A Semi-Closed Form Solution to Power Regression

Yan Zeng

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The problem. The power regression model assumes that the relationship between the response variable Y and the predictor variable X is described by a power function: $Y = aX^b + \varepsilon$. The estimation of the regression coefficients a and b is conducted via the optimization problem

$$\min_{a,b} \sum_{i=1}^{N} (y_i - a x_i^b)^2 w_i, \tag{1}$$

where x_i is the ith observation of X and y_i is the ith observation of Y $(i = 1, \cdots, N)$. $(w_i)_{i=1}^N$ is an arbitrary sequence of weights, with the requirement of $w_i \geq 0$ for each i.

The problem (1) can be solved by using a nonlinear optimizer, like the `nloptr` function of `R`, or the nonlinear least square regression functions of `R` (e.g. `nls` of the package `stats` and `nlsLM` of the package `minpack.lm`). In this note, we provide a semi-closed form solution which can be used to validate the numbers given by an optimizer.

Unconstrained case. Let $f(a,b) = \sum_{i=1}^N (y_i - a x_i^b)^2 w_i$. Then the requirement of $\frac{\partial f}{\partial a} = 0$ gives

$$a = \frac{\sum_{i=1}^N w_i y_i x_i^b}{\sum_{i=1}^N w_i x_i^{2b}} \tag{2}$$

and the requirement of $\frac{\partial f}{\partial b} = 0$ gives

$$a = \frac{\sum_{i=1}^N w_i y_i x_i^{b-1}}{\sum_{i=1}^N w_i x_i^{2b-1}} \tag{3}$$

which, when combined with equation (2), gives the following equation of b to solve:

$$\sum_{i,j=1}^{N} y_i w_i w_j x_i^{b-1} x_j^{2b-1} (x_i - x_j) = 0.$$

Once b is solved, we can obtain a via the closed-form formula (2).

Interval constraints. Suppose a is constrained by the interval $I_a = [l_a, u_a]$ and b is constrained by interval $I_b = [l_b, u_b]$. We make the observation that $\{f(a,b)\}_b$ is a family of parabolic curves parameterized by b, since the optimal point $a^*(b)$ for a given b can be determined in the following way:

Step 1: For any given b, calculate $a = a(b)$ according to the formula (2).

Step 2: the optimal a^* as a function of b is determined by the algorithm below:

$$a^* = a^*(b) = \begin{cases} a(b) & \text{if } a(b) \in I_a \\ l_a & \text{if } a(b) < l_a \\ u_a & \text{if } a(b) > u_a. \end{cases}$$

Then the two-dimension optimization problem (1) is reduced to the one-dimensional optimization problem

$$\min_b f(a^*(b), b) \tag{4}$$

By using optimizers to find the optimal solution b^* for problem (4), we can obtain the optimal solution (a^*, b^*) of problem (1). This semi-closed form solution allows us to plot the objective function as a one-variable function of b, so that minimum point can be visually identified as a check of the result produced by optimizers.

For the more general case of $Y = aX^b + c + \varepsilon$, a similar result can be obtained.