{\sqrt{(n+\frac{1}{2}+a)^2+y^2}(e^{2\pi y}+e^{-2\pi y}+2)} \\ &\leq \int_0^n \frac{8dy}{(n+\frac{1}{2}+a)e^{2\pi y}} = \frac{8}{(n+\frac{1}{2}+a)}\frac{1-e^{-2\pi n}}{2\pi} \to 0 \end{aligned}$$

as $n \to \infty$,

$$|III| \leq \int_{-(n+\frac{1}{2})}^{(n+\frac{1}{2})} \frac{4dx}{\sqrt{(x-a)^2+n^2}(e^{2n\pi}-e^{-2n\pi}-2)} \leq \frac{4(2n+1)}{n(e^{2n\pi}-e^{-2n\pi}-2)} \to 0$$

as $n \to \infty$, and lastly

$$\begin{aligned} |IV| &\leq \int_{-n}^{n} \frac{4dy}{\sqrt{(n+\frac{1}{2}-a)^2+y^2}(e^{2y\pi}+e^{-2y\pi}+2)} = \int_{0}^{n} \frac{8dy}{\sqrt{(n+\frac{1}{2}-a)^2+y^2}(e^{2y\pi}+e^{-2y\pi}+2)} \\ &\leq \int_0^n \frac{8}{(n+\frac{1}{2}-a)e^{2y\pi}} = \frac{8}{(n+\frac{1}{2}-a)}\frac{1-e^{-2\pi n}}{2\pi} \to 0 \end{aligned}$$

as $n \to \infty$. Combined, we have shown $\lim_{n\to\infty}\int_{\gamma_n} f(z)dz = 0$. By Residue Theorem, we have

$$\int_{\gamma_n} f(z)dz = 2\pi i \text{Res}(f;a) + 2\pi i \sum_{k=-n}^{n} \text{Res}(f;k).$$

It's easy to see $\text{Res}(f;a) = \frac{1}{\sin^2(\pi a)}$. Define $h_k(z) = \frac{(z-k)^2}{\sin^2(\pi z)}$. Since $\lim_{z\to k} h_k(z) = \frac{1}{\pi^2}$, h_k is holomorphic in a neighborhood of k and for $\varepsilon > 0$ small enough, and we have

$$\text{Res}(f;k) = \frac{1}{2\pi i}\int_{|z-k|=\varepsilon} \frac{dz}{(z-a)\sin^2(\pi z)} = \frac{1}{2\pi i}\int_{|z-k|=\varepsilon} \frac{h_k(z)}{(z-a)} \cdot \frac{dz}{(z-k)^2} = \frac{d}{dz}\left[\frac{h_k(z)}{z-a}\right]\Bigg|_{z=k}.$$

We note

$$\frac{d}{dz}\left[\frac{h_k(z)}{z-a}\right]\Bigg|_{z=k} = \lim_{z\to k}\frac{h_k'(z)(z-a)-h(z)}{(z-a)^2} = \lim_{z\to k}\frac{h_k'(z)}{z-a} - \frac{1}{(k-a)^2\pi^2},$$

and

$$\begin{aligned}
\lim_{z\to k}\frac{h_k'(z)}{z-a} &= \lim_{z\to k}\frac{2(z-k)\sin(\pi z)-2\pi(z-k)^2\cos(\pi z)}{\sin^3(\pi z)} \\
&= \lim_{z\to k}\frac{2\sin(\pi z)+2\pi(z-k)\cos(\pi z)-4\pi(z-k)\cos(\pi z)+2\pi^2(z-k)^2\sin(\pi z)}{3\sin^2(\pi z)\cdot\pi\cos(\pi z)} \\
&= \lim_{z\to k}\frac{2\sin(\pi z)-2\pi(z-k)\cos(\pi z)}{3\pi\sin^2(\pi z)} \\
&= \lim_{z\to k}\frac{2\pi\cos(\pi z)-2\pi\cos(\pi z)+2\pi^2(z-k)\sin(\pi z)}{6\pi^2\sin(\pi z)\cos(\pi z)} \\
&= 0.
\end{aligned}$$

So for $a\in\mathbb{C}\setminus\mathbb{Z}$,

$$\int_{\gamma_n} f(z)dz = \frac{2\pi i}{\sin^2(\pi a)} - \sum_{k=-n}^{n}\frac{2\pi i}{(k-a)^2\pi^2}.$$

Let $n\to\infty$, we get

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty}\frac{1}{(z-n)^2}.$$

□

Remark 12. *Another choice is to let* $f(z)=\frac{\cot(\pi z)}{(z+a)^2}$, *see Conway [2], page 122, Exercise 6. The trick used in this problem and problem 6 (ii) will be generalized in problem 12.*

(iii) Show that if $\alpha\neq 0$ and $(\beta/\alpha)\neq\pm1,\pm2,\cdots$, then

$$\frac{\pi}{\alpha}\cot\frac{\pi\beta}{\alpha} = \sum_{n=0}^{\infty}\left\{\frac{1}{n\alpha+\beta}-\frac{1}{n\alpha+(\alpha-\beta)}\right\}.$$

From the above equation show that

$$\frac{1}{1\cdot 2}+\frac{1}{4\cdot 5}+\frac{1}{7\cdot 8}+\cdots+\frac{1}{(3n-2)(3n-1)}+\cdots=\frac{\pi}{3\sqrt{3}}.$$

Proof. By the solution of problem 6 (ii), we have

$$\coth z = \frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2+\pi^2n^2}.$$

Since $\coth z = i\cot(iz)$, we have

$$\cot z = i\coth(iz) = \frac{1}{z}-\sum_{n=1}^{\infty}\frac{2z}{\pi^2n^2-z^2} = \frac{1}{z}-\sum_{n=1}^{\infty}\left(\frac{1}{\pi n-z}-\frac{1}{\pi n+z}\right).$$

Therefore

$$
\begin{aligned}
& \frac{\pi}{\alpha}\cot\frac{\pi\beta}{\alpha} \\
= \;& \frac{1}{\beta} - \sum_{n=1}^{\infty}\left(\frac{1}{\alpha n-\beta} - \frac{1}{\alpha n+\beta}\right) = \lim_{N\to\infty}\left[\frac{1}{\beta} - \sum_{n=1}^{N}\left(\frac{1}{\alpha n-\beta} - \frac{1}{\alpha n+\beta}\right)\right] \\
= \;& \lim_{N\to\infty}\left[\frac{1}{\beta} - \left(\frac{1}{\alpha-\beta} + \frac{1}{2\alpha-\beta} + \cdots + \frac{1}{N\alpha-\beta}\right) + \left(\frac{1}{\alpha+\beta} + \frac{1}{2\alpha+\beta} + \cdots + \frac{1}{N\alpha+\beta}\right)\right] \\
= \;& \lim_{N\to\infty}\left[\left(\frac{1}{\beta} - \frac{1}{\alpha-\beta}\right) + \left(\frac{1}{\alpha+\beta} - \frac{1}{2\alpha-\beta}\right) + \cdots + \left(\frac{1}{(N-1)\alpha+\beta} - \frac{1}{N\alpha-\beta}\right) + \frac{1}{N\alpha+\beta}\right] \\
= \;& \sum_{n=0}^{\infty}\left[\frac{1}{n\alpha+\beta} - \frac{1}{n\alpha+(\alpha-\beta)}\right] = \sum_{n=0}^{\infty}\frac{\alpha-2\beta}{(n\alpha+\beta)[n\alpha+(\alpha-\beta)]}.
\end{aligned}
$$

By letting $\alpha = 3$ and $\beta = 1$, we get

$$
\frac{\pi}{3\sqrt{3}} = \frac{\pi}{3}\cot\frac{\pi}{3} = \sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n-1)}.
$$

□

Remark 13. *For a direct proof without using problem 6 (ii), we can let* $f(z) = \frac{\cot(\pi z)}{z^2-a^2}$ *and apply Residue Theorem to it. For details, see Conway [2], page 122, Exercise 8. The line of reasoning used in this approach will be generalized in problem 12 below.*

► **12.** Suppose a meromorphic function $f(z)$ only has finitely many poles $\alpha_1, \alpha_2, \cdots, \alpha_m$ and α_k is not an integer ($1 \le k \le m$). the infinity $z = \infty$ is a zero of $f(z)$ with multiplicity $p \ge 2$. Show that

(1) $\lim_{n\to\infty}\sum_{k=-n}^{n}(-1)^k f(k) = -\pi\sum_{k=1}^{m}\text{Res}(f(z)\cot\pi z, \alpha_k)$;

Proof. Since $z = \infty$ is a zero of $f(z)$ with multiplicity p, $f(z)$ can be written as $\frac{h(z)}{z^p}$ where h is holomorphic in a neighborhood of ∞. Consequently, h is bounded in a neighborhood of ∞. In the below, we shall work with a sequence of neighborhoods of ∞ that shrinks to ∞. So without loss of generality, we can assume h is bounded and denote its bounded by M.

Let γ_n be the rectangular path $[(n+\frac{1}{2})+ni, -(n+\frac{1}{2})+ni, -(n+\frac{1}{2})-ni, (n+\frac{1}{2})-ni, (n+\frac{1}{2})+ni]$. Then

$$
\int_{\gamma_n} f(z)\cot(\pi z)dz = I + II + III + IV,
$$

where

$$
I = \int_{(n+\frac{1}{2})}^{-(n+\frac{1}{2})}\frac{h(x+ni)}{(x+ni)^p}\frac{\frac{e^{i\pi(x+ni)}+e^{-i\pi(x+ni)}}{2}}{\frac{e^{i\pi(x+ni)}-e^{-i\pi(x+ni)}}{2i}}dx = \int_{(n+\frac{1}{2})}^{-(n+\frac{1}{2})}\frac{h(x+ni)}{(x+ni)^p}\frac{e^{i\pi x-n\pi}+e^{-i\pi x+n\pi}}{e^{i\pi x-n\pi}-e^{-i\pi x+n\pi}}idx,
$$

$$
\begin{aligned}
II \;=\;& \int_{n}^{-n}\frac{h(-(n+\frac{1}{2})+yi)}{[-(n+\frac{1}{2})+yi]^p}\frac{\frac{e^{i\pi[-(n+\frac{1}{2})+yi]}+e^{-i\pi[-(n+\frac{1}{2})+yi]}}{2}}{\frac{e^{i\pi[-(n+\frac{1}{2})+yi]}-e^{-i\pi[-(n+\frac{1}{2})+yi]}}{2i}}idy \\
=\;& \int_{n}^{-n}\frac{h(-(n+\frac{1}{2})+yi)}{[-(n+\frac{1}{2})+yi]^p}\frac{(-1)^n e^{-y\pi}(-i)+(-1)^n e^{y\pi}i}{(-1)^n e^{-y\pi}(-i)-(-1)^n e^{y\pi}i}(-1)dy,
\end{aligned}
$$

$$
III = \int_{-(n+\frac{1}{2})}^{(n+\frac{1}{2})}\frac{h(x-ni)}{(x-ni)^p}\frac{e^{i\pi x+n\pi}+e^{-i\pi x-n\pi}}{e^{i\pi x+n\pi}-e^{-i\pi x-n\pi}}idx,
$$

and

$$\mathit{IV} = \int_{-n}^{n} \frac{h((n+\frac{1}{2})+yi)}{[(n+\frac{1}{2})+yi]^p} \frac{(-1)^n e^{-y\pi} i + (-1)^n e^{y\pi}(-i)}{(-1)^n e^{-y\pi} i - (-1)^n e^{y\pi}(-i)} (-1) dy.$$

We note

$$|I| \le \int_{-(n+\frac{1}{2})}^{(n+\frac{1}{2})} \frac{M}{(x^2+n^2)^{\frac{p}{2}}} \frac{e^{-n\pi}+e^{n\pi}}{e^{n\pi}-e^{-n\pi}} dx = \frac{e^{2n\pi}+1}{e^{2n\pi}-1} \frac{2M}{n} \arctan \frac{n+\frac{1}{2}}{n} \to 0$$

as $n \to \infty$,

$$|\mathit{II}| \le \int_{-n}^{n} \frac{M}{\left[(n+\frac{1}{2})^2+y^2\right]^{\frac{p}{2}}} \left| \frac{e^{y\pi}-e^{-y\pi}}{e^{y\pi}+e^{-y\pi}} \right| dy \le \frac{2M}{n+\frac{1}{2}} \arctan \frac{n}{n+\frac{1}{2}} \to 0$$

as $n \to \infty$,

$$|\mathit{III}| \le \int_{-(n+\frac{1}{2})}^{(n+\frac{1}{2})} \frac{M}{(x^2+n^2)^{\frac{p}{2}}} \frac{e^{n\pi}+e^{-n\pi}}{e^{n\pi}-e^{-n\pi}} dx = \frac{e^{n\pi}+e^{-n\pi}}{e^{n\pi}-e^{-n\pi}} \frac{2M}{n} \arctan \frac{n+\frac{1}{2}}{n} \to 0$$

as $n \to \infty$, and

$$|\mathit{IV}| \le \int_{-n}^{n} \frac{M}{\left[(n+\frac{1}{2})^2+y^2\right]^{\frac{p}{2}}} \left| \frac{e^{y\pi}-e^{-y\pi}}{e^{y\pi}+e^{-y\pi}} \right| dy \le \frac{2M}{n+\frac{1}{2}} \arctan \frac{n}{n+\frac{1}{2}} \to 0$$

as $n \to \infty$. Combined, we can conclude $\lim_{n\to\infty} \int_{\gamma_n} f(z)\cot(\pi z) dz = 0$. Meanwhile, for n sufficiently large, by Residue Theorem, we have

$$\int_{\gamma_n} f(z)\cot(\pi z) dz = 2\pi i \sum_{k=1}^{m} \mathrm{Res}(f(z)\cot(\pi z), \alpha_k) + 2\pi i \sum_{k=-n}^{n} \mathrm{Res}(f(z)\cot(\pi z), k).$$

We note

$$\begin{aligned} \mathrm{Res}(f(z)\cot(\pi z), k) &= \frac{1}{2\pi i} \int_{|z-k|=\varepsilon} f(z) \frac{\cos(\pi z)}{\sin(\pi z)} dz \\ &= \frac{1}{2\pi i} \int_{|z-k|=\varepsilon} \frac{(-1)^k f(z)}{z-k} \cdot \frac{\cos(\pi z)(z-k)}{\sin[\pi(z-k)]} dz \\ &= \frac{f(k)}{\pi}. \end{aligned}$$

So

$$\int_{\gamma_n} f(z)\cot(\pi z) dz = 2\pi i \sum_{k=1}^{m} \mathrm{Res}(f(z)\cot(\pi z), \alpha_k) + 2\pi i \sum_{k=-n}^{n} \frac{f(k)}{\pi}.$$

Letting $n \to \infty$ gives us

$$\lim_{n\to\infty} \sum_{k=-n}^{n} f(k) = -\pi \sum_{k=1}^{n} \mathrm{Res}(f(z)\cot(\pi z), \alpha_k).$$

□

(2) $\lim_{n\to\infty} \sum_{k=-n}^{n} (-1)^k f(k) = -\pi \sum_{k=1}^{m} \mathrm{Res}\left(\frac{f(z)}{\sin \pi z}, \alpha_k\right)$;

Proof. As argued in the solution of (1), we can assume $f(z) = \frac{h(z)}{z^p}$ where h is holomorphic in a neighborhood of ∞ and has bound M in that neighborhood. Let γ_n denote the same rectangular path as in our solution of (1). Then

$$\int_{\gamma_n} \frac{f(z)}{\sin(\pi z)} dz = I + \mathit{II} + \mathit{III} + \mathit{IV},$$

where

$$I=\int_{n+\frac{1}{2}}^{-(n+\frac{1}{2})}\frac{h(x+ni)}{(x+ni)^p}\frac{2idx}{e^{i\pi x-n\pi}-e^{-i\pi x+n\pi}},$$

$$II=\int_{n}^{-n}\frac{h(-(n+\frac{1}{2})+yi)}{[-(n+\frac{1}{2})+yi]^p}\frac{2i}{e^{-y\pi}(-1)^n(-i)-e^{y\pi}(-1)^n i}idy,$$

$$III=\int_{-(n+\frac{1}{2})}^{n+\frac{1}{2}}\frac{h(x-ni)}{(x-ni)^p}\frac{2idx}{e^{i\pi x+n\pi}-e^{-i\pi x-n\pi}},$$

and

$$IV=\int_{-n}^{n}\frac{h((n+\frac{1}{2})+yi)}{\left[(n+\frac{1}{2})+yi\right]^p}\frac{2i}{e^{-\pi y}(-1)^n i-e^{\pi y}(-1)^n(-i)}idy.$$

We note

$$|I|\le\frac{2M}{e^{n\pi}-e^{-n\pi}}\int_{-(n+\frac{1}{2})}^{n+\frac{1}{2}}\frac{dx}{(x^2+n^2)^{\frac{p}{2}}}\le\frac{4M}{e^{n\pi}-e^{-n\pi}}\int_{0}^{n+\frac{1}{2}}\frac{dx}{x^2+n^2}=\frac{2M}{(e^{n\pi}-e^{-n\pi})n}\arctan\frac{n+\frac{1}{2}}{n}\to 0$$

as $n\to\infty$, and

$$|II|\le\int_{-n}^{n}\frac{1}{\left[(n+\frac{1}{2})^2+y^2\right]^{\frac{p}{2}}}\frac{2Mdy}{e^{y\pi}+e^{-y\pi}}\le\int_{-n}^{n}\frac{Mdy}{(n+\frac{1}{2})^2+y^2}=\frac{2M}{n+\frac{1}{2}}\arctan\frac{n}{n+\frac{1}{2}}\to 0$$

as $n\to\infty$. By similar argument, we can prove $\lim_{n\to\infty} III=\lim_{n\to\infty} IV=0$. Combined, we can conclude that $\lim_{n\to\infty}\int_{\gamma_n}\frac{f(z)}{\sin(\pi z)}dz=0$. Meanwhile, for n sufficiently large, by Residue Theorem, we have

$$\int_{\gamma_n}\frac{f(z)}{\sin(\pi z)}dz=2\pi i\sum_{k=1}^{m}\text{Res}\left(\frac{f(z)}{\sin(\pi z)},\alpha_k\right)+2\pi i\sum_{k=-n}^{n}\text{Res}\left(\frac{f(z)}{\sin(\pi z)},k\right).$$

It's easy to see

$$\text{Res}\left(\frac{f(z)}{\sin(\pi z)},k\right)=\frac{1}{2\pi i}\int_{|z-k|=\varepsilon}\frac{(-1)^k}{z-k}\cdot\frac{f(z)(z-k)}{\sin[\pi(z-k)]}dz=\frac{(-1)^k f(k)}{\pi}.$$

Therefore,

$$\int_{\gamma_n}\frac{f(z)}{\sin(\pi z)}dz=2\pi i\sum_{k=1}^{m}\text{Res}\left(\frac{f(z)}{\sin(\pi z)},\alpha_k\right)+2\pi i\sum_{k=-n}^{n}\frac{(-1)^k f(k)}{\pi}.$$

Let $n\to\infty$, we get $\lim_{n\to\infty}\sum_{k=-n}^{n}(-1)^k f(k)=-\pi\sum_{k=1}^{m}\text{Res}\left(\frac{f(z)}{\sin(\pi z)},\alpha_k\right)$. □

(3) Find the sums of the following series by using (1) and (2).
(i) $\sum_{n=-\infty}^{\infty}\frac{1}{(a+n)^2}$, where a is not an integer.

Solution. Let $f(z)=\frac{1}{(z+a)^2}$. Then by result of (1),

$$\sum_{n=-\infty}^{\infty}\frac{1}{(n+a)^2}=-\pi\text{Res}\left(\frac{\cot(\pi z)}{(z+a)^2},-a\right)=-\pi\frac{d}{dz}[\cot(\pi z)]|_{z=-a}=\frac{\pi^2}{\sin^2(\pi a)}.$$

This result is verified by problem 11 (ii). □

(ii) $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+a^2}$, where a is a non-zero real number.

Solution. Let $f(z) = \frac{1}{z^2+a^2}$. Then by result of (2),

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2+a^2} = -\pi \left[\text{Res}\left(\frac{1}{(z^2+a^2)\sin(\pi z)}, ai \right) + \text{Res}\left(\frac{1}{(z^2+a^2)\sin(\pi z)}, -ai \right) \right]$$
$$= -\frac{\pi}{ai \cdot \sin(\pi ai)} = \frac{\pi \text{csch}(\pi a)}{a}.$$

This result can be verified by Mathematica. □

► **13.** Find the residues of the following functions at their isolated singularities (including the infinity point if it is an isolated singularity).

(i) $\frac{1}{z^2-z^4}$;

Solution. $f(z) = \frac{1}{z^2-z^4}$ has poles 0, 1, and -1. ∞ is a removable singularity.

$$\text{Res}(f;0) = \frac{1}{2\pi i}\int_{|z|=\rho} \frac{dz}{z^2(1-z^2)} = \frac{d}{dz}\left(\frac{1}{1-z^2}\right)|_{z=0} = \frac{-(-2z)}{(1-z^2)^2}|_{z=0} = 0,$$

$$\text{Res}(f;1) = \frac{1}{2\pi i}\int_{|z-1|=\rho} \frac{dz}{(1-z)(1+z)z^2} = -\frac{1}{(1+z)z^2}|_{z=1} = -\frac{1}{2},$$

and

$$\text{Res}(f;-1) = \frac{1}{2\pi i}\int_{|z+1|=\rho} \frac{dz}{(1+z)(1-z)z^2} = \frac{1}{2}.$$

□

(ii) $\frac{z^2+z+2}{z(z^2+1)^2}$;

Solution. We note

$$f(z) := \frac{z^2+z+2}{z(z^2+1)^2} = \frac{2}{z} + \frac{-1+i}{4(z-i)^2} + \frac{-4-i}{4(z-i)} + \frac{-1-i}{4(z+i)^2} + \frac{-4+i}{4(z+i)}.$$

Therefore, the function $f(z)$ has poles 0, i and $-i$. ∞ is a removable singularity of $f(z)$. Furthermore, we have

$$\text{Res}(f;0) = 2,\ \text{Res}(f;i) = \frac{-4-i}{4} = -1 - \frac{i}{4},\ \text{Res}(f;-i) = \frac{-4+i}{4} = -1 + \frac{i}{4}.$$

□

(iii) $\frac{z^{n-1}}{z^n+a^n}$, where $a \neq 0$ and n is a positive integer;

Solution. Suppose $a_1, \cdots, a_n$ are the roots of the equation $z^n + a^n = 0$. Then each a_i is a pole of order 1 for the function $f(z) = \frac{z^{n-1}}{z^n+a^n}$. ∞ is a removable singularity of $f(z)$. Then

$$\text{Res}(f;a_i) = \frac{a_i^{n-1}}{\prod_{j=1,j\neq i}^n (a_i - a_j)} = \lim_{z\to a_i} \frac{a_i^{n-1}(z-a_i)}{\prod_{j=1}^n (z-a_j)} = \lim_{z\to a_i} \frac{a_i^{n-1}(z-a_i)}{z^n+a^n} = \frac{1}{n}.$$

□

(iv) $\frac{1}{\sin z}$;

Solution. $f(z) = \frac{1}{\sin z}$ has $\pi\mathbb{Z}$ as poles of order 1. ∞ is not an isolated singularity of $f(z)$. And

$$\text{Res}(f,n\pi) = \frac{1}{2\pi i}\int_{|z-n\pi|=\varepsilon} \frac{dz}{\sin z} = \frac{1}{2\pi i}\int_{|z-n\pi|=\varepsilon} \frac{1}{z-n\pi}\frac{(-1)^n(z-n\pi)}{\sin(z-n\pi)}dz = (-1)^n.$$

□

(v) $z^3 \cos \frac{1}{z-2}$;

Solution. First, we note as $\cos\left(\frac{1}{z-2}\right) = \frac{1}{2}\left(e^{\frac{i}{z-2}} + e^{-\frac{i}{z-2}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(z-2)^{2n}}$. So $f(z) := z^3 \cos\left(\frac{1}{z-2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n z^3}{(z-2)^{2n}}$. This shows $z = 2$ is an essential singularity of $f(z)$, and

$$\mathrm{Res}(f;2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2\pi i} \int_{|z-2|=\varepsilon} \frac{z^3}{(z-2)^{2n}} dz = (-1)\cdot 3z^2|_{z=2} + (-1)^2 \cdot \frac{1}{3!} \cdot 6|_{z=2} = -11.$$

Also, ∞ is an isolated singularity of $f(z)$, and we have

$$\begin{aligned} \mathrm{Res}(f;\infty) &= -\frac{1}{2\pi i}\int_{|z|=R} z^3 \cos\left(\frac{1}{z-2}\right) dz = -\frac{1}{2\pi i}\int_{|z-2|=R} z^3 \cos\left(\frac{1}{z-2}\right) dz \\ &= -\frac{1}{2\pi i}\int_{|\zeta|=\frac{1}{R}} \left(\frac{1}{\zeta}+2\right)^3 \cos\zeta \left(-\frac{d\zeta}{\zeta^2}\right) = \frac{1}{2\pi i}\int_{|\zeta=\varepsilon|} \frac{(\zeta+2)^3}{\zeta^5} \cos\zeta d\zeta \\ &= \frac{1}{4!}\frac{d^4}{d\zeta^4}\left[(\zeta+2)^3 \cos\zeta\right]|_{\zeta=0} = -\frac{8}{3}. \end{aligned}$$

□

(vi) $\frac{e^z}{z(z+1)}$.

Solution. Let $f(z) = \frac{e^z}{z(z+1)} = e^z\left(\frac{1}{z} - \frac{1}{z+1}\right)$. Then $\mathrm{Res}(f;0) = 1$, $\mathrm{Res}(f;-1) = -e^{-1}$, and $\mathrm{Res}(f;\infty) = 1 - e^{-1}$. □

► **14.** Suppose that $f(z)$ and $g(z)$ are holomorphic at $z = a$, $f(a) \neq 0$ and $z = a$ is a zero of $g(z)$ with multiplicity 2. Find $\mathrm{Res}(f(z)/g(z), a)$.

Solution. We can write $g(z)$ as $(z-a)^2 h(z)$ where h is holomorphic and $h(a) \neq 0$. Then $h(a) = \frac{g(a)}{(z-a)^2}$ and

$$h'(a) = \frac{g'(a)(z-a) - 2g(a)}{(z-a)^3}.$$

Therefore

$$\begin{aligned} \mathrm{Res}\left(\frac{f(z)}{g(z)}, a\right) &= \mathrm{Res}\left(\frac{1}{(z-a)^2}\cdot\frac{f(z)}{h(z)}, a\right) = \frac{d}{dz}\left[\frac{f(z)}{h(z)}\right]\Bigg|_{z=a} = \frac{f'(a)h(a) - f(a)h'(a)}{h^2(a)} \\ &= \frac{f'(a)\frac{g(a)}{(z-a)^2} - f(a)\frac{g'(a)(z-a)-2g(a)}{(z-a)^3}}{\frac{g^2(a)}{(z-a)^4}} \\ &= \frac{g(a)(z-a)[f'(a)(z-a) + 2f(a)] - f(a)g'(a)(z-a)^2}{g^2(a)}. \end{aligned}$$

□

► **15.** Evaluate the following integrals:

Remark 14. *We make an observation of some simple rules that facilitate the evaluation of integrals via Residue Theorem.*

Rule 1 (rule for integrand function). The poles of the integrand function should be easy to find, such that the integrand function can be easily represented as $\frac{f(z)}{(z-a)^n}$*, where* f *is holomorphic.*

The holomorphic function $f(z)$ *can be multi-valued function like logarithm function and power function. When it's difficult to make* f *holomorphic in the desired region, check if it is the real or imaginary part of a holomorphic function.*

Rule 2 (rule for integration path). The choice of integration path is highly dependent on the properties of the integrand function, where symmetry and multi-valued functions (log and power functions) are often helpful.

Oftentimes, the upper and lower limits of integration either form a full circle (i.e. $[0, 2\pi]$) so that substitution for trigonometric functions can be easily done or have ∞ as one of the end points.

(i) $\int_0^{\infty} \frac{x^2 dx}{(x^2+1)^2}$;

Solution. Let $\gamma_1 = \{z : -R \le \text{Re}z \le R, \text{Im}z = 0\}$, $\gamma_2 = \{z : |z| = R, 0 \le \arg z \le \pi\}$, and $f(z) = \frac{z^2}{(z^2+1)^2}$. Then

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = 2\pi i \cdot \text{Res}(f, i) = 2\pi i \cdot \lim_{z \to i} \frac{d}{dz}\left[\frac{z^2}{(z+i)^2}\right] = \frac{\pi}{2},$$

and

$$\left|\int_{\gamma_2} f(z)dz\right| = \left|\int_0^{\pi} \frac{R^2 e^{2\theta i}}{(R^2 e^{2\theta i}+1)^2} Re^{i\theta} \cdot id\theta\right| \le \int_0^{\pi} \frac{R^3}{(R^2-1)^2} d\theta \to 0$$

as $R \to \infty$. So by letting $R \to \infty$, we have $\int_0^{\infty} \frac{x^2 dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2} = \frac{\pi}{4}$. □

(ii) $\int_0^{\frac{\pi}{2}} \frac{dx}{a+\sin^2 x}$, $a > 0$;

Solution. We note $\sin^2 x = \frac{1-\cos 2x}{2}$, so

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a+\sin^2 x} = \frac{1}{2}\int_0^{\pi} \frac{dx}{a+\sin^2 x} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(2a+1)-\cos\theta}.$$

Let $b = 2a+1$, then

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a+\sin^2 x} = \frac{1}{2}\int_{|z|=1} \frac{\frac{dz}{iz}}{b-\frac{1}{2}(z+z^{-1})} = \frac{1}{2i}\int_{|z|=1} \frac{dz}{-\frac{1}{2}z^2+bz-\frac{1}{2}}.$$

The equation $-\frac{1}{2}z^2+bz-\frac{1}{2} = 0$ has two roots: $z_1 = b-\sqrt{b^2-1}$ and $z_2 = b+\sqrt{b^2-1}$. Since $b > 1$, we have $z_1 \in D(0,1)$ and $z_2 \notin D(0,1)$. Therefore,

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a+\sin^2 x} = i\int_{|z|=1} \frac{dz}{(z-z_1)(z-z_2)} = i \cdot 2\pi i \cdot \frac{1}{z_1-z_2} = \frac{\pi}{\sqrt{b^2-1}} = \frac{\pi}{2\sqrt{a(a+1)}}.$$

□

Remark 15. *If $R(x,y)$ is a rational function of two variables x and y, for $z = e^{i\theta}$, we have*

$$R(\sin\theta, \cos\theta) = R\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right), \ d\theta = \frac{dz}{iz}.$$

Therefore

$$\int_0^{2\pi} R(\sin\theta, \cos\theta)d\theta = \int_{|z|=1} R\left(\frac{1}{2}(z+\frac{1}{z}), \frac{1}{2i}(z-\frac{1}{z})\right)\frac{dz}{iz}.$$

Remark 16. *When the integrand function is a rational function of trigonometric functions, in view of the previous remark, it is desirable to have $[0, 2\pi]$ as the integration interval. For this reason, we first use symmetry to expand the integration interval from $[0, \frac{\pi}{2}]$ to $[0, \pi]$, and then use double angle formula to expand $[0, \pi]$ to $[0, 2\pi]$. This solution is motivated by Rule 2.*

(iii) $\int_0^{\infty} \frac{x \sin x}{x^2+1} dx$;

Solution. Let $\gamma_1 = \{z : -R \leq \text{Re}z \leq R, \text{Im}z = 0\}$, $\gamma_2 = \{z : |z| = R, 0 \leq \arg z \leq \pi\}$, and $f(z) = \frac{ze^{iz}}{z^2+1}$. Then

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = 2\pi i \text{Res}(f, i) = \int_{|z-i|=\varepsilon} \frac{ze^{iz}dz}{(z-i)(z+i)} = 2\pi i \cdot \frac{ie^{-1}}{2i} = \frac{\pi}{e}i,$$

and

$$\left|\int_{\gamma_2} f(z)dz\right| = \left|\int_0^\pi \frac{Re^{iRe^{i\theta}}}{R^2e^{2\theta i}+1} Re^{i\theta} \cdot id\theta\right| \leq \int_0^\pi \frac{R^2e^{-R\sin\theta}}{R^2-1} d\theta \to 0$$

as $R \to \infty$ by Lebesgue's Dominated Convergence Theorem. Therefore by letting $R \to \infty$, we have

$$\int_{-\infty}^\infty \frac{xe^{ix}}{x^2+1} dx = \int_{-\infty}^\infty \frac{x\cos x}{x^2+1} dx + i\int_{-\infty}^\infty \frac{x\sin x}{x^2+1} dx = \frac{\pi}{e}i,$$

which implies $\int_0^\infty \frac{x\sin x}{x^2+1} dx = \frac{\pi}{2e}$. □

(iv) $\int_0^\infty \frac{\log x}{(x^2+1)^2} dx$;

Solution. Let r, R be two positive numbers such that $r < 1 < R$. Let $\gamma_1 = \{z : r \leq |z| \leq R, \arg z = 0\}$, $\gamma_2 = \{z : r \leq |z| \leq R, \arg z = \pi\}$, $\gamma_R = \{z : |z| = R, 0 \leq \arg z \leq \pi\}$, and $\gamma_r = \{z : |z| = r, 0 \leq \arg z \leq \pi\}$. Define $f(z) = \frac{\log z}{(1+z^2)^2}$ where $\log z$ is defined on $\mathbb{C} \setminus [0, \infty)$ and takes the principle branch of Log$z$ with $\arg z \in (0, 2\pi)$ (see page 23).

By Residue Theorem,

$$\int_{\gamma_1+\gamma_R+\gamma_2-\gamma_r} f(z)dz = 2\pi i \text{Res}(f, i) = 2\pi i \lim_{z\to i} \frac{d}{dz}\left[\frac{\log z}{(z+i)^2}\right] = -\frac{\pi}{2} + \frac{\pi^2}{4}i.$$

We note

$$\left|\int_{\gamma_R} f(z)dz\right| = \left|\int_0^\pi \frac{\log(Re^{i\theta})}{(R^2e^{2i\theta}+1)^2} Re^{i\theta} \cdot id\theta\right| \leq \int_0^\pi \frac{R\sqrt{(\log R)^2+\theta^2}}{(R^2-1)^2} d\theta \leq \frac{\pi R(\log R + \pi)}{(R^2-1)^2} \to 0$$

as $R \to \infty$,

$$\left|\int_{\gamma_r} f(z)dz\right| = \left|\int_0^\pi \frac{\log(re^{i\theta})}{(r^2e^{2i\theta}+1)^2} re^{i\theta} \cdot id\theta\right| \leq \int_0^\pi \frac{r\sqrt{(\log r)^2+\theta^2}}{(1-r^2)^2} d\theta \leq \frac{\pi r(-\log r + \pi)}{(1-r^2)^2} \to 0$$

as $r \to 0$, and

$$\int_{\gamma_2} f(z)dz = \int_{-R}^{-r} \frac{\log x}{(1+x^2)^2} dx = \int_r^R \frac{\log(-x)}{(1+x^2)^2} dx = \int_r^R \frac{\log x + \pi i}{(1+x^2)^2} dx.$$

Therefore by letting $r \to 0$ and $R \to \infty$, we have

$$\int_0^\infty \frac{2\log x + \pi i}{(1+x^2)^2} dx = -\frac{\pi}{2} + \frac{\pi^2}{4}i,$$

i.e. $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx = -\frac{\pi}{4}$. □

(v) $\int_0^\infty \frac{x^{1-\alpha}}{1+x^2} dx$, $0 < \alpha < 2$;

Solution. Let γ_1, γ_2, γ_r and γ_R be defined as in (iv). Define $f(z) = \frac{z^{1-\alpha}}{1+z^2}$, where $z^{1-\alpha} = e^{(1-\alpha)\log z}$ is defined on $\mathbb{C} \setminus [0, \infty)$ with $\log z$ taking the principle branch of Log$z$. By Residue Theorem,

$$\int_{\gamma_1+\gamma_R+\gamma_2-\gamma_r} f(z)dz = 2\pi i \text{Res}(f, i) = 2\pi i \cdot \frac{e^{(1-\alpha)\log z}}{z+i}|_{z=i} = \pi e^{(1-\alpha)\frac{\pi}{2}i}.$$

We note

$$\left|\int_{\gamma_R} f(z)dz\right| = \left|\int_0^\pi \frac{R^{1-\alpha}e^{i\theta(1-\alpha)}}{1+R^2e^{2i\theta}} \cdot Re^{i\theta} \cdot id\theta\right| \le \frac{R^{2-\alpha}}{R^2-1}\pi \to 0$$

as $R \to \infty$,

$$\left|\int_{\gamma_r} f(z)dz\right| = \left|\int_0^\pi \frac{r^{1-\alpha}e^{i\theta(1-\alpha)}}{1+r^2e^{2i\theta}} \cdot re^{i\theta} \cdot id\theta\right| \le \frac{r^{2-\alpha}}{1-r^2}\pi \to 0$$

as $r \to 0$, and

$$\int_{\gamma_2} f(z)dz = \int_{-R}^{-r} \frac{x^{1-\alpha}}{1+x^2}dx = \int_r^R \frac{e^{(1-\alpha)\log(-x)}}{1+x^2}dx = \int_r^R \frac{x^{1-\alpha}e^{i(1-\alpha)\pi}}{1+x^2}dx.$$

Therefore by letting $r \to 0$ and $R \to \infty$, we have

$$\int_0^\infty \frac{x^{1-\alpha}(1+e^{i(1-\alpha)\pi})}{1+x^2}dx = \int_0^\infty \frac{x^{1-\alpha}}{1+x^2}dx \cdot 2\cos\frac{(1-\alpha)\pi}{2}e^{i(1-\alpha)\frac{\pi}{2}} = \pi e^{i(1-\alpha)\frac{\pi}{2}},$$

i.e. $\int_0^\infty \frac{x^{1-\alpha}}{1+x^2}dx = \frac{\pi}{2}\csc\left(\frac{\alpha\pi}{2}\right)$. □

(vi) $\int_0^\infty \frac{dx}{1+x^n}$, where n is an integer and $n \ge 2$;

Solution. This problem is a special of problem (x). See the solution there. □

(vii) $\int_0^\pi \frac{d\theta}{a+\cos\theta}$, where a is a constant and $a > 1$;

Solution. We use the method outlined in the remark of Problem (ii).

$$\int_0^\pi \frac{d\theta}{a+\cos\theta} = \frac{1}{2}\int_{-\pi}^\pi \frac{d\theta}{a+\cos\theta} = \frac{1}{2}\int_{|z|=1} \frac{\frac{dz}{iz}}{a+\frac{z+z^{-1}}{2}} = \frac{1}{2i}\int_{|z|=1} \frac{dz}{\frac{1}{2}z^2+az+\frac{1}{2}}.$$

The equation $\frac{1}{2}z^2 + az + \frac{1}{2} = 0$ has two roots: $z_1 = -a + \sqrt{a^2-1}$ and $z_2 = -a - \sqrt{a^2-1}$. Clearly, $|z_1| = \frac{1}{a+\sqrt{a^2-1}} < 1$ and $|z_2| > 1$. So

$$\int_0^\pi \frac{d\theta}{a+\cos\theta} = \frac{1}{i}\int_{|z|=1} \frac{dz}{(z-z_1)(z-z_2)} = 2\pi \cdot \frac{1}{z_1-z_2} = \frac{\pi}{\sqrt{a^2-1}}.$$

□

Remark 17. *The expansion of integration interval from* $[0,\pi]$ *to* $[-\pi,\pi]$ *is motivated by Rule 2.*

(viii) $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$;

Solution. Using the result on Dirichlet integral: $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$, we have

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \int_0^\infty \sin^2 x d\left(-\frac{1}{x}\right) = \int_0^\infty \frac{2\sin x\cos x dx}{x} = \int_0^\infty \frac{\sin(2x)}{2x}d(2x) = \frac{\pi}{2}.$$

□

(ix) $\int_0^\infty \frac{x^p}{x^2+2x\cos\lambda+1}dx$, where $-1 < p < 1$ and $-\pi < \lambda < \pi$;

Solution. It is easy to see that when $\lambda = p = 0$, the integral is equal to 1. So without loss of generality, we only consider the cases where λ and p are not simultaneously equal to 0.

Choose r, R so that $0 < r < R$, and r is sufficiently small and R is sufficiently large. Define $\gamma_1 = \{z : r \le |z| \le R, \arg z = 0\}$, $\gamma_2 = \{z : r \le |z| \le R, \arg z = 2\pi\}$, $\gamma_r = \{z : |z| = r, 0 < \arg z < 2\pi\}$, and $\gamma_R = \{z : |z| = R, 0 < \arg z < 2\pi\}$. Let $\rho = \cos\lambda$, then $\rho \in (-1, 1]$. Finally, define $f(z) = \frac{z^p}{z^2+2z\cos\lambda+1} = \frac{e^{p\log z}}{z^2+2\rho z+1}$.

For R sufficiently large and r sufficiently small, the two roots of $z^2+2rz+1=0$ are contained in $\{z : r < |z| < R\}$. These two roots are, respectively, $z_1 = -\rho + i\sqrt{1-r^2} = -\cos\lambda + i|\sin\lambda|$ and $z_2 = -\rho - i\sqrt{1-r^2} = -\cos\lambda - i|\sin\lambda|$. So $|z_1| = |z_2| = 1$, and $\arg z_1 + \arg z_2 = 2\pi$. Denote $\arg z_1$ by θ. If $\lambda \neq 0$, we have by Residue Theorem

$$\begin{aligned}\int_{\gamma_1+\gamma_R-\gamma_2-\gamma_r} f(z)dz &= 2\pi i[\mathrm{Res}(f,z_1)+\mathrm{Res}(f,z_2)] = 2\pi i\left[\frac{e^{p\log z_1}}{z_1-z_2}+\frac{e^{p\log z_2}}{z_2-z_1}\right] \\ &= \frac{\pi}{|\sin\lambda|}[e^{p\theta i}-e^{p(2\pi-\theta)i}] = -\frac{2\pi i}{\sin\lambda}\sin(\lambda p)e^{p\pi i}.\end{aligned}$$

If $\lambda = 0$, we have by Residue Theorem

$$\int_{\gamma_1+\gamma_R-\gamma_2-\gamma_r} f(z)dz = 2\pi i\mathrm{Res}(f,-1) = -2p\pi i \cdot e^{p\pi i}.$$

Meanwhile, we have the estimates

$$\left|\int_{\gamma_R} f(z)dz\right| \leq \frac{R^p}{R^2-2R-1}\cdot R\cdot 2\pi \to 0$$

as $R\to\infty$,

$$\left|\int_{\gamma_r} f(z)dz\right| \leq \frac{r^p}{1-2\rho r-r^2}\cdot r\cdot 2\pi \to 0$$

as $r \to 0$, and

$$\int_{\gamma_2} f(z)dz = -\int_0^R \frac{(xe^{2\pi i})^p dx}{x^2+2\rho x+1} = -\int_0^R \frac{x^p dx}{x^2+2\rho x+1}e^{2p\pi i}.$$

Since $1-e^{2p\pi i} = 2\sin^2(p\pi) - 2\sin(p\pi)\cos(p\pi)i = -2i\sin(p\pi)e^{p\pi i}$, by letting $R\to\infty$ and $r\to 0$, we have

$$\int_0^\infty \frac{x^p dx}{x^2+2x\cos\lambda+1} = \begin{cases}\frac{\pi\sin(\lambda p)}{\sin\lambda\sin(p\pi)} & \text{if } \lambda\neq 0\\ p\pi\csc(p\pi) & \text{if } \lambda = 0 \text{ and } p\neq 0.\end{cases}$$

If we take the convention that $\frac{\alpha}{\sin\alpha} = 1$ for $\alpha = 0$, then the above three formulas can be unified into a single one: $\frac{\pi\sin(\lambda p)}{\sin\lambda\sin(p\pi)}$. □

Remark 18. *We choose the above integration path in order to take advantage of the multi-valued function x^p (Rule 1). The no symmetry in $x^2+2x\cos\lambda+1$ is handled by using the full circle instead of half circle.*

(x) $\int_0^\infty \frac{1}{1+x^p}dx$, where $p > 1$;

Solution. Choose two positive numbers r and R such that $0<r<1<R$. Let $\gamma_1 = \{z : r\leq |z|\leq R, \arg z = 0\}$, $\gamma_2 = \{z : r\leq|z|\leq R, \arg z = 2\pi\}$, $\gamma_R = \{z : |z| = R, 0 < \arg z < 2\pi\}$ and $\gamma_r = \{z : |z| = r, 0<\arg z < 2\pi\}$. Define $f(z) = \frac{z^{1/p}}{p(z+1)z}$ where $z^{1/p} = e^{\frac{\log z}{p}}$ is defined on $\mathbb{C}\setminus[0,\infty)$. Note by substituting $y^{\frac{1}{p}}$ for x, we get

$$\int_0^\infty \frac{dx}{1+x^p} = \int_0^\infty \frac{y^{\frac{1}{p}}dy}{p(y+1)y}.$$

By Residue Theorem,

$$\int_{\gamma_1+\gamma_R-\gamma_2-\gamma_r} f(z)dz = 2\pi\mathrm{Res}(f,-1) = 2\pi i\cdot\frac{(-1)^{\frac{1}{p}}}{-p} = -\frac{2\pi i}{p}e^{\frac{\log e^{\pi i}}{p}} = -2i\alpha e^{\alpha i},$$

where $\alpha = \frac{\pi}{p}$. We have the estimates

$$\left|\int_{\gamma_R} f(z)dz\right| = \left|\int_0^{2\pi}\frac{e^{\frac{1}{p}\log(Re^{i\theta})}}{p(Re^{i\theta}+1)Re^{i\theta}}Re^{i\theta}\cdot id\theta\right| \leq \frac{2\pi R^{\frac{1}{p}}}{p(R-1)}\to 0$$

as $R \to \infty$,

$$\left|\int_{\gamma_r} f(z)dz\right| = \left|\int_0^{2\pi} \frac{e^{\frac{1}{p}\log(re^{i\theta})}}{p(re^{i\theta}+1)re^{i\theta}} re^{i\theta} \cdot id\theta\right| \le \frac{2\pi r^{\frac{1}{p}}}{p(1-r)} \to 0$$

as $r \to 0$, and

$$\int_{\gamma_2} f(z)dz = -\int_r^R \frac{(xe^{2\pi i})^{\frac{1}{p}} dx}{p(x+1)x} = -\int_r^R \frac{x^{\frac{1}{p}} dx}{p(x+1)x} \cdot e^{2\alpha i}.$$

Therefore by letting $r \to 0$ and $R \to \infty$, we have

$$\int_0^\infty \frac{dx}{1+x^p} = \int_0^\infty \frac{x^{\frac{1}{p}} dx}{p(x+1)x} = \frac{-2i\alpha e^{\alpha i}}{1-e^{2\alpha i}} = \frac{-2i\alpha e^{\alpha i}}{-2\sin\alpha e^{(\frac{\pi}{2}+\alpha)i}} = \frac{\alpha}{\sin\alpha} = \frac{\pi}{p}\csc\left(\frac{\pi}{p}\right).$$

□

Remark 19. *The roots of $1+x^p=0$ are not very expressible. So following Rule 1, we make the change of variable $x^p=y$. Then we choose the above integration path to take advantage of the multi-valued function $y^{\frac{1}{p}}$. Since there is no symmetry in the denominator $(y+1)y$, we used a full circle instead of a half circle. The change of variable $x^p=y$ is equivalent to integration on an arc of angle $2\pi/p$ instead of 2π.*

(xi) $\int_0^1 \frac{x^{1-p}(1-x)^p}{1+x^2}dx$, where $-1<p<2$;

Solution. (Oops! looks like my solution has a bug. Catch it if you can.) By the change of variable $x=\frac{1}{y+1}$, we have

$$\int_0^1 \frac{x^{1-p}(1-x)^p}{1+x^2}dx = \int_0^\infty \frac{y^p dy}{(y+1)^3+(y+1)}.$$

The equation $(y+1)^3+(y+1)=0$ has three roots: $z_0=-1$, $z_1=-1+i$, and $z_2=-1-i$. Define $f(z)=\frac{z^p}{(z+1)^3+(z+1)}$. Then

$$\begin{aligned}\sum_{i=0}^2 Res(f,z_i) &= \left.\frac{1}{(z+1)(z-z_1)}\right|_{z=z_2} + \left.\frac{1}{(z+1)(z-z_2)}\right|_{z=z_1} + \left.\frac{1}{(z-z_1)(z-z_2)}\right|_{z=z_0} \\ &= \frac{1}{-i\cdot(-2i)} + \frac{1}{i\cdot 2i} + \frac{1}{-i\cdot i} \\ &= 0.\end{aligned}$$

Let R and r be two positive numbers with $R>r$. Define $\gamma_1=\{z : r<|z|\le R, \arg z=0\}$, $\gamma_2=\{z : r<|z|\le R, \arg z=2\pi\}$, $\gamma_R=\{z : |z|=R, 0<\arg z<2\pi\}$, and $\gamma_r=\{z : |z|=r, 0<\arg z<2\pi\}$. Then by Residue Theorem

$$\int_{\gamma_1+\gamma_R-\gamma_2-\gamma_r} f(z)dz = 2\pi i\sum_{i=0}^2 Res(f,z_i)=0.$$

We note

$$\left|\int_{\Gamma_R} f(z)dz\right| = \left|\int_0^{2\pi} \frac{(Re^{i\theta})^p}{(Re^{i\theta}+1)^3+(Re^{i\theta}+1)} Re^{i\theta}\cdot id\theta\right| \le \frac{2\pi R^{p+1}}{R^3-3R^2-4R-2} \to 0$$

as $R\to\infty$, and

$$\left|\int_0^{2\pi} \frac{(re^{i\theta})^p}{(re^{i\theta}+1)^3+(re^{i\theta}+1)} re^{i\theta}\cdot id\theta\right| \le \frac{2\pi r^{p+1}}{2-4r-3r^2-r^3} \to 0$$

as $r\to 0$. So by letting $r\to 0$ and $R\to\infty$, we have

$$\int_0^\infty \frac{y^p}{(y+1)^3+(y+1)}dy + 0 - \int_0^\infty \frac{y^p}{(y+1)^3+(y+1)}dy\cdot e^{2p\pi i} - 0 = 0.$$

Therefore..., "Houston, we got a problem here." □

(xii) $\int_0^\infty \frac{\log x}{x^2+2x+2}dx$;

Solution. Choose r, R such that $0 < r < R$ and r is sufficiently small and R is sufficiently large. Let $\gamma_1 = \{z : r \le |z| \le R, \arg z = 0\}$, $\gamma_2 = \{z : r \le |z| \le R, \arg z = \pi\}$, $\gamma_R = \{z : |z| = R, 0 < \arg z < \pi\}$ and $\gamma_r = \{z : |z| = r, 0 < \arg z < \pi\}$. Define $f(z) = \frac{\log z}{z^2+2z+2}$. By Residue Theorem,

$$\int_{\gamma_1+\gamma_R+\gamma_2-\gamma_r} f(z)dz = 2\pi i \cdot \mathrm{Res}(f, -1+i) = \pi\left(\frac{\log 2}{2} + \frac{3}{4}\pi i\right).$$

We note

$$\left|\int_{\gamma_R} f(z)dz\right| = \left|\int_0^{2\pi} \frac{\log(Re^{i\theta})}{R^2e^{2i\theta}+2Re^{i\theta}+2} Re^{i\theta} \cdot id\theta\right| \le \frac{\log R + 2\pi}{R^2-2R-2}2\pi R \to 0$$

as $R \to \infty$,

$$\left|\int_{\gamma_r} f(z)dz\right| = \left|\int_0^{2\pi} \frac{\log(re^{i\theta})}{r^2e^{2i\theta}+2re^{i\theta}+2} re^{i\theta} \cdot id\theta\right| \le \frac{\log r + 2\pi}{2-2r-r^2}2\pi r \to 0$$

as $r \to 0$, and

$$\int_{\gamma_2} f(z)dz = \int_r^R \frac{\log(-x)}{x^2-2x+2}dx = \int_r^R \frac{\log x + \pi i}{x^2-2x+2}dx.$$

Therefore by letting $r \to 0$ and $R \to \infty$, we have

$$\int_0^\infty \left[\frac{\log x}{x^2-2x+2} + \frac{\log x}{x^2+2x+2}\right] dx + i\int_0^\infty \frac{\pi dx}{x^2-2x+2} = \frac{\pi}{2}\log 2 + \frac{3}{4}\pi^2 i,$$

i.e. $\int_0^\infty \left[\frac{\log x}{x^2-2x+2} + \frac{\log x}{x^2+2x+2}\right] dx = \frac{\pi}{2}\log 2$.

Meanwhile, we note $\int_0^\infty \left[\frac{\log x}{x^2-2x+2} - \frac{\log x}{x^2+2x+2}\right] dx = \int_0^\infty \log x \frac{4x}{(x^2+2)^2-4x^2}dx = \int_0^\infty \frac{\log y}{y^2+4}dy$. To calculate the value of $\int_0^\infty \frac{\log y}{y^2+4}dy$, we apply again Residue Theorem. Let's define $g(z) = \frac{\log z}{z^2+4}$ and then we have

$$\int_{\gamma_1+\gamma_R+\gamma_2-\gamma_r} g(z)dz = 2\pi i \cdot \mathrm{Res}(g, 2i) = \frac{\pi}{2}\left[\log 2 + \frac{\pi}{2}i\right].$$

We note

$$\left|\int_{\gamma_R} g(z)dz\right| = \left|\int_0^{2\pi} \frac{\log(Re^{i\theta})}{R^2e^{2i\theta}+4} Re^{i\theta} \cdot id\theta\right| \le \frac{\log R + 2\pi}{R^2-4}2\pi R \to 0$$

as $R \to \infty$,

$$\left|\int_{\gamma_r} g(z)dz\right| = \left|\int_0^{2\pi} \frac{\log(re^{i\theta})}{r^2e^{2i\theta}+4} re^{i\theta} \cdot id\theta\right| \le \frac{\log r + 2\pi}{4-r^2}2\pi r \to 0$$

as $r \to 0$, and

$$\int_{\gamma_2} g(z)dz = \int_r^R \frac{\log(-x)}{x^2+4}dx = \int_r^R \frac{\log x + \pi i}{x^2+4}dx.$$

So by letting $r \to 0$ and $R \to \infty$, we have

$$2\int_0^\infty \frac{\log x}{x^2+4}dx + \pi i\int_0^\infty \frac{dx}{x^2+4} = \frac{\pi}{2}\log 2 + \frac{\pi^2}{4}i.$$

Hence $\int_0^\infty \frac{\log x}{x^2+4}dx = \frac{\pi}{4}\log 2$. Solving the equation

$$\begin{cases} \int_0^\infty \frac{\log x}{x^2-2x+2}dx + \int_0^\infty \frac{\log x}{x^2+2x+2}dx = \frac{\pi}{2}\log 2 \\ \int_0^\infty \frac{\log x}{x^2-2x+2}dx - \int_0^\infty \frac{\log x}{x^2+2x+2}dx = \frac{\pi}{4}\log 2, \end{cases}$$

we get

$$\int_0^\infty \frac{\log x}{x^2+2x+2}dx = \frac{\pi}{8}\log 2.$$

□

Remark 20. *Note how we handled the combined difficulty caused by no symmetry in the denominator of the integrand function and log function: because x^2+2x+2 is not symmetric, we want to use full circle; but this will cause the integrals produced by different integration paths to cancel with each other. Therefore we are forced to choose half circle and use two equations to solve for the desired integral.*

(xiii) $\int_0^\infty \frac{\sqrt{x}\log x}{x^2+1}dx$

Solution. We choose r and R such that $0<r<R$, and r is sufficiently small and R is sufficiently large. Let $\gamma_1=\{z: r\le |z|\le R, \arg z=0\}$, $\gamma_2=\{z: r\le |z|\le R, \arg z=\pi\}$, $\gamma_R=\{z:|z|=R, 0<\arg z<\pi\}$, and $\gamma_r=\{z:|z|=r, 0<\arg z<\pi\}$. Define $f(z)=\frac{\sqrt{z}\log z}{z^2+1}$. Then

$$\int_{\gamma_1+\gamma_R+\gamma_2-\gamma_r} f(z)dz = 2\pi i \mathrm{Res}(f,i) = \frac{\pi^2}{2} i e^{\frac{\pi}{4}i} = \frac{\pi^2}{2\sqrt{2}}(-1+i).$$

We note

$$\left|\int_{\gamma_R} f(z)dz\right| = \left|\int_0^{2\pi} \frac{(Re^{i\theta})^{\frac{1}{2}}\log(Re^{i\theta})}{R^2e^{2i\theta}+1} Re^{i\theta}\cdot id\theta\right| \le \frac{2\pi R\sqrt{R}(\log R+2\pi)}{R^2-1}\to 0$$

as $R\to\infty$,

$$\left|\int_{\gamma_r} f(z)dz\right| = \left|\int_0^{2\pi} \frac{(re^{i\theta})^{\frac{1}{2}}\log(re^{i\theta})}{r^2e^{2i\theta}+1} re^{i\theta}\cdot id\theta\right| \le \frac{2\pi r\sqrt{r}(\log r+2\pi)}{1-r^2}\to 0$$

as $r\to 0$, and

$$\int_{\gamma_2} f(z)dz = \int_r^R \frac{\sqrt{-x}\log(-x)}{x^2+1}dx = \int_r^R \frac{\sqrt{x}e^{\frac{\pi}{2}i}(\log x+\pi i)}{x^2+1}dx = e^{\frac{\pi}{2}i}\int_r^R \frac{\sqrt{x}\log x}{x^2+1}dx + e^{\frac{\pi}{2}i}\pi i\int_r^R \frac{\sqrt{x}}{x^2+1}dx.$$

By letting $r\to 0$ and $R\to\infty$, we have

$$(1+i)\int_0^\infty \frac{\sqrt{x}\log x}{x^2+1}dx - \pi\int_0^\infty \frac{\sqrt{x}}{x^2+1}dx = \frac{\pi^2}{2\sqrt{2}}(-1+i).$$

Therefore, $\int_0^\infty \frac{\sqrt{x}\log x}{x^2+1}dx = \frac{\pi^2}{2\sqrt{2}}$. □

(xiv) $\int_0^\infty \log\left(\frac{e^x+1}{e^x-1}\right)dx$;

Solution. We note the power series expansion $\ln(1-z) = -\sum_{n=1}^\infty \frac{z^n}{n}$ and $\ln(1+z)=\sum_{n=1}^\infty(-1)^{n+1}\frac{z^n}{n}$ hold in $|z|<1$ and the convergence is uniform on $|z|\le 1-\varepsilon$, for any $\varepsilon\in(0,1)$. Therefore, for any $\delta>0$, we have

$$\begin{aligned}\int_\delta^\infty \log\left(\frac{e^x+1}{e^x-1}\right)dx &= \int_\delta^\infty \left[\sum_{n=1}^\infty (-1)^{n+1}\frac{e^{-nx}}{n} + \sum_{n=1}^\infty \frac{e^{-nx}}{n}\right]dx \\ &= 2\int_\delta^\infty \left[\sum_{k=1}^\infty \frac{e^{-(2k-1)x}}{2k-1}\right]dx = 2\sum_{k=1}^\infty \frac{e^{-(2k-1)\delta}}{(2k-1)^2}.\end{aligned}$$

Therefore

$$\int_0^\infty \log\left(\frac{e^x+1}{e^x-1}\right)dx = \lim_{\delta\to 0} 2\sum_{k=1}^\infty \frac{e^{-(2k-1)\delta}}{(2k-1)^2} = 2\sum_{k=1}^\infty \frac{1}{(2k-1)^2},$$

where the last equality can be seen as an example of Lebesgue's Dominated Convergence Theorem applied to counting measure. By Problem 11 (ii) with $z=\frac{1}{2}$, we know $\sum_{k=1}^\infty \frac{1}{(2k-1)^2}=\frac{\pi^2}{8}$. So the integral is equal to $\frac{\pi^2}{4}$. □

(xv) $\int_0^\infty \frac{x}{e^x+1}dx$;

Solution. Since we have argued rather rigorously in (xiv), we will argue non-rigorously in this problem.

$$\begin{aligned}\int_0^\infty \frac{x}{e^x+1}dx &= \int_0^\infty \frac{e^{-x}x}{1+e^{-x}}dx = \int_0^\infty e^{-x}x\sum_{n=0}^\infty(-1)^n e^{-nx}dx \\ &= \sum_{n=0}^\infty (-1)^n\int_0^\infty e^{-(n+1)x}xdx = \sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^2}.\end{aligned}$$

Using the function $\frac{\pi}{\pi^2\sin\pi z}$ and imitating what we did in Problem 12, we can easily get $\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$. Therefore $\int_0^\infty \frac{x}{e^x+1}dx = \frac{\pi^2}{12}$. □

(xvi) $\int_0^{\frac{\pi}{2}} \log(\sin\theta)d\theta$;

Solution. We provide two solutions, which are essentially the same one.

Solution 1. In problem (xvii), we shall prove (let $a=1$)

$$\int_0^\pi \frac{x\sin x}{2-2\cos x}dx = \pi\log 2.$$

Note

$$\int_0^\pi \frac{x\sin x}{2-2\cos x}dx = \int_0^\pi \frac{x\cdot 2\sin\frac{x}{2}\cos\frac{x}{2}dx}{2\cdot 2\sin^2\frac{x}{2}} = 2\int_0^{\frac{\pi}{2}}\frac{\theta\cos\theta d\theta}{\sin\theta} = -2\int_0^{\frac{\pi}{2}}\log(\sin\theta)d\theta.$$

So $\int_0^{\frac{\pi}{2}}\log(\sin\theta)d\theta = -\frac{\pi}{2}\log$.

Solution 2. We provide a heuristic proof which discloses the essence of the calculation. Note how Rule 1-3 lead us to this solution.

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\log(\sin\theta)d\theta &= \int_0^{\frac{\pi}{2}}\log(\cos\alpha)d\alpha = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\log(\cos\alpha)d\alpha \\ &= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\log\left(\frac{1+\cos 2\alpha}{2}\right)d\alpha \\ &= \frac{1}{8}\int_{-\pi}^{\pi}\log(1+\cos\beta)d\beta - \frac{\pi}{4}\log 2.\end{aligned}$$

(Note how Rule 2 leads us to extended the integration interval from $[0,\frac{\pi}{2}]$ to $[-\pi,\pi]$.) Motivated by the similar difficulty explained in the remark of Problem (xvii), we try Rule 1 and note

$$\int_{-\pi}^{\pi}\log(1+e^{i\beta})d\beta = \int_{|z|=1}\frac{\log(1+z)}{iz}dz = 2\pi\log 1 = 0.$$

For the above application of Cauchy's integral formula to be rigorous, we need to take a branch of logarithm function that is defined on $\mathbb{C}\setminus(-\infty,0]$ and take a small bypass around 0 for the integration contour, and then take limit.

Using the above result, we have

$$0 = \operatorname{Re}\int_{-\pi}^{\pi}\log(1+e^{i\beta})d\beta = \frac{1}{2}\int_{-\pi}^{\pi}\log\left[(1+\cos\beta)^2+\sin^2\beta\right]d\beta = \frac{1}{2}\int_{-\pi}^{\pi}\left[\log 2+\log(1+\cos\beta)\right]d\beta.$$

Therefore $\int_{-\pi}^{\pi}\log(1+\cos\beta)d\beta = -2\pi\log 2$ and

$$\int_0^{\frac{\pi}{2}}\log(\sin\theta)d\theta = \frac{1}{8}\int_{-\pi}^{\pi}\log(1+\cos\beta)d\beta - \frac{\pi}{4}\log 2 = -\frac{\pi}{2}\log 2.$$

□

(xvii) $\int_0^\pi \frac{x\sin x}{1-2a\cos x+a^2}dx$, where $a>0$;

Solution. We first assume $a>1$. Then by integration-by-parts formula we have

$$\begin{aligned}\int_0^\pi \frac{x\sin x}{1-2a\cos x+a^2}dx &= \frac{1}{2}\int_{-\pi}^{\pi}\frac{x\sin x dx}{1-2a\cos x+a^2}\\ &= \frac{1}{2}\int_{-\pi}^{\pi}\frac{xd(1-2a\cos x+a^2)}{(1-2a\cos x+a^2)\cdot 2a}\\ &= \frac{1}{4a}\left[2\pi\log(1+a)^2-\int_{-\pi}^{\pi}\log(1-2a\cos x+a^2)dx\right].\end{aligned}$$

Since $a>1$, $a-\zeta\in\{z:\mathrm{Re}z>0\}$ for any $\zeta\in\partial D(0,1)$. Therefore we can take a branch of the logarithm function such that $\log(a-z)$ is a holomorphic function on $D(0,1)$ and is continuous on $\bar{D}(0,1)$. For example, we can take the branch $\log z$ such that it is defined on $\mathbb{C}\setminus(-\infty,0]$ with $\log(e^{i\pi})=\pi$ and $\log(e^{-i\pi})=-\pi$. By Cauchy's integral formula,

$$\int_{-\pi}^{\pi}\log(a-e^{i\theta})d\theta=\int_{|z|=1}\frac{\log(a-z)}{iz}dz=2\pi\log a.$$

Therefore, $\int_{-\pi}^{\pi}\log(1-2a\cos x+a^2)dx = 2\mathrm{Re}\int_{-\pi}^{\pi}\log(a-e^{i\theta})d\theta = 4\pi\log a$. Plug this equality into the calculation of the original integral, we get $\frac{\pi}{a}\log\left(\frac{1+a}{a}\right)$.

If $0<a<1$, we have

$$\int_0^\pi\frac{x\sin x}{1-2a\cos x+a^2}dx=\frac{1}{a^2}\int_0^\pi\frac{x\sin x}{1-\frac{2}{a}\cos x+\frac{1}{a^2}}dx=\frac{1}{a^2}\cdot\frac{\pi}{\frac{1}{a}}\log\left(\frac{1+\frac{1}{a}}{\frac{1}{a}}\right)=\frac{\pi}{a}\log(a+1).$$

Assuming the integral as a function of a is continuous at 1, we can conclude for $a=1$, the integral is equal to $\pi\log 2$. The result agrees with that of **Mathematica**. To prove the continuity rigorously, we split the integral into $\int_0^{\frac{\pi}{2}}\frac{x\sin x dx}{1-2a\cos x+a^2}$ and $\int_{\frac{\pi}{2}}^{\pi}\frac{x\sin x dx}{1-2a\cos x+a^2}$. For the second integral, we have ($x\in[\frac{\pi}{2},\pi]$)

$$\frac{x\sin x}{1-2a\cos x+a^2}=\frac{x\sin x}{|a-e^{ix}|}\le\frac{x\sin x}{|a-e^{i\frac{\pi}{2}}|}\le x\sin x.$$

So by Lebesgue's dominated convergence theorem, the second integral is a continuous function of a for $a\in(0,\infty)$. For the first integral, we note $(1-2a\cos x+a^2)$ takes its minimum at $\cos x$. So

$$\frac{x\sin x}{1-2a\cos x+a^2}\le\frac{x}{\sin x}.$$

Again by Lebesgue's dominated convergence theorem, the first integral is a continuous function of a for $a\in(0,\infty)$. Combined, we conclude the integral

$$\int_0^\pi\frac{x\sin x}{1-2a\cos x+a^2}dx$$

as a function of $a\in(0,\infty)$ is a continuous function. □

Remark 21. *We did not take the transform* $\cos x=\frac{z+z^{-1}}{2}$ *because* $\log(1-2a\cos x+a^2)$ *would then become* $\log[w(a)-w(z)]-\log(2a)$, *where* $w(z)=\frac{z+z^{-1}}{2}$ *maps* $D(0,1)$ *to* $U:=\mathbb{C}\setminus[-1,1]$ *and we cannot find a holomorphic branch of logarithm function on* U.

The application of integration-by-parts formula and the extension of integration interval from $[0,\pi]$ *to* $[-\pi,\pi]$ *are motivated by Rule 2. But here a new trick emerged: if the integrand function or part of it could come from the real or imaginary part of a holomorphic function, apply Cauchy's integral formula (or integration theorem) to that holomorphic function and try to relate the result to our original integral (Rule 1).*

(xviii) $\int_0^\infty \frac{\log(1+x^2)}{1+x^2}dx$

Solution. The function $\log(z+i)$ is well-defined on $\mathbb{C}^+ := \{z : \mathrm{Im} z \geq 0\}$. We take the branch of $\log(z+i)$ that evaluates to $\log 2 + \frac{\pi}{2}i$ at i. Define $\gamma_R = \{z : |z| = R, 0 \leq \arg z \leq \pi\}$. Then for $R > 1$, Residue Theorem implies

$$\int_{-R}^{R} \frac{\log(z+i)}{1+z^2}dz + \int_{\gamma_R} \frac{\log(z+i)}{1+z^2}dz = 2\pi i \cdot \frac{i+i}{i+i} = \pi \log 2 + \frac{\pi^2}{2}i.$$

Note

$$\left|\int_{\gamma_R} \frac{\log(z+i)}{1+z^2}dz\right| = \left|\int_0^\pi \frac{\log(Re^{i\theta}+1)}{1+R^2e^{2i\theta}}Re^{i\theta}\cdot id\theta\right| \leq \frac{\pi R[\log(R+1)+2\pi]}{R^2-1} \to 0$$

as $R \to \infty$. So $\int_{-\infty}^\infty \frac{\log(x+i)}{1+x^2}dx = \pi \log 2 + \frac{\pi^2}{2}i$. Hence

$$\pi \log 2 = \mathrm{Re} \int_{-\infty}^\infty \frac{\log(x+i)}{1+x^2}dx = \int_{-\infty}^\infty \frac{\log\sqrt{x^2+1}}{1+x^2}dx = \int_0^\infty \frac{\log(1+x^2)}{1+x^2}dx.$$

□

Remark 22. *The main difficulty of this problem is that the poles of* $\frac{1}{1+x^2}$ *coincide with the branch points of* $\log(x^2+1)$*, so that we cannot apply Residue Theorem directly. An attempt to avoid this difficulty might be writing* $\frac{\log(x^2+1)}{1+x^2}$ *as the sum of* $\frac{\log(x+i)}{1+x^2}$ *and* $\frac{\log(x-i)}{1+x^2}$ *and integrate them separately along different paths which do not contain their respective branch points. But* $\log(x+i)$ *and* $\log(x-i)$ *cannot be defined simultaneously in a region of a Riemann surface which contains both integration paths. But the observation that Re*$[\log(x \pm i)] = \log\sqrt{x^2+1}$ *gives us a simple solution (Rule 1).*

► **16.** Explain whether the function defined by the series

$$-\frac{1}{z} - 1 - z - z^2 - \cdots$$

in $0 < |z| < 1$ can be analytically continued to the function defined by the series

$$\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

in $|z| > 1$.

Solution. The power series $\sum_{n=0}^\infty z^n$ is divergent on every point of $\partial D(0,1)$: if $z = 1$, $\sum_{n=0}^\infty z^n = 1+1+1+\cdots$; if $z \neq 1$ and $z = e^{i\theta}$,

$$\sum_{n=1}^N z^n = \sum_{n=0}^N \cos(n\theta) + i\sum_{n=0}^N \sin(n\theta) = \frac{\sin\frac{(N+1)\theta}{2}\cos\frac{N\theta}{2}}{\sin\frac{\theta}{2}} + i\frac{\sin\frac{(N+1)\theta}{2}\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}}.$$

Similarly, the series $\sum_{n=0}^\infty z^{-n-2}$ is divergent on every point of $\partial D(0,1)$. So the functions represented by these two power series cannot be analytic continuation of each other. □

► **17.** Show that the function defined by the series

$$1 + \alpha z + \alpha^2 z^2 + \cdots + \alpha^n z^n + \cdots$$

and the function defined by

$$\frac{1}{1-z} - \frac{(1-\alpha)z}{(1-z)^2} + \frac{(1-\alpha)^2 z^2}{(1-z)^3} - \cdots$$

are analytic continuations of each other.

Proof. It's clear we need the assumption that $\alpha \neq 0$. The series $\sum_{n=0}^{\infty}(\alpha z)^n$ is convergent in $U_1 = \{z : |z| < \frac{1}{|\alpha|}\}$. The series $\frac{1}{1-z}\sum_{n=0}^{\infty}(-1)^n \frac{[(1-\alpha)z]^n}{(1-z)^n}$ is convergent in $U_2 = \{z : |1-\alpha||z| < |1-z|\}$. On $U_3 = U_1 \cap U_2$, both of the series represent the analytic function $\frac{1}{1-\alpha z}$. So the functions represented by these two series are analytic continuation of each other. □

► **18.** Show that the function $f_1(z)$ defined by the series

$$z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \cdots$$

in $|z| < 1$, and the function $f_2(z)$ defined by the series

$$\ln 2 - \frac{1-z}{2} - \frac{(1-z)^2}{2 \cdot 2^2} - \frac{(1-z)^3}{3 \cdot 2^3} - \cdots$$

in $|z-1| < 2$ are analytic continuations of each other.

Proof. On the line segment $\{z : 0 < \text{Re}z < 1,\ \text{Im}z = 0\}$, Taylor series of $\ln(1+z)$ is $z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \cdots$. On the same line segment, $0 < \frac{1-z}{2} < \frac{1}{2}$. So Taylor series of $\ln(1-x)$ ($|x| \leq 1, x \neq 1$) gives

$$\ln 2 - \frac{1-z}{2} - \frac{(1-z)^2}{2 \cdot 2^2} - \frac{(1-z)^3}{3 \cdot 2^3} - \cdots = \ln 2 + \ln\left(1 - \frac{1-z}{2}\right) = \ln(1+z).$$

Since the two holomorphic functions represented by these two series agree on a line segment, they must agree on the intersection of their respective domains (Theorem 2.13). Therefore they are analytic continuations of each other. □

► **19.** Prove that the power series $\sum_{n=0}^{\infty} z^{2^n}$ cannot be analytically continued to the outside of its circle of convergence.

Proof. Clearly the series is convergent in $D(0,1)$ and is divergent for $z = 1$. So its radius of convergent $R = 1$. Let $f(z) = \sum_{n=0}^{\infty} z^{2^n}$. Then $f(z)$ is analytic in $D(0,1)$ and $z = 1$ is a singularity of $f(z)$. We note $f(z) = z + \sum_{n=1}^{\infty} z^{2^n} = z + \sum_{n=1}^{\infty} z^{2 \cdot 2^{n-1}} = z + \sum_{n=1}^{\infty}(z^2)^{2^{n-1}} = z + f(z^2)$. Therefore, we have

$$f(z) = z + f(z^2) = z + z^2 + f(z^4) = z + z^2 + z^4 + f(z^8) = \cdots.$$

So the roots of equations $z^2 = 1$, $z^4 = 1$, $z^8 = 1, \cdots$, $z^{2^n} = 1$, $\cdots$, etc. are all singularities of f. These roots form a dense subset of $\partial D(0,1)$, so $f(z)$ can not be analytically continued to the outside of its circle of convergence $D(0,1)$. □

4 Riemann Mapping Theorem

► **3.** Show that if the entire function $f(z) = \sum_{n=0}^{\infty} c_n z^n$ assumes real values on the real axis, then $c_n (n = 0, 1, \cdots)$ are real numbers.

Proof. Let $g(z) = \sum_{n=0}^{\infty} \bar{c}_n z^n$. Then $g(z)$ is holomorphic and has the same radius of convergence as $f(z)$. $\forall z_* \in \mathbb{R} \cap$ convergence domain of $g(z)$, we have

$$g(z_*) = \sum_{n=0}^{\infty} \bar{c}_n z_*^n = \sum_{n=0}^{\infty} \bar{c}_n \left(\bar{\bar{z}}_*\right)^n = \bar{f}(\bar{z}_*) = \bar{f}(z_*) = f(z_*),$$

where the next to last equality is due to the fact that $z_* \in \mathbb{R}$ and the last equality is due to the fact that $f(\mathbb{R}) \subset \mathbb{R}$. By Theorem 2.13, $f \equiv g$ in the intersection of their respective domains of convergence. So $c_n = \bar{c}_n$, i.e. $c_n (n = 0, 1, \cdots)$ are real numbers. □

▶ **11.** In the Riemann Mapping Theorem, suppose that z_0 is a real number and U is symmetric about the real axis. Show that f satisfies the symmetric property $f(\bar{z}) = \overline{f(z)}$ by using the uniqueness of the theorem.

Proof. Let $g(z) = \bar{f}(\bar{z})$ (first take conjugate of z, then take conjugate of $f(\bar{z})$). Then

$$\frac{\partial g(z)}{\partial \bar{z}} = \frac{\partial \bar{f}(\bar{z})}{\partial \bar{z}} = \text{conjugate}\left(\frac{\partial f(\bar{z})}{\partial z}\right) = 0.$$

So g is a holomorphic function. Clearly g is univalent. Since U is symmetric w.r.t. real axis, $g(U) = \bar{f}(\bar{U}) = \bar{f}(U) = \text{conjugate}(D(0,1)) = D(0,1)$. Furthermore, $g(z_0) = \bar{f}(\bar{z}_0) = \bar{f}(z_0) = 0$, and

$$\begin{aligned}
g'(z_0) &= \lim_{z \to z_0} \frac{\bar{f}(\bar{z}) - \bar{f}(\bar{z}_0)}{z - z_0} \\
&= \lim_{z \to z_0} \text{conjugate}\left(\frac{f(\bar{z}) - f(\bar{z}_0)}{\bar{z} - \bar{z}_0}\right) \\
&= \text{conjugate}(f'(\bar{z}_0)) \\
&= \text{conjugate}(f'(z_0)) \\
&= f'(z_0) > 0.
\end{aligned}$$

By the uniqueness of Riemann Mapping Theorem, $g \equiv f$. □

5 Differential Geometry and Picard's Theorem

6 A First Taste of Function Theory of Several Complex Variables

There are no exercise problems for this chapter.

7 Elliptic Functions

There are no exercise problems for this chapter.

8 The Riemann Zeta Function and The Prime Number Theory

There are no exercise problems for this chapter.

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