Construction of Markov Processes

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Abstract

Summary of various construction methods, based on 龚光鲁、钱敏平 [1]。

1 Basic methods

1. *Given transition function.*
(a) *Kolmogorov's extension theorem* $\Longrightarrow$

$$\begin{cases} \text{realization on } C\text{-space: Kolmogorov's metric condition; Dynkin-Kinney uniform } o(\delta) \text{ condition} \\ \text{realization on } D\text{-space: Kolmogorov's metric condition; Dynkin-Kinney uniform } o(1) \text{ condition.} \end{cases}$$

(b) *Weak convergence on metric space.* A more intuitive and unified approach to the realization of processes on C-space or D-space. In particular, this method re-derives Kolmogorov condition(s) and Dynkin-Kinney condition(s).

2. *Given infinitesimal generator.*
(a)*Hille-Yosida theorem.* For given boundary condition, find resolvent operator and use Hille-Yosida theorem to find semigroup. Usually we cannot find resolvent explicitly but can only prove abstractly the conditions of Hille-Yosida theorem are satisfied. The difficulty lies in the existence of resolvent and the estimate of its norm. Sometimes, we can use Riesz representation theorem.
(b) *When the generator is a differential operator.* Analytic construction of transition function by solving parabolic PDEs (Kolmogorov's backward equation). Then we return to method 1.

3. *When the process is a diffusion with given drift coefficient b and diffusion coefficient A.* Itô's construction via stochastic integration (strong solution and pathwise uniqueness). The necessary information is the drift coefficient b and the *square root* σ of the diffusion coefficient A.

Remark: Abstractly speaking, the (strong) Markov property of SDE solution can be formulated precisely as follows. Suppose we have a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, on which an m-dimensional *standard* $(\mathcal{F}_t)$-Brownian motion W is defined. Then we can consider an n-dimensional SDE driven by W, $dX_t = f(t, X.)dW_t + g(t, X.)dt$, where f and g are two predictable functions with values in $n \times m$ matrices and n-vector (cf. Revuz and Yor [3], Chapter IX, Definition 1.2). If X^x is a solution with $X_0 = x$, the distribution $X^x(P)$ of X^x, denoted by P^x, induces a probability measure on $C(\mathbb{R}_+, \mathbb{R}^n)$. The (strong) Markov property then means the coordinate process defined on $C(\mathbb{R}_+, \mathbb{R}^n)$ is a (strong) Markov process with respect to its natural filtration, under the family of measures $(P^x)_{x\in\mathbb{R}^n}$ (cf. Revuz and Yor [3], Chapter IX, Theorem 1.9). Usually, we need the SDE to be homogenous, i.e. $f(t, X.) = \sigma(X_t)$ and $g(t, X.) = b(X_t)$. If $f(t, X.) = \sigma(t, X_t)$ and $g(t, X.) = b(t, X_t)$, then X is an inhomogeneous Markov process, since the original SDE becomes homogeneous after adding one more equation $dX_t^{n+1} = dt$.

4. *Given formal infinitesimal generator $\hat{\mathcal{A}}$.* Stroock-Varadhan martingale problem. This approach deals directly with diffusion coefficient A, instead of its square root σ. When $A = \sigma\sigma^T$, martingale problem $\pi(A, b)$ is equivalent to the existence and uniqueness in law of the weak solution of SDE(σ, b) (cf. 龚光鲁 [2, page 247], Theorem 3.4). One way to find σ is to write A in the form MNM^T, where M is an orthogonal matrix and N is a diagonal matrix, and then set $\sigma = MN^{\frac{1}{2}}$.

2 Summary in a graph

Classical diffusion processes: drift coefficient $\mathbf{b}(x)$ and diffusion coefficient $\mathbf{A}(x) \Longrightarrow$ formal infinitesimal generator $\hat{\mathcal{A}} = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}$

$$\Longrightarrow \begin{cases} \text{Semi-group approach via the Hille-Yosida theorem;} \\ \text{Kolmogorov's backward equation } \frac{dP_t f}{dt} = \hat{\mathcal{A}} P_t f \Longrightarrow \text{transition function;} \\ \text{Itô's construction via stochastic integration;} \\ \text{Stroock-Varadhan martingale problem approach.} \end{cases}$$

References

[1] 钱敏平、龚光鲁：《随机过程论（第二版）》。北京：北京大学出版社，1997.10。1

[2] 龚光鲁：《随机微分方程引论（第二版）》。北京：北京大学出版社，1995。1

[3] D. Revuz and M. Yor. *Continous martingales and Brownian motion*. Third edition. Springer-Verlag, Berline, 1998. 1