Markov Processes: Theorems and Problems

Solution of Exercise Problems

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Contents

This is a solution manual for the book *Markov processes: Theorems and problems*, by Evgenii B. Dynkin and Aleksandr A. Yushkevich, translated from Russian by James S. Wood. If you have any comments or find any typos/errors, please email me at yz44@cornell.edu.

This version of solution manual omits the problems from Chapter 1, 3, 4 and problems 2.25, 2.26.

Chapter 1

A Criterion of Recurrence

Chapter 2

Probabilistic Solutions of Certain Equations

2.1.

Proof. To prove it analytically, we note

$$
\begin{aligned}
& \int_{\mathbb{R}^l} p(t,x)p(s,y-x)dx \\
= & \int_{\mathbb{R}^l} \frac{e^{-\frac{x^2}{2t}}e^{-\frac{(y-x)^2}{2s}}}{(4\pi^2 st)^{\frac{l}{2}}}dx \\
= & \int_{\mathbb{R}^l} \frac{e^{-\frac{(x-\frac{t}{s+t}y)^2+\frac{st}{(s+t)^2}y^2}{2ts/(s+t)}}}{(4\pi^2 st)^{\frac{l}{2}}}dx \\
= & \frac{e^{-\frac{y^2}{2(s+t)}}}{(2\pi(s+t))^{\frac{l}{2}}}\int_{\mathbb{R}^l} \frac{e^{-\frac{(x-\frac{t}{s+t}y)^2}{2ts/(s+t)}}}{(2\pi\frac{ts}{s+t})^{\frac{l}{2}}}dx \\
= & p(t+s,y).
\end{aligned}
$$

To prove it probabilistically, note for any bounded measurable function $f \in b\mathcal{B}(\mathbb{R}^l)$, we have

$$
\begin{aligned}
& \int f(y)p(t+s,y)dy \\
= & E[f(X_{t+s}-X_0)] \\
= & E[f(X_{t+s}-X_s+X_s-X_0)] \\
= & E[\int f(\xi+X_s-X_0)p(t,\xi)d\xi] \\
= & \int\int f(\xi+\eta)p(s,\eta)d\eta p(t,\xi)d\xi \\
= & \int\int f(y)p(t,\xi)p(s,y-\xi)d\xi dy \\
= & \int f(y)[\int p(t,x)p(s,y-x)dx]dy.
\end{aligned}
$$

Since f is arbitrarily chosen, we must have $p(t+s,y) = \int p(t,x)p(s,y-x)dx$. □

2.2.

Proof. If we define τ_0 as the first hitting time at endpoint 0 and τ_a the first hitting time at endpoint a, then $p(a;x) = P_x(\tau_a < \tau_0)$, $q(a;x) = P_x(\tau_0 < \tau_a)$, and $m(a;x) = E[\tau_a \wedge \tau_0]$.

First, $p(a;x_1) = 0$ and $p(a;x_n) = 1$. For any x_i between x_1 and x_n, by virtue of strong Markov property and symmetry, we have

$$p(a;x_i) = \frac{1}{2}p(a;x_{i-1}) + \frac{1}{2}p(a;x_{i+1}),$$

that is, $p(a;x_i) - p(a;x_{i-1}) = p(a;x_{i+1}) - p(a;x_i)$. So any three neighboring points of the graph lie on a single line. Consequently, all points of the graph lie on one line, whose slope is $(p(a;x_n) - p(a;x_1))/(x_n - x_1) = \frac{1}{a}$. So, $p(a;x_i) = \frac{x_i}{a}$.

Alternative solution: Imitating the reasoning on page 66, we note $p(a;0) = 0$, $p(a;a) = 1$, $p(a;\frac{1}{2}a) = \frac{1}{2}$, $p(a;\frac{1}{4}a) = \frac{1}{2}\cdot 0 + \frac{1}{2}p(a;\frac{1}{2}a) = \frac{1}{4}$, $p(a;\frac{3}{4}a) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot 1 = \frac{3}{4}$. Continue with this procedure, we can prove $p(a;\frac{k}{2^n}a) = \frac{k}{2^n}$. If we suppose $p(a;x)$ is continuous, then by taking limit, we have $p(a,x) = \frac{x}{a}$.

Remark: The essential trick can be summarized as *strong Markov property gives mean value property.*

□

2.3.

Proof. We already show in Problem 2.2 that $p(a;x) = \frac{x}{a}$. So the monotonicity follows easily. If we want to follow the hint instead, the argument goes as follows. For any $0 \le x < y \le a$, we have

$$\begin{aligned} p(a;x) &= P_x(\tau_a < \tau_0) = P_x(P_x(\tau_a \circ \theta_{\tau_y} < \tau_0 \circ \theta_{\tau_y} | \mathcal{F}_{\tau_y}) 1_{\{\tau_y < \tau_0\}}) \\ &= P_x(P_y(\tau_a < \tau_0) 1_{\tau_y < \tau_0}) = p(a;y)p(y;x) < p(a;y). \end{aligned}$$

So $p(a;x)$ as a function of x is monotone increasing. □

2.4.

Proof. Already given by solution of Problem 2.2. □

2.5.

Proof. Intuitively, for any $x < \frac{a}{2}$, we have the following equality:

$$\begin{aligned} & \text{average exit time from } (0,a) \text{ starting at } \tfrac{a}{2} \\ = & \text{ average exit time from } (x, a-x) \text{ starting at } \tfrac{a}{2} \\ & +\frac{1}{2} \times \text{average exit time from } (0,a) \text{ starting at } x \\ & +\frac{1}{2} \times \text{average exit time from } (0,a) \text{ starting at } a-x. \end{aligned}$$

By symmetry $m(a;x) = m(a;a-x)$. So we have from above equality

$$m(a;\frac{a}{2}) = m(a-2x;\frac{a}{2}-x) + m(a;x),$$

i.e. $m(a;x) = c_1(a/2)^2 - c_1(a/2-x)^2 = c_1x(a-x)$. For $x > \frac{a}{2}$, symmetry yields $m(a;x) = m(a;a-x) = c_1(a-x)x$.

To prove the above argument rigorously, we define $\tau = \tau_0 \wedge \tau_a$ and $\bar{\tau} = \tau_x \wedge \tau_{a-x}$, then

$$\begin{aligned} E_{\frac{a}{2}}[\tau] &= E_{\frac{a}{2}}[\bar{\tau} + \tau\circ\theta_{\bar{\tau}}] = E_{\frac{a}{2}}[\bar{\tau}] + E_{\frac{a}{2}}[E_{\frac{a}{2}}[\tau\circ\theta_{\bar{\tau}}|\mathcal{F}_{\bar{\tau}}]] = E_{\frac{a}{2}}[\bar{\tau}] + E_{\frac{a}{2}}[E_{B(\bar{\tau})}[\tau]] \\ &= E_{\frac{a}{2}}[\bar{\tau}] + P_{\frac{a}{2}}(\tau_x < \tau_{a-x})E_x[\tau] + P_{\frac{a}{2}}(\tau_x > \tau_{a-x})E_{a-x}[\tau]. \end{aligned}$$

Remark: The essential trick can be summarized as follows: *suppose particle reaches Γ_1 by passing through Γ_2, then the information of first hitting time τ_{Γ_1} can help us understand the first hitting time τ_{Γ_2}, and vice versa.* The same trick is used in Problem 2.19.

□

2.6.

Proof. Define $\tau_0 = \inf\{t > 0 : x(t) = 0\}$ and $\tau_n = \inf\{t > 0 : x(t) = n\}$. Then $\{\text{the particle hits } 0\} = \{\tau_0 < \infty\}$ is measurable, and

$$P_x(\text{the particle hits } 0) = P_x(\tau_0 < \infty) = P_x(\cup_{n=1}^{\infty}\{\tau_0 < \tau_n\}) \geq P_x(\tau_0 < \tau_n) = m(n; x) = \frac{n-x}{n}.$$

Taking limit gives $P_x(\text{the particle hits } 0) \geq 1$. Meanwhile, for any given ω, when $n > x$, $\tau_n(\omega)$ is monotone increasing. If $\lim_{n\to\infty} \tau_n(\omega) < \infty$, then the particle goes to infinity during a finite amount of time, which is contradictory with the continuity of particle path. So $\lim_{n\to\infty} \tau_n(\omega) = \infty$. Therefore

$$E_x[\tau_0] = \lim_{n\to\infty} E_x[\tau_0 \wedge \tau_n] = \lim_{n\to\infty} c_1 x(n-x) = \infty.$$

□

2.7.

Proof. For any interval $\Gamma \subset [0, \infty)$, denote by Γ' the reflection of Γ at zero, then we have

$$P_x(y(t) \in \Gamma) = P_x(x(t) \in \Gamma \cup \Gamma') = \int_\Gamma \frac{e^{-\frac{(\xi-x)^2}{2t}}}{\sqrt{2\pi t}} d\xi + \int_{\Gamma'} \frac{e^{-\frac{(\xi-x)^2}{2t}}}{\sqrt{2\pi t}} d\xi = \int_\Gamma \frac{e^{-\frac{(\xi-x)^2}{2t}}}{\sqrt{2\pi t}} d\xi + \int_\Gamma \frac{e^{-\frac{(\xi+x)^2}{2t}}}{\sqrt{2\pi t}} d\xi.$$

Therefore $p(t, x, y) = \frac{1}{\sqrt{2\pi t}}(e^{-\frac{(y-x)^2}{2t}} + e^{-\frac{(y+x)^2}{2t}})$.

Remark: Basically, $y(t) = |x(t)|$, where x is a Wiener process. □

2.8.

Proof. Fix an interval $\Gamma \subset (0, a)$. It can be obtained by reflecting a set Γ_1 at point a. More precisely, $\Gamma_1 = 2a - \Gamma := \{2a - x : x \in \Gamma\}$. Γ_1 can be further obtained by reflection a set Γ_2 at point 0, i.e. $\Gamma_2 = -\Gamma_1 := \{-x : x \in \Gamma_1\}$. Γ_2 can be further obtained by reflection a set Γ_3 at point a: $\Gamma_3 = 2a - \Gamma_2$. In general, the above procedure produces a sequence of sets: $\Gamma_{2n-1} = 2na - \Gamma$, $\Gamma_{2n} = \Gamma - 2na$, $n = 1, 2, \cdots$.

Similarly, Γ can also be obtained by reflecting a set Γ_1' at 0, i.e. $\Gamma_1' = -\Gamma$, and Γ_1' can be further obtained by reflection $\Gamma_2' = 2a - \Gamma_1' = 2a + \Gamma$ at point a. In general, the procedure produces a sequence of sets:

$\Gamma'_{2n-1} = -\Gamma - 2(n-1)a$, $\Gamma'_{2n} = 2na + \Gamma$, $n = 1, 2, \cdots$. Combined, we conclude $P_x(y(t) \in \Gamma)$ equals to

$$
\begin{aligned}
& \sum_{n=1}^{\infty} [P_x(x(t) \in \Gamma_{2n-1}) + P_x(x(t) \in \Gamma_{2n}) + P_x(x(t) \in \Gamma'_{2n-1}) + P_x(x(t) \in \Gamma'_{2n})] + P_x(x(t) \in \Gamma) \\
= & \sum_{n=1}^{\infty} [P_x(x(t) \in 2na - \Gamma) + P_x(x(t) \in \Gamma - 2na) + P_x(x(t) \in -\Gamma - 2(n-1)a) + P_x(x(t) \in \Gamma + 2na)] \\
& + P_x(x(t) \in \Gamma) \\
= & P_x(x(t) \in \Gamma) + \sum_{n=1}^{\infty} [P_0(x(t) \in \Gamma - 2na + x) + P_0(x(t) \in \Gamma - 2na - x)] \\
& + \sum_{n=1}^{\infty} [P_0(x(t) \in \Gamma + 2(n-1)a + x) + P_0(x(t) \in \Gamma + 2na - x)] \\
= & P_0(x(t) \in \Gamma - x) + \sum_{n=-1}^{\infty} [P_0(x(t) \in \Gamma + 2na + x) + P_0(x(t) \in \Gamma + 2na - x)] \\
& + \sum_{n=1}^{\infty} [P_0(x(t) \in \Gamma + 2(n-1)a + x) + P_0(x(t) \in \Gamma + 2na - x)] \\
= & \sum_{n=-\infty}^{\infty} [P_0(x(t) \in \Gamma + 2na + x) + P_0(x(t) \in \Gamma + 2na - x)] \\
= & \sum_{n=-\infty}^{\infty} \int_{\Gamma} \frac{1}{2\sqrt{\pi t}} [e^{\frac{(y-x+2na)^2}{2t}} + e^{\frac{(y+x+2na)^2}{2t}}] dy.
\end{aligned}
$$

□

2.9.

Proof. For every sample path of $x(t)$ that starts at x, hits 0 by time t, and falls into Γ at time t, there is a corresponding sample path, obtained through reflection at 0, that starts at $-x$ and falls into Γ at time t. So heuristically, $P_x(\tau \le t, x(t) \in \Gamma) = P(t, -x, \Gamma)$. Formally, by Equation (42),

$$
\begin{aligned}
P_x(\tau \le t, x(t) \in \Gamma) &= \int_0^t P(t-s, 0, \Gamma) P_x(\tau \in ds) = \int_0^t P(t-s, 0, \Gamma) P_{-x}(\tau \in dx) \\
&= P_{-x}(\tau \le t, x(t) \in \Gamma) = P_{-x}(x(t) \in \Gamma).
\end{aligned}
$$

Remark: The hint of this problem is the reflection principle. □

2.10.

Proof. Define $\tau = \inf\{t > 0 : x(t) = 0\}$. Then for any interval $\Gamma \subset (0, \infty)$, by Problem 2.9,

$$
\begin{aligned}
P_x(z(t) \in \Gamma) &= P_x(\tau > t, x(t) \in \Gamma) = P_x(x(t) \in \Gamma) - P_x(\tau \le t, x(t) \in \Gamma) \\
&= \int_{\Gamma} \frac{e^{-\frac{(\xi-x)^2}{2t}}}{\sqrt{2\pi t}} d\xi - \int_{\Gamma} \frac{e^{-\frac{(\xi+x)^2}{2t}}}{\sqrt{2\pi t}} d\xi = \int_{\Gamma} \left[\frac{e^{-\frac{(\xi-x)^2}{2t}} - e^{-\frac{(\xi+x)^2}{2t}}}{\sqrt{2\pi t}} \right] d\xi.
\end{aligned}
$$

□

2.11.

Proof. By Problem 2.9, we have

$$\begin{aligned}
P_x(\tau \le t) &= P_x(\tau \le t, x(t) > 0) + P_x(\tau \le t, x(t) \le 0) \\
&= P(t, -x, (0,\infty)) + P_x(x(t) \le 0) \\
&= \int_0^\infty \frac{e^{-\frac{(\xi+x)^2}{2t}}}{\sqrt{2\pi t}} dt + \int_{-\infty}^0 \frac{e^{-\frac{(\xi-x)^2}{2t}}}{\sqrt{2\pi t}} d\xi \\
&= 1 - \int_{-\infty}^{\frac{x}{\sqrt{t}}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du + \int_{-\infty}^{-\frac{x}{\sqrt{t}}} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du.
\end{aligned}$$

So taking derivative w.r.t. t gives u $P_x(\tau \in dt) = \frac{xe^{-\frac{x^2}{2t}}}{\sqrt{2\pi t^3}}$.

Alternative solution: Following the hint, we have

$$\begin{aligned}
P_x(\tau \le t) &= P_x(z(t) = 0) = 1 - P_x(z(t) \in (0,\infty)) = 1 - \int_0^\infty p(t,x,y)dy \\
&= 1 - \int_0^\infty \frac{1}{\sqrt{2\pi t}}[e^{-\frac{(y-x)^2}{2t}} - e^{-\frac{(y+x)^2}{2t}}]dy \\
&= 1 - \int_{-\frac{x}{\sqrt{t}}}^0 \frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}} d\xi + \int_{\frac{x}{\sqrt{t}}}^\infty \frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}} d\xi.
\end{aligned}$$

Taking derivative gives the density.

Remark: 1. Note this method of calculating first hitting time density: get information about first exit time by considering process with absorbing boundary. This trick is used again in Problem 2.16. Spitzer [1] also employed this trick (page 191-192, second problem).

2. We prove Problem 2.6 again by using Equation (43).

(i) $P_x(\tau < \infty) = 1$: $P_x(\tau < \infty) = \int_0^\infty \frac{xe^{-\frac{x^2}{2t}}}{\sqrt{2\pi t^3}} dt = -2\int_0^\infty \frac{xe^{-\frac{x^2}{2t}}}{\sqrt{2\pi}} dt^{-\frac{1}{2}} = -2\int_\infty^0 \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du = 1$.

(ii) $E_x[\tau] = \infty$: $E_x[\tau] = \int_0^\infty \frac{xte^{-\frac{x^2}{2t}}}{\sqrt{2\pi t^3}} dt = x\int_0^\infty \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dt = x\int_0^\infty \frac{e^{-\frac{u}{2}}}{\sqrt{2\pi x^2/u}}(-1)\frac{x^2}{u^2} du = x^2\int_0^\infty \frac{e^{-\frac{u}{2}}}{\sqrt{2\pi u^3}} du = \infty$, since near 0 the integral is divergent.

□

2.12.

Proof. We first draw a graph to illustrate the relation among the stopping times mentioned in this problem and problem 2.13.

0	a	0	a	0		a	0	a	0
τ_0	σ_1	τ_1	σ_2	τ_2	$\cdots$	σ_n	τ_n	σ_{n+1}	τ_{n+1}
	ρ_0	π_1	ρ_1	π_2	$\cdots$	ρ_{n-1}	π_n	ρ_n	π_{n+1}

All the difference $\sigma_{i+1} - \tau_i$ and $\tau_i - \sigma_i$ are i.i.d. with $\tau_i - \sigma_i$ being the amount of time the particle needs to move from a to 0 and $\sigma_{i+1} - \tau_i$ the amount of time the particle needs to move from 0 to a. So by symmetry of Brownian motion, for any n, $\tau_n - \tau_0$ is the amount of the time the particle needs to move from 0 to $2na$ and $\sigma_n - \tau_0$ is the amount of time the particle needs to move from 0 to $(2n-1)a$. Therefore τ_n is distributed the same as the time of first arrival from the point $-2na - x$ at 0, and σ_n is distributed the same as the time of first arrival from $2na + x$ at a. □

2.13.

Proof. This is clear from the graph in the solution for Problem 2.12. □

2.14.

Proof. We prove the following property by mathematical induction:

$$\begin{aligned} & P_x(A,\{\tau_0 \le t \text{ or } \rho_0 \le t\}) \\ = & \sum_{n=0}^{N-1}[P_x(A,\tau_n \le t)+P_x(A,\rho_n \le t)] - \sum_{n=1}^{N}[P_x(A,\sigma_n \le t)+P_x(A,\pi_n \le t)] + P_x(A,\{\sigma_N \le t, \pi_N \le t\}). \end{aligned}$$

For $N=1$, we note

$$\begin{aligned} & P_x(A,\{\tau_0 \le t \text{ or } \rho_0 \le t\}) \\ = & P_x((A\cap\{\tau_0 \le t\})\cup(A\cap\{\rho_0 \le t\})) \\ = & P_x(A,\tau_0 \le t)+P_x(A,\rho_0 \le t) - P_x(A,\{\sigma_1 \le \text{ or } \pi_1 \le t\}) \\ = & P_x(A,\tau_0 \le t)+P_x(A,\rho_0 \le t) - P_x(A,\sigma_1 \le t) - P_x(A,\pi_1 \le t) + P_x(A,\{\sigma_1 \le t \text{ or } \pi_1 \le t\}). \end{aligned}$$

Assume the claimed property is true for $N \le k$. Since

$$\begin{aligned} & P_x(A,\{\sigma_k \le t, \pi_k \le t\}) \\ = & P_x(A,\{\tau_k \le t \text{ or } \rho_k \le t\}) \\ = & P_x(A,\tau_k \le t)+P_x(A,\rho_k \le t) - P_x(A,\{\tau_k \le t, \rho_k \le t\}) \\ = & P_x(A,\tau_k \le t)+P_x(A,\rho_k \le t) - P_x(A,\{\sigma_{k+1} \le t \text{ or } \pi_{k+1} \le t\}) \\ = & P_x(A,\tau_k \le t)+P_x(A,\rho_k \le t) - P_x(A,\sigma_{k+1} \le t) - P_x(A,\pi_{k+1} \le t) + P_x(A,\{\sigma_{k+1} \le t, \pi_{k+1} \le t\}), \end{aligned}$$

we can see the property is also true for $N=k+1$. Now let $N\to\infty$ and we are done. □

2.15.

Proof. For any $x \in (0,a)$ and interval $\Gamma \subset (0,a)$, we have

$$\begin{aligned} & P_x(z(t)\in\Gamma) = P_x(x(t)\in\Gamma, \tau_0 > t, \rho_0 > t) \\ = & P_x(x(t)\in\Gamma) - P_x(x(t)\in\Gamma, \{\tau_0 \le t \text{ or } \rho_0 \le t\}) \\ = & \int_\Gamma \frac{e^{-\frac{(\xi-x)^2}{2t}}}{\sqrt{2\pi t}}d\xi - \sum_{n=0}^{\infty}[P_x(x(t)\in\Gamma,\tau_n \le t)+P_x(x(t)\in\Gamma,\rho_n \le t)] \\ & + \sum_{n=1}^{\infty}[P_x(x(t)\in\Gamma,\sigma_n \le t)+P_x(x(t)\in\Gamma,\pi_n \le t)]. \end{aligned}$$

As shown in Problem 2.12, τ_n is distributed the same as the time of first arrival from the point $-2na-x$ at 0, and σ_n is distributed the same as the time of first arrival from $2na+x$ at a. Similarly, ρ_n is distributed the same as the time of first arrival from $x-2na$ at a, and π_n is distributed the same as the time of first arrival from $x-2na$ at 0. By Equation (42), for any $n \ge 0$,

$$\begin{aligned} P_x(x(t)\in\Gamma,\tau_n \le t) &= \int_0^t P(t-s,0,\Gamma)P_x(\tau_n \in ds) = \int_0^t P(t-s,0,\Gamma)P_{-2na-x}(\tau_0 \in ds) \\ &= P_{-2na-x}(\tau_0 \le t, x(t)\in\Gamma) = P_{-2na-x}(x(t)\in\Gamma), \end{aligned}$$

$$\begin{aligned} P_x(x(t)\in\Gamma,\rho_n \le t) &= \int_0^t P(t-s,a,\Gamma)P_x(\rho_n \in ds) = \int_0^t P(t-s,a,\Gamma)P_{x-2na}(\rho_0 \in ds) \\ &= \int_0^t P(t-s,a,\Gamma)P_{(2n+2)a-x}(\rho_0 \in ds) \\ &= P_{(2n+2)a-x}(x(t)\in\Gamma, \rho_0 \le t) \\ &= P_{(2n+2)a-x}(x(t)\in\Gamma). \end{aligned}$$

And for any $n \geq 1$,

$$\begin{aligned} P_x(x(t)\in\Gamma, \sigma_n \leq t) &= \int_0^t P(t-s,a,\Gamma)P_x(\sigma_n\in ds) = \int_0^t P(t-s,a,\Gamma)P_{2na+x}(\rho_0\in ds) \\ &= P_{2na+x}(\rho_0\leq t, x(t)\in\Gamma) = P_{2na+x}(x(t)\in\Gamma), \end{aligned}$$

$$\begin{aligned} P_x(x(t)\in\Gamma, \pi_n \leq t) &= \int_0^t P(t-s,0,\Gamma)P_x(\pi_n\in ds) = \int_0^t P(t-s,0,\Gamma)P_{x-2na}(\tau_0\in ds) \\ &= P_{x-2na}(\tau_0\leq t, x(t)\in\Gamma) = P_{x-2na}(x(t)\in\Gamma). \end{aligned}$$

Therefore

$$\begin{aligned} P_x(z(t)\in\Gamma) &= \int_\Gamma \frac{e^{-\frac{(\xi-x)^2}{2t}}}{\sqrt{2\pi t}}d\xi - \sum_{n=0}^\infty [P_{-2na-x}(x(t)\in\Gamma) + P_{(2n+2)a-x}(x(t)\in\Gamma)] \\ &\quad + \sum_{n=1}^\infty [P_{2na+x}(x(t)\in\Gamma) + P_{x-2na}(x(t)\in\Gamma)] \\ &= \int_\Gamma \frac{e^{-\frac{(\xi-x)^2}{2t}}}{\sqrt{2\pi t}}d\xi - \sum_{n=0}^\infty [\int_\Gamma \frac{e^{-\frac{(\xi+x+2na)^2}{2t}}}{\sqrt{2\pi t}}d\xi + \int_\Gamma \frac{e^{-\frac{(\xi+x-(2n+2)a)^2}{2t}}}{\sqrt{2\pi t}}d\xi] \\ &\quad + \sum_{n=1}^\infty [\int_\Gamma \frac{e^{-\frac{(\xi-x-2na)^2}{2t}}}{\sqrt{2\pi t}}d\xi + \int_\Gamma \frac{e^{-\frac{(\xi-x+2na)^2}{2t}}}{\sqrt{2\pi t}}d\xi] \\ &= \sum_{n=-\infty}^\infty \int_\Gamma \frac{e^{-\frac{(\xi-x+2na)^2}{2t}}}{\sqrt{2\pi t}}d\xi - \sum_{n=-\infty}^\infty \int_\Gamma \frac{e^{-\frac{(\xi+x+2na)^2}{2t}}}{\sqrt{2\pi t}}d\xi. \end{aligned}$$

So $p(t,x,y) = \frac{1}{\sqrt{2\pi t}}\sum_{n=-\infty}^\infty (e^{-\frac{(\xi-x+2na)^2}{2t}} - e^{\frac{(\xi+x+2na)^2}{2t}})$.

Remark: Note the trick suggested by the hint to Problem 2.9. □

2.16.

Proof.

$$\begin{aligned} P_x(\tau\leq t) &= P_x(z(t)=0 \; or \; a) = 1 - \int_0^a p(t,x,y)dy \\ &= 1 - \int_0^a \frac{1}{\sqrt{2\pi t}}\sum_{n=-\infty}^\infty [e^{-\frac{(y-x+2na)^2}{2t}} - e^{-\frac{(y+x+2na)^2}{2t}}]dy \\ &= 1 - \sum_{-\infty}^\infty \left[\int_{\frac{-x+2na}{\sqrt{t}}}^{\frac{a-x+2na}{\sqrt{t}}} \frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}d\xi - \int_{\frac{x+2na}{\sqrt{t}}}^{\frac{a+x+2na}{\sqrt{t}}} \frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}d\xi\right] \\ &= 1 - \sum_{n=-\infty}^\infty \left[\Phi\left(\frac{a-x+2na}{\sqrt{t}}\right) - \Phi\left(\frac{-x+2na}{\sqrt{t}}\right) - \Phi\left(\frac{a+x+2na}{\sqrt{t}}\right) + \Phi\left(\frac{x+2na}{\sqrt{t}}\right)\right], \end{aligned}$$

where Φ is the distribution function of a standard normal random variable. Since $\frac{d}{dt}\Phi(\frac{\alpha}{\sqrt{t}}) = \frac{e^{-\frac{\alpha^2}{2t}}}{\sqrt{2\pi t^3}}(-\frac{\alpha}{2})$, taking derivative of the above equality gives us

$$\begin{aligned}
&P_x(\tau \in dt) \\
= \quad & \frac{1}{2\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} \Big[(a-x+2na)e^{-\frac{(a-x+2na)^2}{2t}} - (-x+2na)e^{-\frac{(-x+2na)^2}{2t}} \\
& -(a+x+2na)e^{-\frac{(a+x+2na)^2}{2t}} + (x+2na)e^{-\frac{(x+(2n-1)a)^2}{2t}} \Big] \\
= \quad & \frac{1}{2\sqrt{2\pi t^3}} \left(\sum_{-\infty}^{\infty} (a-x-2na)e^{-\frac{(x+(2n-1)a)^2}{2t}} + \sum_{-\infty}^{\infty} (x+2na)e^{-\frac{(x+2na)^2}{2t}} \right. \\
& \left. + \sum_{n=-\infty}^{\infty} (x+2na)e^{-\frac{(x+2na)^2}{2t}} + \sum_{-\infty}^{\infty} (x+2na)e^{-\frac{-(x+2na)^2}{2t}} \right) \\
= \quad & \frac{1}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} \left[(x+2na)e^{-\frac{(x+2na)^2}{2t}} + (2na+a-x)e^{-\frac{(2na+a-x)^2}{2t}} \right].
\end{aligned}$$

In particular, we have

$$\begin{aligned}
p(\frac{a}{2}, t) \quad &= \quad \frac{1}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} 2a(\frac{1}{2} + 2n)e^{-\frac{(\frac{a}{2}+2na)^2}{2t}} \\
&= \quad \frac{a}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} (1+4n)e^{-\frac{(1+4n)^2}{8t}a^2} \\
&= \quad \frac{a}{\sqrt{2\pi t^3}} \left[\sum_{n=0}^{\infty} (1+4n)e^{-\frac{(1+4n)^2}{8t}a^2} - \sum_{n=1}^{\infty} (4n-1)e^{-\frac{(4n-1)^2}{8t}a^2} \right] \\
&= \quad \frac{a}{\sqrt{2\pi t^3}} \sum_{k=0}^{\infty} (-1)^k (2k+1)e^{-\frac{(2k+1)^2}{8t}a^2}.
\end{aligned}$$

□

2.17.

Proof.

$$
\begin{aligned}
& \int_0^\infty te^{-\lambda t} p(\frac{a}{2}, t)dt \\
= & \int_0^\infty te^{-\lambda t} \frac{a}{\sqrt{2\pi t^3}} \sum_{k=0}^\infty (-1)^k (2k+1) e^{-\frac{(2k+1)^2 a^2}{8t}} dt \\
= & \sum_{k=0}^\infty (-1)^k (2k+1) \int_0^\infty \frac{ae^{-\lambda t}}{\sqrt{2\pi t}} e^{-\frac{(2k+1)^2 a^2}{8t}} dt \\
= & \sum_{k=0}^\infty (-1)^k \sqrt{\frac{2}{\pi}} (2k+1) a \int_0^\infty e^{-\lambda u^2 - \frac{(2k+1)^2 a^2}{8u^2}} du \\
= & \sum_{k=0}^\infty \frac{(-1)^k a}{\sqrt{2\lambda}} (2k+1) e^{-a(2k+1)\sqrt{\frac{\lambda}{2}}} \\
= & \frac{1}{\sqrt{2\lambda}} \sum_{k=0}^\infty (-1)^k (2k+1) a e^{-\frac{1}{2}\sqrt{2\lambda}(2k+1)a} \\
= & \frac{1}{\lambda} \sum_{k=0}^\infty (-1)^k \frac{\sqrt{2\lambda}}{2} (2k+1) a e^{-\frac{\sqrt{2\lambda}}{2}(2k+1)a} \\
= & \frac{1}{\lambda} \frac{d}{dt} \left(e^{-\frac{\sqrt{2\lambda}}{2} at} \sum_{k=0}^\infty (-1)^{k-1} e^{-a\sqrt{2\lambda}tk} \right) \Bigg|_{t=1} \\
= & \frac{1}{\lambda} \frac{d}{dt} \left(-e^{-\frac{\sqrt{2\lambda}}{2} at} \frac{1}{1+e^{-a\sqrt{2\lambda}t}} \right) \Bigg|_{t=1} \\
= & \frac{ae^{-\frac{\sqrt{2\lambda}}{2} a}}{\sqrt{2\lambda}} \frac{1-e^{-\sqrt{2\lambda}a}}{(1+e^{-a\sqrt{2\lambda}})^2}.
\end{aligned}
$$

□

2.18.

Proof. By Problem 2.17,

$$E_{\frac{a}{2}}[\tau] = \lim_{\lambda \to 0} \frac{ae^{-a\sqrt{\frac{\lambda}{2}}}(1-e^{-a\sqrt{2\lambda}})}{\sqrt{2\lambda}(1+e^{-a\sqrt{2\lambda}})^2} = \lim_{u\to 0} a\frac{1-e^{-au}}{4u} = \frac{a^2}{4}.$$

By Problem 2.5, we conclude $c_1 = 1$. □

2.19.

Proof. We follow the hint closely. Note $m_0[\tau_2] = m_0[\tau_1 + \tau_2 \circ \theta_{\tau_1}] = m_0[\tau_1] + \int_{\partial B(0,r)} E_x[\tau_2]\mu(dx)$, where μ is the uniform distribution on $\partial B(0,r)$. So

$$E_0[\tau_1] = E_0[\tau_2] - \int_{\partial B(0,r)} E_x[\tau_2]\mu(dx) = r^2 - \int_{\partial B(0,r)} (x_1+r)(r-x_1)\mu(dx) = \int_{\partial B(0,r)} x_1^2 \mu(dx) = \frac{r^2}{2}.$$

Remark: The trick suggested by the hint is also used in Problem 2.5. □

2.20.

Proof. Let τ_1 be the time of first visit of $x(t)$ to the sphere $\sum_{i=1}^l x_i^2 = r^2$ and let τ_2 be the time of first visit of $x(t)$ to one of the hyper-planes $x_l = \pm r$. Then

$$E_0[\tau_2] = E_0[\tau_1] + E_0[E_0[\tau_2 \circ \theta_{\tau_1} | \mathcal{F}_{\tau_1}]] = E_0[\tau_1] + \int_{\partial B(0,r)} E_x[\tau_2]\mu(dx),$$

where $\mu(dx)$ is the uniform distribution on $\partial B(0,r)$. Therefore

$$E_0[\tau_1] = E_0[\tau_2] - \int_{\partial B(0,r)} E_x[\tau_2]\mu(dx) = r^2 - \int_{\partial B(0,r)} (x_l + r)(r - x_l)\mu(dx) = \frac{r^2}{l}.$$

□

2.21.

Proof. Let $A = \{\omega :$ there exists positive times t arbitrarily close to zero such that $x(t) \in \Gamma_t\}$, then $A \in \mathcal{F}_0$. By the zero-one law, either $P(A) = 0$ or $P(A) = 1$. We choose a decreasing sequence $\{t_n\}_{n\geq 1}$ so that $\lim_{n\to\infty} t_n = 0$, then

$$P_0(A) \geq P_0(\cap_{m=1}^{\infty} \cup_{n\geq m}^{\infty} \{x(t_n) \in \Gamma_{t_n}\}) \geq \lim_{m\to\infty} P_0(x(t_n) \in \Gamma_{t_n}) \geq \varepsilon.$$

So $P_0(A) = 1$. □

2.22.

Proof. Let $\Gamma_t = (\sqrt{t}, \infty)$ and $\Gamma'_t = (-\infty, -\sqrt{t})$, then

$$P_0(x(t) \in \Gamma_t) = P_0(x(t) \in \Gamma'_t) = \int_{\sqrt{t}}^{\infty} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx = \int_1^{\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du = 1 - \Phi(1) > 0,$$

where Φ is the distribution function of a standard normal random variable. So by Problem 2.21, for almost all sample path ω, $x(t)$ oscillates between $\sqrt{t}$ and $-\sqrt{t}$ as $t \to 0$. This implies $\frac{x(t)-x(0)}{t}$ oscillates between $t^{-\frac{1}{2}}$ and $-t^{-\frac{1}{2}}$ as $t \to 0$. So the ratio $\frac{x(t)-x(0)}{t}$ has a probability one of assuming all the real values in any interval $(0, \varepsilon)$ $(\varepsilon > 0)$. In particular, this implies x is not differentiable at time 0 for almost all sample paths. Since $x(t + \cdot) - x(t)$ is again a Brownian motion, we can conclude for any $t \geq 0$, x is not differentiable at time t for almost all sample paths. □

2.23.

Proof. This problem is to provide some details for the footnote on page 63. Define

$$A = \{\omega : x.(\omega) \text{ arrives at } a \text{ after a positive amount of time}\}.$$

To see A is measurable, note $A = \{\sigma < \infty\}$, where $\sigma = \inf\{t > 0 : x_t(\omega) = 0\}$ is a stopping time. To show $P_a(A) = 0$, define $\sigma_n = \inf\{t > 0 : x_t \in \partial B(a, \frac{1}{n})\}$, where $B(a, r)$ is the ball centered at a with radius r. By continuity of sample path, $\sigma_n > 0$ P_a-a.s. We have also $\sigma_n \downarrow 0$ P_a-a.s. Otherwise, there would exist $t_0 > 0$, so that $x_t = a$ on $[0, t_0)$ with positive probability, which is impossible. Then

$$\begin{aligned} & P_a(A) \\ = \; & P_a(\{\omega : \exists t > 0, \; s.t. \; x_t(\omega) = 0\}) \\ = \; & P_a(\cup_{n=1}^{\infty} \{\omega : \exists t > \sigma_n(\omega), x_t(\omega) = a\}) \\ \leq \; & \sum_{n=1}^{\infty} P_a(\{\omega : \exists t > \sigma_n(\omega), x_t(\omega) = a\}) \\ \leq \; & \sum_{n=1}^{\infty} P_a(P_a(\omega : \exists t > 0, (x \circ \theta_{\sigma_n})_t(\omega) = a | \mathcal{F}_{\sigma_n})) \\ = \; & \sum_{n=1}^{\infty} \int_{\partial B(a, \frac{1}{n})} P_x(\{\omega : \exists t > 0, x_t(\omega) = a\}) \mu(dx) \\ = \; & 0, \end{aligned}$$

where μ is the uniform distribution on $\partial B(a, \frac{1}{n})$, and the last equality comes from the result on page 63 (i.e. a Wiener path on a plane or in a space has a probability one of never hitting a fixed point a different from the initial point of the path).

Remark: To see the measurability of the set $\{\omega : \exists t > 0, x_t(\omega) = a, t > \sigma_n(\omega)\}$, note it is $\{\sigma_n' < \infty\}$ where $\sigma_n' = \inf\{t > \sigma_n : x_t = a\}$ is a stopping time. □

2.24.

Proof. Let $K_n = B(a, \frac{1}{n})$, then $P_a(x(\sigma) = a) = P_a(\cap_{n=1}^{\infty} \{x(\sigma) \in K_n\}) = \lim_{n\to\infty} P_a(x(\sigma) \in K_n)$. Since $P_a(x(\sigma) = a) = 0$ by Problem 2.23, for n large enough, $P_a(x(\sigma) \in K_n) < \frac{1}{2}$. □

2.27.

Proof. If a is irregular, then with probability one, the particle will lie during some positive time interval $(0, \sigma)$ inside G and not intersect with the half line l determined by the line segment. By rotational invariance of $x(t)$, this result will be equally true for any half line obtained by the rotation of l about the point a. In particular, the half line l' which is collinear with l, but with opposite direction. This implies for a positive amount of time, $x(t)$ does not intersect with the line $L = l \cup l'$ except at time 0. Decomposing $x(t)$ along the direction L and the direction perpendicular to L, we get two independent one-dimensional Brownian motion, $x_1(t)$ and $x_2(t)$. With probability one, $x_2(t)$ is non-zero during some positive time interval $(0, \sigma)$. By the zero-one law,

$$P_a(x_2(t) \text{ is positive after an arbitrarily small positive amount of time}) = 0 \; \textit{or} \; 1,$$

and

$$P_a(x_2(t) \text{ is negative after an arbitrarily small positive amount of time}) = 0 \; \textit{or} \; 1.$$

But we just showed the disjoint sum of these two events has probability one. So symmetry yields both of them have probability $\frac{1}{2}$. Contradiction. □

2.28.

Proof. The proof is similar to that of Problem 2.27. □

2.29.

Proof. $A_r = \{\exists t > 0, \text{so that } |x(t)| < r\}$. So $\lim_{r\to 0} A_r = \cap_{r\in\mathbb{Q}_+} A_r = \{\text{for any n}, \exists t_n > 0, \text{ so that } |x(t_n)| < \frac{1}{n}\}$. But $\{\exists t > 0, \text{so that } x(t) = 0\} \subsetneqq \{\text{for any n}, \exists t_n > 0, \text{ so that } |x(t_n)| < \frac{1}{n}\}$. For example, the continuous function $f \in C(\mathbb{R}_+, \mathbb{R}^2)$ that goes to 0 in a spiral but never hits 0 belongs to the second set, but not the first one. □

2.30.

Proof. Since G is bounded, there exists $R > 0$, so that $B(x, R) \supset G$, $\forall x \in G$. For any $x \in G$, the first exist time from x to the outside of G is smaller or equal to the first exit time from x to the outside of $B(x, R)$. The expected value of the latter is $\frac{1}{2}R^2$ by Problem 2.19. So $m(x) \le \frac{1}{2}R^2$. □

2.31.

Proof. Let $\tau = \inf\{t \ge 0 : x(t) \in G\}$, then

$$\begin{aligned} m(x) &= E_x[\tau 1_{\{\tau\le\epsilon\}}] + E_x[\tau 1_{\{\tau>\epsilon\}}] \\ &\le \epsilon P_x(\tau \le \epsilon) + E_x[(\epsilon + \tau\circ\theta_\epsilon)1_{\{\tau>\epsilon\}}] \\ &= \epsilon + E_x[1_{\{\tau>\epsilon\}} E_{x(\epsilon)}[\tau]] \\ &\le \epsilon + P_x(\tau > \epsilon)\sup_{y\in G} m(y). \end{aligned}$$

By definition of regularity, $\lim_{x\to 0,\, x\in G} P_x(\tau > \epsilon) = 0$. So the above inequality implies $\lim_{x\to 0,\, x\in G} m(x) = 0$. □

2.32.

Proof. Without loss of generality, suppose $K = B(0, r)$. Define $\tau = \inf\{t \ge 0 : x(t) \in \partial K\}$ and $\tau' = \inf\{t \ge 0 : x(t) \in \partial B(0, |x|)\}$. Then

$$E_0[\tau] = E_0[\tau' + \tau\circ\theta_{\tau'}] = E_0[\tau'] + E_0[E_{x(\tau')}[\tau]].$$

By Problem 2.20, $\frac{1}{l}r^2 = \frac{1}{l}|x|^2 + \int_{\partial B(0,|x|)} E_y[\tau]\mu(dy)$, where μ is the uniform distribution on $\partial B(0, |x|)$. By symmetry, $E_y[\tau] = E_x[\tau] = m(x)$ for any $y \in \partial B(0, |x|)$. So $\frac{1}{l}(r^2 - |x|^2) = m(x)$. In the two-dimensional case, $m(x) = \frac{1}{2}(r + |x|)(r - |x|)$. □

2.33.

Proof. The hint is detailed enough. So we will omit the proof. Note the key to the whole proof is the inequality $\mu_y(\Gamma) > c\mu_x(\Gamma)$. This is a special case of Harnack's inequality: if $u(x)$ is twice differentiable, harmonic and nonnegative, Ω is a bounded domain contained in the domain G of u, then there is a constant A which is independent of u such that $\sup_{x\in\Omega} u(x) \le A \inf_{x\in\Omega} u(x)$. To make this result applicable to our case, we need to use a smooth function φ_n to approximate 1_Γ on the circle, and then we can take limit. □

Chapter 3

The Optimal Stopping Problem

Chapter 4

Boundary Conditions

Bibliography

[1] Spitzer, Frank: Some theorems concerning 2-dimensional Brownian motion. *Trans. Amer. Math. Soc.* **87**, 1958, 187-197.