Frequently Used Results from Measure Theory

Yan Zeng

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Abstract

Results for the measure-theoretical foundations of stochastic processes, based on Shiryaev [1], Chapter 2.

1 Monotone Class Theorem

Definition 1. Let Ω be a set. A class $\mathcal{P}$ of subsets of Ω is called a *π-system* if it's closed under intersection. A class $\mathcal{L}$ of subsets of Ω is called a *λ-system* if it satisfies:
(a) $\Omega \in \mathcal{L}$;
(b) If $A, B \in \mathcal{L}$ and $A \subset B$, then $B \setminus A \in \mathcal{L}$;
(c) If $A_n \in \mathcal{L}$ and $A_n \uparrow A$, then $A \in \mathcal{L}$.

Theorem 1. (Dynkin's $\pi - \lambda$ Theorem) If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system that contains $\mathcal{P}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.

Theorem 2. (Monotone Class Theorem) Suppose $\mathcal{P}$ is a π-system on Ω, and $\mathcal{H}$ is a family of real-valued functions defined on Ω, such that
(a) $1 \in \mathcal{H}$;
(b) $\forall A \in \mathcal{P}$, $1_A \in \mathcal{H}$;
(c) $f_n \in \mathcal{H}$, $0 \leq f_n \uparrow f$, and f is finite (resp. bounded)$\Longrightarrow f \in \mathcal{H}$.
Then $\mathcal{H}$ contains all the $(\Omega, \sigma(\mathcal{P}))$-measurable (resp. -bounded) functions.

2 Kolmogorov's Extension Theorem

Theorem 3. Suppose E is a Polish space (i.e. a complete separable metric space) and $\mathcal{E}$ is the Borel σ-field on E. For any $t_1 < t_2 < \cdots < t_n$, let $P_{t_1,\cdots,t_n}$ be a probability measure on $(\prod_{i=1}^n E, \otimes_{i=1}^n \mathcal{E})$, then the following conditions are equivalent:
(a) there is a probability measure P on $(E^{\mathbb{R}_+}, \mathcal{E}^{\mathbb{R}_+})$ such that $\forall B \in \otimes_{i=1}^n \mathcal{E}$,

$$P(\omega \in E^{\mathbb{R}_+} : (\omega_{t_1}, \cdots, \omega_{t_n}) \in B) = P_{t_1,\cdots,t_n}(B);$$

(b) $(P_{t_1,\cdots,t_n})_{t_1,\cdots,t_n}$ satisfies the following consistency condition: for any two finite subsets T_1, T_2 of $\mathbb{R}_+$, if $T_1 \subset T_2$, then

$$P_{T_1}(A) = P_{T_2}(A \times \otimes_{i \in T_2 \setminus T_1} \mathcal{E}),\ \forall A \in \otimes_{i \in T_1} \mathcal{E}.$$

3 Regular Conditional Distribution

Definition 2. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ a measurable map, and $\mathcal{G}$ a sub σ-field of $\mathcal{F}$. $\mu : \Omega \times \mathcal{E} \to [0, 1]$ is said to be a *regular conditional distribution for X given $\mathcal{G}$*, if
(a) for each $A \in \mathcal{E}$, $\omega \to \mu(\omega, A)$ is a version of $P(X \in A|\mathcal{G})$;
(b) for a.s. ω, $A \to \mu(\omega, A)$ is a probability measure on $(E, \mathcal{E})$.

Definition 3. A measurable space $(E, \mathcal{E})$ is said to be a *Borel space*, if there is a one-to-one map ϕ from E into $\mathbb{R}$ so that ϕ and ϕ^{-1} are both measurable.

Theorem 4. Every Borel subset of a Polish space is a Borel space.

Theorem 5. Regular conditional distributions exist if $(E, \mathcal{E})$ is a Borel space.

Theorem 6. Suppose X and Y are measurable functions from $(\Omega, \mathcal{F})$ to a Borel space $(E, \mathcal{E})$ and $\mathcal{G} = \sigma(Y)$, then there is a function $\mu : E \times \mathcal{E} \to [0, 1]$ so that
(a) for each $A \in \mathcal{E}$, $\mu(\cdot, A)$ is $\mathcal{E}$-measurable and $\mu(Y(\omega), A)$ is a version of

$$P(X \in A|\mathcal{G}) = P(X \in A|Y);$$

(b) for a.s. ω, $A \to \mu(Y(\omega), A)$ is a probability measure on $(E, \mathcal{E})$.

References

[1] A. N. Shiryaev.*Probability*, 2nd edition. Springer, New York, 1995. 1