Maximum Likelihood Estimation of Two-Parameter Weibull Distribution

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1 The Problem

Recall two-parameter Weibull distribution has a density function of the form

$$f(y;\alpha,\beta)=\alpha\beta y^{\beta-1}\exp\{-\alpha y^{\beta}\}1_{\{y>0\}},$$

where α, $\beta>0$. For the maximum likelihood estimation of the parameters, note the logarithm of n sample likelihood function is

$$\log\prod_{i=1}^{n}f(y_i;\alpha,\beta)=n(\log\alpha+\log\beta)+\sum_{i=1}^{n}[(\beta-1)\log y_i-\alpha y_i^{\beta}].$$

Therefore, the likelihood equation is

$$\begin{cases}\frac{1}{\hat\alpha}-\frac{1}{n}\sum_{i=1}^{n}y_i^{\hat\beta}=0\\ \frac{1}{\hat\beta}+\frac{1}{n}\sum_{i=1}^{n}\log y_i-\frac{\hat\alpha}{n}\sum_{i=1}^{n}y_i^{\hat\beta}\log y_i=0.\end{cases}$$

Use the first equation to express $\hat\alpha$ in terms of $\hat\beta$ and plug it into the second equation, we have

$$1+\hat\beta\left[\frac{1}{n}\sum_{i=1}^{n}\log y_i-\frac{\sum_{i=1}^{n}y_i^{\hat\beta}\log y_i}{\sum_{i=1}^{n}y_i^{\hat\beta}}\right]=0. \tag{1}$$

Question: When does equation (1) has a positive solution?

2 Answer

Proposition 2.1. *Equation (1) has a positive solution if and only if y_i's are not identical.*

Proof. If all y_i's are equal to the same number, clearly equation (1) does not have a solution. This proves the necessity.

To prove the sufficiency, we use the substitution $y_i=e^{\theta_i}$ and define

$$h(\beta):=\beta\left[\frac{1}{n}\sum_{i=1}^{n}\theta_i-\frac{\sum_{i=1}^{n}\theta_i e^{\beta\theta_i}}{\sum_{i=1}^{n}e^{\beta\theta_i}}\right].$$

Note $\lim_{\beta\to0+}h(\beta)=0$, so if we can show $\lim_{\beta\to\infty}h(\beta)=-\infty$, by intermediate value theorem, there must exist $\beta_0\in(0,\infty)$ such that $h(\beta_0)=-1$, i.e. equation (1) has a solution. Indeed, let $M=\max_{1\le i\le n}\theta_i$ and assume there are exactly $k(< n)$ θ_i's equal to M. Without loss of generality, we assume $\theta_1 = \cdots = \theta_k = M$. Then for $\beta > 0$

$$
\begin{aligned}
h(\beta) &= \beta \left[\frac{kM + \sum_{i=k+1}^n \theta_i}{n} - \frac{kMe^{\beta M} + \sum_{i=k+1}^n \theta_i e^{\beta\theta_i}}{ke^{\beta M} + \sum_{i=k+1}^n e^{\beta\theta_i}} \right] \\
&= \beta \left[\frac{\sum_{i=k+1}^n \theta_i}{n} + \frac{k}{n}M - \frac{kM + \sum_{i=k+1}^n \theta_i e^{-\beta(M-\theta_i)}}{k + \sum_{i=1}^n e^{-\beta(M-\theta_i)}} \right] \\
&= \beta \left\{ \frac{\sum_{i=k+1}^n \theta_i}{n} + \frac{-kM(n-k) + n\sum_{i=k+1}^n \left(\frac{k}{n}M - \theta_i\right) e^{-\beta(M-\theta_i)}}{n[k + \sum_{i=1}^n e^{-\beta(M-\theta_i)}]} \right\}.
\end{aligned}
$$

Note the second term in the bracket tends to $-\frac{n-k}{n}M$ as $\beta \to \infty$. So the terms in the bracket tend to $\frac{\sum_{i=k+1}^n (\theta_i - M)}{n} < 0$ as $\beta \to \infty$, which implies $\lim_{\beta\to\infty} h(\beta) = -\infty$. This proves the sufficiency. $\square$

Remark 1. *From the above proof, we might heuristically choose* $\frac{n}{\sum_{i=k+1}^n (M-\theta_i)}$ *as the initial guess for the root finder. In practice, we choose* $\max\left\{1, \frac{n}{\sum_{i=k+1}^n (M-\theta_i)}\right\}$ *as the initial guess. For one set of data we tried,* β *is around 0.33 and the number* $\frac{n}{\sum_{i=k+1}^n (M-\theta_i)}$ *is around 0.12.*

Remark 2. *Numerical experiments show Matlab's* **fzero** *tests endpoints of the interval that includes root by trying points on both sides the previous trial. With the result in this note, we can find the interval fairly quickly by looking at one side. One preliminary experiment shows with initial guess 1, Matlab* **fzero** *needs 31 function evaluation while the crude "homemade" bisection root finder only needs 20 evaluations.*