# Pinned Diffusions

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**Abstract**

In this note, we explain the idea of pinned diffusions, in particular, Brownian bridge. The note is intended to be self-contained, and is based on a recent paper by Qian and Zheng [1]. For the prerequisites, we refer to Revuz and Yor [2], Chapter III, Section 1.

## 1 Definition and basic properties

Let $(X_t, P^x)$ be a (time homogeneous) diffusion process with its natural filtration $(\mathcal{F}_t)_{t\geq 0}$ and state space $E$. We assume $E$ is a Polish space (for example, a complete Riemannian manifold), equipped with its Borel $\sigma$-field $\mathcal{E}$. Suppose the transition probability function $P_t(x, dy)$ possesses a positive, continuous density function $p(t, x, y)$ for all $t > 0$, with respect to a $\sigma$-finite measure $\mu$ on $E$ (in many applications it will be a weighted Riemann-Lebesgue measure on $E$).

Without loss of generality, we assume $(X_t, P^x)$ is a canonical realization. Fix $T > 0$, we are to define a measure $P_T^{x,y}$ on $\Omega = C[0, T]$, such that under $P_T^{x,y}$, the coordinate process $X$ has the intuitive interpretation of a diffusion process pinned to $x$ at time 0 and to $y$ at time $T$.

The first way to define $P_T^{x,y}$ is to introduce a non-homogeneous transition density function

$$H_{T,y}(s, z; t, w) = \frac{p(z, t-s, w)p(w, T-t, y)}{p(z, T-s, y)} \tag{1}$$

for all $0 \leq s < t < T$. Then the corresponding transition probability function is defined by

$$Q_{s,t}^{T,y}(A) = \int_A H_{T,y}(s, z; t, w)\mu(dw) \text{ for } 0 \leq s < t < T. \tag{2}$$

It's not hard to check $Q_{s,t}^{T,y}$ satisfies all the conditions for transition probability function (cf. Revuz and Yor [2], p. 80, Definition (1.2)). Therefore, by Kolmogorov's Extension Theorem, we can construct $P_T^{x,y}$ on $\Omega = C[0, T]$ for each initial position $x \in E$, such that $(X_t, P_T^{x,y})$ is a Markov process with transition function (2) and initial measure $\delta_x$ (cf. Revuz and Yor [2], p. 82, Theorem 1.5).

To see at an intuitive level how we guess out formula (1) and why it defines a pinned diffusion, note formula (1) can be heuristically written as, by Markov property,

$$\begin{aligned}
& \frac{P(X_t = w|X_s = z)P(X_T = y|X_t = w)}{P(X_T = y|X_s = z)} \\
= \quad & \frac{P(X_t = w, X_s = z)P(X_T = y|X_t = w, X_s = z)}{P(X_T = y, X_s = z)} \\
= \quad & \frac{P(X_T = y, X_t = w, X_s = z)}{P(X_T = y, X_s = z)} \\
= \quad & P(X_t = w|X_T = y, X_s = z).
\end{aligned}$$

There is an alternative treatment of pinned diffusions. It is more intuitive and is more commonly seen in literature (for example, Brownian bridge as conditioned Brownian motion, cf. Revuz and Yor [2], p. 41, exercise 3.16). Plus, it illustrates one of the many applications of regular conditional distributions.

**Definition 1.** We define $P_T^{x,y}$ as the regular conditional distribution $P^x(\cdot|X_T = y)$.[1]

**Proposition 1.1.** *The process $(X_t, P_T^{x,y})$ is a Markov process and has the transition function defined by formula (2).*

*Proof.* For any $0 = t_0 < t_1 < \cdots < t_n < T$ and $B_i \in \mathcal{E}$ $(0 \le i \le n)$,

$$\begin{aligned}
& P_T^{x,y}(X_{t_0} \in B_0, \cdots, X_{t_n} \in B_n) \\
= \; & P^x(X_{t_0} \in B_0, \cdots, X_{t_n} \in B_n | X_T = y) \\
= \; & 1_{B_0}(x) \int_{B_1 \times \cdots \times B_n} \frac{f(x; x_1, \cdots, x_n, y)}{p(x, T, y)} \mu(dx_1) \cdots \mu(dx_n)
\end{aligned} \tag{3}$$

where $f(x; x_1, \cdots, x_n, y)$ is the joint density function of $(X_{t_1}, \cdots, X_{t_n}, X_T)$ under $P^x$. By the Markov property of $(X_t, P^x)$, we have

$$\begin{aligned}
& \frac{f(x; x_1, \cdots, x_n, y)}{p(x, T, y)} \\
= \; & \frac{p(x, t_1, x_1) p(x_1, t_2 - t_1, x_2) \cdots p(x_n, T - t_n, y)}{p(x, T, y)} \\
= \; & \frac{p(x, t_1, x_1) p(x_1, T - t_1, y)}{p(x, T, y)} \frac{p(x_1, t_2 - t_1, x_2) p(x_2, T - t_2, y)}{p(x_1, T - t_1, y)} \cdots \\
& \frac{p(x_{n-1}, t_n - t_{n-1}, x_n) p(x_n, T - t_n, y)}{p(x_{n-1}, T - t_{n-1}, y)} \\
= \; & H_{T,y}(0, x; t_1, x_1) \cdots H_{T,y}(t_{n-1}, x_{n-1}; t_n, x_n).
\end{aligned} \tag{4}$$

Plug this back into our original equation, we have

$$\begin{aligned}
& P_T^{x,y}(X_{t_0} \in B_0, \cdots, X_{t_n} \in B_n) \\
= \; & 1_{B_0}(x) \int_{B_1 \times \cdots \times B_n} H_{T,y}(0, x; t_1, x_1) \cdots H_{T,y}(t_{n-1}, x_{n-1}; t_n, x_n) \\
& \mu(dx_1) \cdots \mu(dx_n).
\end{aligned}$$

By Proposition (1.4) of Revuz and Yor [2], p. 81, we proved our proposition. □

In applications, the following characterization, while equivalent to Proposition 1.1, is more convenient to use.

**Proposition 1.2.** *For all $t < T$,*

$$\left. \frac{dP_T^{x,y}}{dP^x} \right|_{\mathcal{F}_t} = \frac{p(X_t, T - t, y)}{p(x, T, y)}.$$

*Remark:* The above characterization provides a third way to define $P_T^{x,y}$.

*Proof.* By formulas (3) and (4), we have

$$\begin{aligned}
& P_T^{x,y}(X_{t_0} \in B_0, \cdots, X_{t_n} \in B_n) \\
= \; & 1_{B_0}(x) \int_{B_1 \times \cdots \times B_n} \frac{p(x, t_1, x_1) p(x_1, t_2 - t_1, x_2) \cdots p(x_n, T - t_n, y)}{p(x, T, y)} \\
& \mu(dx_1) \cdots \mu(dx_n) \\
= \; & E^x \left[ X_{t_0} \in B_0, \cdots, X_{t_n} \in B_n; \frac{p(X_{t_n}, T - t_n, y)}{p(x, T, y)} \right].
\end{aligned}$$

This implies the desired property.

□

[1] For the existence of regular conditional distribution, we refer to the relevant result in Shiryaev [4], where $X$ is the identity mapping, $Y = X_T$ and $E$ is a Polish space.

## 2 Applications

For an application of Brownian bridge, as well as other classical tricks in stochastic calculus, we refer to Rogers [3]. (To check Brownian bridge has the representation

$$x + \frac{s}{t}(y-x) + \left(W_s - \frac{s}{t}W_t\right),$$

we note they are both Gaussina processes and they have the same covariance function.)

The key to Rogers's proof can be summarized as follows: use Lamperti's transform to simplify diffusion coefficient, use Girsanov's formula to simplify drift coefficient, use Brownian bridge to make explicit calculation possible, and finally use Ito's formula to derive Kolmogorov's forward/backward PDE's. All these tricks are widely used in stochastic calculus.

For some recent interesting application of pinned diffusions, we refer to Qian and Zheng [1], Lemma 2.3 and Theorem 2.4.

## References

[1] Z. M. Qian and W. A. Zheng. A representation formula for transition probability densities of diffusions and applications. *Stochastic Process. Appl.* 111 (2004), no. 1, 57–76. 1, 3

[2] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, 3rd edition. Springer, New York, 1999. 1, 2

[3] L. C. G. Rogers: Smooth transition densities for one-dimensional diffusions, Bull. London Math. Soc., 17, 157-161 (1985). 3

[4] A. N. Shiryaev.*Probability*, 2nd edition. Springer, New York, 1995. 2