Interest rate modeling: A summary

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Abstract

Summary of interest rate modeling as presented in Shreve [1], [2]. Sections marked with * have not had their notes recorded.

Contents

1 Discrete-Time Model

We build the interest rate model under the risk-neutral measure. We first describe the evolution of the interest rate under the risk-neutral measure and then determine the prices of zero-coupon bonds and all other fixed income assets by using the risk-neutral pricing formula. This construction guarantees that all discounted asset prices are martingales, and so all discounted portfolio processes are martingales (when all coupons and other payouts of cash are reinvested in the portfolio).

Zero-coupon bond prices. $B_{n,m} = \widetilde{\mathbb{E}}_n\left[\frac{D_m}{D_n}\right]$.

No arbitrage property of bond portfolio. Let $\Delta_{n,m}$ be the number of m-maturity zero-coupon bonds held by the agent between times n and $n+1$ ($n<m$). His wealth at time $n+1$ is then given by

$$X_{n+1} = \Delta_{n,n+1} + \sum_{m=n+2}^{N} \Delta_{n,m} B_{n+1,m} + (1+R_n)\left(X_n - \sum_{m=n+1}^{N} \Delta_{n,m} B_{n,m}\right). \tag{1}$$

By using the definition of zero-coupon bond, we can prove the discounted wealth process $D_n X_n$ is a martingale under the risk-neutral measure $\widetilde{\mathbb{P}}$.

Money market account in terms of zero-coupon bond. In the wealth equation (1), an agent is permitted to invest in the money market and zero-coupon bonds of all maturities. However, investing at time n in the zero-coupon bond with maturity $n+1$ is the same as investing in the money market:

$$\frac{1}{B_{n,n+1}} = \frac{1}{\widetilde{\mathbb{E}}\left[\frac{D_{n+1}}{D_n}\right]} = \frac{1}{\widetilde{\mathbb{E}}\left[\frac{1}{1+R_n}\right]} = \frac{1}{\frac{1}{1+R_n}} = 1 + R_n. \tag{2}$$

So it is actually unnecessary to include the money market account among the traded assets.

Forward price $\mathrm{For}_{n,m}$. For an asset with price process S_0, S_1, $\cdots$, S_N and $0 \le n \le m \le N$, the m-forward price at time n of the asset is

$$\mathrm{For}_{n,m} = \frac{S_n}{B_{n,m}}. \tag{3}$$

This is seen from the following *static hedge*: at time n, borrow to buy one unit of the asset at the cost of S_n and short $\frac{S_n}{B_{n,m}}$ units of bonds with maturity m to offset the borrowing cost. At time m, receive the forward price $\mathrm{For}_{n,m}$ for the exchange of the asset, and pay $\frac{S_n}{B_{n,m}}$ to cover the short bond position. To avoid arbitrage, we must have $\mathrm{For}_{n,m} = \frac{S_n}{B_{n,m}}$.

Forward interest rate $F_{n,m}$ at time n for investing at time m. Consider the following investment:

• At time n: short 1 unit of m-maturity zero-coupon bond and buy $\frac{B_{n,m}}{B_{n,m+1}}$ units of $(m+1)$-maturity zero-coupon bond. Total cash flow is

$$1 \times B_{n,m} - \frac{B_{n,m}}{B_{n,m+1}} \times B_{n,m+1} = 0.$$

• At time m: Cover the short bond position. Total cash flow is -1.

• At time $m+1$: Receive payment from the long bond position. Total cash flow is $\frac{B_{n,m}}{B_{n,m+1}}$.

Thus, between times m and $m+1$, it is as if the agent has invested at the interest rate

$$F_{n,m} = \frac{B_{n,m}}{B_{n,m+1}} - 1 = \frac{B_{n,m} - B_{n,m+1}}{B_{n,m+1}}.$$

Moreover, this interest rate for investing between times m and $m+1$ was "locked in" by the portfolio set up at time n. In particular,

$$F_{m,m} = \frac{B_{m,m}}{B_{m,m+1}} - 1 = \frac{1}{\frac{1}{1+R_m}} - 1 = R_m,$$

where the second "=" comes from equation (2).

As will be shown below, forward rate $F_{n,m}$ has the interpretation of "forward price of the rate R_m delivered at time $m+1$".

Forward rate agreement (FRA). Let $0 \le n \le m \le N-1$ be given. The no-arbitrage price at time n of a contract that pays R_m at time $m+1$ is

$$B_{n,m+1} F_{n,m} = B_{n,m} - B_{n,m+1}.$$

Intuition of the RHS: The payoff $R_m = (1 + R_m) - 1$ at time $m + 1$ is equivalent to an investment of 1 at time m (which gives the payoff $1 + R_m$ at time $m + 1$), subtracting a payoff of 1 at time $m + 1$. The first part has time n price $B_{n,m}$ while the second part has time n price $B_{n,m+1}$.

Intuition of the LHS: We note $R_m = F_{m,m}$. Using the $(m + 1)$-forward measure, the wanted price is

$$B_{n,m+1}\widetilde{\mathbb{E}}_n^{m+1}[F_{m,m}] = B_{n,m+1}F_{n,m},$$

where the last "=" comes from the observation that the forward rate process $F_{n,m} = \frac{B_{n,m}}{B_{n,m+1}} - 1(0 \leq n \leq m)$ is a martingale under the $(m + 1)$-forward measure.

Using formula (3) for the forward price of an asset, the forward price at time n of the contract that delivers R_m at time $m + 1$ is $F_{n,m}$. This is another way to regard the concept of "locking in" an interest rate.

m**-period swap rate** SR_m. This is the fixed rate that makes the time-zero no-arbitrage price of the interest rate swap equal to zero. Note the time-zero price of the floating leg is $1 - B_{0,m}$. So the m-period swap rate is

$$SR_m = \frac{1 - B_{0,m}}{\sum_{n=1}^{m} B_{0,n}}.$$

Forward measures. The equivalent martingale measure $\widetilde{\mathbb{P}}^m$ that uses $(B_{n,m})_{0\leq n\leq m}$ as the numeraire. Used to simplify the pricing of interest rate caps and floors. Its Radon-Nikodym derivative process is

$$Z_{n,m} = \frac{B_{n,m}/B_{0,m}}{1/D_n} = \frac{D_n B_{n,m}}{B_{0,m}}.$$

m**-futures price process Fut**$_{n,m}$. By generating a cash flow through market making, a futures contract delivers asset at market price at maturity while has zero value during the contract's life time.

$$\begin{cases} \text{Fut}_{m,m} = S_m; \\ \frac{1}{D_n}\widetilde{\mathbb{E}}_n\left[\sum_{k=n}^{m-1}(\text{Fut}_{k+1,m} - \text{Fut}_{k,m})\right] = 0, \ 0 \leq n \leq m-1. \end{cases}$$

It can be proved that futures price process is a martingale under the risk-neutral measure: $\text{Fut}_{n,m} = \widetilde{\mathbb{E}}_n[S_m], n = 0, 1, \cdots, m$. As a result, we have the following comparison

$$\begin{cases} \text{Forward price:} \quad \text{For}_{n,m} = \frac{S_n}{B_{n,m}} = \frac{\widetilde{\mathbb{E}}_n[D_m S_m]}{\widetilde{\mathbb{E}}_n[D_m]} \\ \text{Futures price:} \quad \text{Fut}_{n,m} = \widetilde{\mathbb{E}}_n[S_m] \end{cases}$$

2 Continuous-Time Models

Yield curve. The curve $\{y(0,T)\}_{0<T<\infty}$ such that $B(0,T) = e^{-y(0,T)\times T}$.

Term structure. The family of curves $\{y(t,T)\}_{0\leq t<T<\infty}$ such that $B(t,T) = e^{-y(t,T)(T-t)}$.

Short rate. The idealization corresponding to the shortest-maturity yield or perhaps the overnight rate offered by the government, depending on the particular application:

$$R(t) = \lim_{T\downarrow t} y(t,T) = -\lim_{T\downarrow t} \frac{\ln B(t,T) - \ln B(t,t)}{T - t} = -\left.\frac{\partial \ln B(t,T)}{\partial T}\right|_{T=t}.$$

2.1 Affine-Yield Models

The multi-factor models in this section do not immediately provide a mechanism for evolution of the prices of tradeable assets. In these models, we begin with the evolution of abstract "factors," and from these the interest rate is obtained. But the interest rate is not the price of an asset, and we cannot infer a market price of risk from the interest rate alone. Therefore, we build these models under the risk-neutral measure from the outset. Zero-coupon bond prices are given by the risk-neutral pricing formula. After these models are built, they are calibrated to market prices for zero-coupon bonds and probably also some fixed income derivatives. The actual probability measure and the market prices of risk never enter the picture.

There are essentially three different two-factor affine-yield models, which are

• The *two-factor Vasicek* model. In this model, both factors have constant diffusion terms, and hence are Gaussian processes, taking negative values with positive probability.

• The *two-factor CIR* model. In this model, both factors appear under the square root in diffusion terms, and hence must be nonnegative at all times.

• The *two-factor mixed term-structure* model. In this model, one factor appears under the square root in the diffusion terms, and only this factor is nonnegative at all times, whereas the other factor can become negative.

For each of these types of models, there is a canonical model. Two-factor affine yield models seen in literature can always be obtained from one of the three canonical models by changing variables. It is desirable when calibrating a model to first change the variables to put the model into a form having the minimum number of parameters. The canonical models have the minimum number of parameters.

2.1.1 Two-Factor Vasicek Model

The general form. The factors $X_1(t)$ and $X_2(t)$ are given by

$$\begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} \right) dt + \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} d\widetilde{B}_1(t) \\ d\widetilde{B}_2(t) \end{bmatrix}$$

where the processes $\widetilde{B}_1(t)$ and $\widetilde{B}_2(t)$ are Brownian motions under a risk-neutral measure $\widetilde{\mathbb{P}}$ with constant correlation $\nu \in (-1, 1)$. The constants σ_1 and σ_2 are assumed to be strictly positive. We further assume that the matrix

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

has strictly positive eigenvalues λ_1 and λ_2, which causes the factors $X_1(t)$ and $X_2(t)$ to be mean reverting. Finally, we assume the short rate is an affine function of the factors,

$$R(t) = \epsilon_0 + \epsilon_1 X_1(t) + \epsilon_2 X_2(t),$$

where ϵ_0, ϵ_1, and ϵ_2 are constants.

The canonical form. The general form above has overparametrization (i.e., different choices of parameters can lead to the same distribution for the process $R(t)$). The canonical form is

$$\begin{bmatrix} dY_1(t) \\ dY_2(t) \end{bmatrix} = - \begin{bmatrix} \lambda_1 & 0 \\ \lambda_{21} & \lambda_2 \end{bmatrix} \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} dt + \begin{bmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t) \end{bmatrix}$$

where $\lambda_1, \lambda_2 > 0$, $\widetilde{W}_1(t)$ and $\widetilde{W}_2(t)$ are independent Brownian motions. The short rate is assumed to follow

$$R(t) = \delta_0 + \delta_1 Y_1(t) + \delta_2 Y_2(t).$$

Bond prices. Assume bond price $B(t,T)$ has the form $f(t, Y_1(t), Y_2(t))$. Let $D(t) = e^{-\int_0^t R(u)du}$ be the discount factor. Then the martingale property of $D(t)B(t,T)$ dictates the dt term in $d(D(t)B(t,T))$ must vanish. This condition gives a PDE for f:

$$\begin{cases} -(\delta_0 + \delta_1 y_1 + \delta_2 y_2) f(t, y_1, y_2) + f_t(t, y_1, y_2) - \lambda_1 y_1 f_{y_1}(t, y_1, y_2) - \lambda_{21} y_1 f_{y_2}(t, y_1, y_2) - \lambda_2 y_2 f_{y_2}(t, y_1, y_2) \\ +\frac{1}{2} f_{y_1 y_1}(t, y_1, y_2) + \frac{1}{2} f_{y_2 y_2}(t, y_1, y_2) = 0, \text{ for } t \in [0, T),\ y_1 \in \mathbb{R},\ y_2 \in \mathbb{R} \\ f(T, y_1, y_2) = 1, \text{ for } y_1 \in \mathbb{R},\ y_2 \in \mathbb{R}. \end{cases}$$

The solution to f's PDE is of the affine-yield form

$$B(t, T; y_1, y_2) = f(t, y_1, y_2) = e^{-y_1 C_1(T-t) - y_2 C_2(T-t) - A(T-t)},$$

where

$$C_1(\tau) = \begin{cases} \frac{1}{\lambda_1}\left(\delta_1 - \frac{\lambda_{21}\delta_2}{\lambda_2}\right)(1-e^{-\lambda_1\tau}) + \frac{\lambda_{21}\delta_2}{\lambda_2(\lambda_1-\lambda_2)}\left(e^{-\lambda_2\tau} - e^{-\lambda_1\tau}\right) & \lambda_1 \neq \lambda_2 \\ \frac{1}{\lambda_1}\left(\delta_1 - \frac{\lambda_{21}\delta_2}{\lambda_1}\right)(1-e^{-\lambda_1\tau}) + \frac{\lambda_{21}\delta_2}{\lambda_1}\tau e^{-\lambda_1\tau} & \lambda_1 = \lambda_2, \end{cases}$$

$$C_2(\tau) = \frac{\delta_2}{\lambda_2}\left(1 - e^{-\lambda_2\tau}\right),$$

and

$$A(\tau) = \int_0^\tau \left[-\frac{1}{2}C_1^2(u) - \frac{1}{2}C_2^2(u) + \delta_0\right] du.$$

Short rate and long rate. The long rate $L(t)$ is the yield at time t on the zero-coupon bond with relative maturity $\bar{\tau}$:

$$L(t) \triangleq -\frac{1}{\bar{\tau}}\ln B(t, t+\bar{\tau}) = \frac{1}{\bar{\tau}}[C_1(\bar{\tau})Y_1(t) + C_2(\bar{\tau})Y_2(t) + A(\bar{\tau})].$$

Because the canonical factors do not have an economic interpreation, we may wish to use $R(t)$ and $L(t)$ as the model factors. Note once we have a model for evolution of the short rate $R(t)$ under the risk-neutral measure, then for each $t \geq 0$ the price of the $(t+\bar{\tau})$-maturity zero-coupon bond is determined by the risk-neutral pricing formula, and hence the short-rate model alone determines the long rate. We cannot therefore write down an arbitrary SDE for the long rate.

The model in terms of short rate and long rate is described by

$$\begin{aligned} \begin{bmatrix} dR(t) \\ dL(t) \end{bmatrix} &= \begin{bmatrix} \delta_1 & \delta_2 \\ \frac{1}{\bar{\tau}}C_1(\bar{\tau}) & \frac{1}{\bar{\tau}}C_2(\bar{\tau}) \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ \lambda_{21} & \lambda_2 \end{bmatrix} \begin{bmatrix} \delta_1 & \delta_2 \\ \frac{1}{\bar{\tau}}C_1(\bar{\tau}) & \frac{1}{\bar{\tau}}C_2(\bar{\tau}) \end{bmatrix}^{-1} \begin{bmatrix} \delta_0 \\ \frac{1}{\bar{\tau}}A(\bar{\tau}) \end{bmatrix} dt \\ &\quad - \begin{bmatrix} \delta_1 & \delta_2 \\ \frac{1}{\bar{\tau}}C_1(\bar{\tau}) & \frac{1}{\bar{\tau}}C_2(\bar{\tau}) \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ \lambda_{21} & \lambda_2 \end{bmatrix} \begin{bmatrix} \delta_1 & \delta_2 \\ \frac{1}{\bar{\tau}}C_1(\bar{\tau}) & \frac{1}{\bar{\tau}}C_2(\bar{\tau}) \end{bmatrix}^{-1} \begin{bmatrix} R(t) \\ L(t) \end{bmatrix} dt \\ &\quad + \begin{bmatrix} \delta_1 & \delta_2 \\ \frac{1}{\bar{\tau}}C_1(\bar{\tau}) & \frac{1}{\bar{\tau}}C_2(\bar{\tau}) \end{bmatrix} \begin{bmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t) \end{bmatrix}. \end{aligned}$$

Gaussian factor processes. The canonical two-factor Vasicek model in vector notation is

$$dY(t) = -\Lambda Y(t) + d\widetilde{(t)},$$

where

$$Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix},\ \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ \lambda_{21} & \lambda_2 \end{bmatrix},\ \widetilde{W}(t) = \begin{bmatrix} \widetilde{W}_1(t) \\ \widetilde{W}_2(t) \end{bmatrix}.$$

There is a closed-form solution to this matrix differential equation

$$Y(t) = e^{-\Lambda t}Y(0) + \int_0^t e^{-\Lambda(t-u)} d\widetilde{W}(u).$$

The processes $Y_1(t)$ and $Y_2(t)$ are Gaussian, and so $R(t) = \delta_0 + \delta_1 Y_1(t) + \delta_2 Y_2(t)$ is normally distributed and mean-reverting.

2.1.2 Two-Factor CIR Model

The canonical form. The factors $Y_1(t)$ and $Y_2(t)$ are given by

$$\begin{bmatrix} dY_1(t) \\ dY_2(t) \end{bmatrix} = \left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} - \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}\right) dt + \begin{bmatrix} \sqrt{Y_1(t)} & 0 \\ 0 & \sqrt{Y_2(t)} \end{bmatrix} \begin{bmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t) \end{bmatrix}$$

with $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\lambda_{11} > 0$, $\lambda_{22} > 0$, $\lambda_{12} \leq 0$, $\lambda_{21} \leq 0$. These conditions on parameters guarantee that, if we start with $Y_1(0) \geq 0$ and $Y_2(0) \geq 0$, we shall have $Y_1(t) \geq 0$ and $Y_2(t) \geq 0$ for all $t \geq 0$ almost surely.

The Brownian motions $\widetilde{W}_1(t)$ and $\widetilde{W}_2(t)$ are assumed to be independent. We do not need this assumption to guarantee nonnegativity of $Y_1(t)$ and $Y_2(t)$ but rather to obtain the affine-yield result.

Finally, the short rate $R(t)$ is defined as

$$R(t) = \delta_0 + \delta_1 Y_1(t) + \delta_2 Y_2(t),\ \delta_0 \geq 0, \delta_1 > 0, \delta_2 > 0.$$

Bond prices. Assume bond price $B(t,T)$ has the form $f(t, Y_1(t), Y_2(t))$. Then the martingale property of $D(t)B(t,T)$ dictates the dt term in $d(D(t)B(t,T))$ must vanish. This condition gives a PDE for f:

$$\begin{cases} -(\delta_0 + \delta_1 y_1 + \delta_2 y_2) f(t, y_1, y_2) + f_t(t, y_1, y_2) \\ +(\mu_1 - \lambda_{11} y_1 - \lambda_{12} y_2) f_{y_1}(t, y_1, y_2) + (\mu_2 - \lambda_{21} y_1 - \lambda_{22} y_2) f_{y_2}(t, y_1, y_2) \\ +\frac{1}{2} y_1 f_{y_1 y_1}(t, y_1, y_2) + \frac{1}{2} y_2 f_{y_2 y_2}(t, y_1, y_2) = 0, \text{ for } t \in [0, T),\ y_1 \geq 0,\ y_2 \geq 0 \\ f(T, y_1, y_2) = 1, \text{ for all } y_1 \in \mathbb{R},\ y_2 \in \mathbb{R}. \end{cases}$$

The solution to this PDE is of the affine-yield form

$$B(t, T; y_1, y_2) = f(t, y_1, y_2) = e^{-y_1 C_1(T-t) - y_2 C_2(T-t) - A(T-t)}$$

where the parameters can be found numerically by solving the following ODE:

$$\begin{cases} C_1(0) = C_2(0) = A(0) = 0 \\ C_1'(\tau) = -\lambda_{11} C_1(\tau) - \lambda_{21} C_2(\tau) - \frac{1}{2} C_1^2(\tau) + \delta_1 \\ C_2'(\tau) = -\lambda_{12} C_1(\tau) - \lambda_{22} C_2(\tau) - \frac{1}{2} C_2^2(\tau) + \delta_2 \\ A'(\tau) = \mu_1 C_1(\tau) + \mu_2 C_2(\tau) + \delta_0. \end{cases}$$

2.1.3 Mixed Model

The canonical form. The factors are given as

$$\begin{bmatrix} dY_1(t) \\ dY_2(t) \end{bmatrix} = \left(\begin{bmatrix} \mu \\ 0 \end{bmatrix} - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} \right) dt + \begin{bmatrix} \sqrt{Y_1(t)} & 0 \\ \sigma_{21}\sqrt{Y_1(t)} & \sqrt{\alpha + \beta Y_1(t)} \end{bmatrix} \begin{bmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t) \end{bmatrix}$$

where $\mu \geq 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\alpha \geq 0$, $\beta \geq 0$, and $\sigma_{21} \in \mathbb{R}$. The Brownian motion $\widetilde{W}_1(t)$ and $\widetilde{W}_2(t)$ are independent. We assume $Y_1(0) \geq 0$, and we have $Y_1(t) \geq 0$ for all $t \geq 0$ almost surely. On the other hand, even if $Y_2(0)$ is positive, $Y_2(t)$ can take negative values for $t > 0$.

The short rate is defined by

$$R(t) = \delta_0 + \delta_1 Y_1(t) + \delta_2 Y_2(t).$$

Bond prices. Assume bond price $B(t,T)$ has the form $f(t, Y_1(t), Y_2(t))$. Then the martingale property of $D(t)B(t,T)$ dictates the dt term in $d(D(t)B(t,T))$ must vanish. This condition gives a PDE for f:

$$\begin{aligned} & \left[-(\delta_0 + \delta_1 y_1 + \delta_2 y_2) + \frac{\partial}{\partial t} + (\mu - \lambda_1 y_1) \frac{\partial}{\partial y_1} - \lambda_2 y_2 \frac{\partial}{\partial y_2} \right] f \\ & + \left[\frac{1}{2} \left(2\sigma_{21} y_1 \frac{\partial^2}{\partial y_1 \partial y_2} + y_1 \frac{\partial^2}{\partial y_1^2} + (\sigma_{21}^2 y_1 + \alpha + \beta y_1) \frac{\partial^2}{\partial y_2^2} \right) \right] f \\ = \ & 0. \end{aligned}$$

If we suppose $f(t, y_1, y_2) = e^{-y_1 C_1(T-t) - y_2 C_2(T-t) - A(T-t)}$, the we have the following system of ODEs

$$\begin{cases} C_1(0) = C2(0) = A(0) = 0 \\ C_1' = -\lambda_1 C_1 - \frac{1}{2} C_1^2 - \sigma_{21} C_1 C_2 - \frac{1}{2}(\sigma_{21}^2 + \beta) C_2^2 + \delta_1, \\ C_2' = -\lambda_2 C_2 + \delta_2, \\ A' = \mu C_1 - \frac{1}{2} \alpha C_2^2 + \delta_0. \end{cases}$$

2.2 Heath-Jarrow-Morton Model

The Heath-Jarrow-Morton (HJM) model takes its state at each time to be the forward curve at that time. For fixed t, one calls the function $T \mapsto f(t,T)$, defined for $T \geq t$, the *forward rate curve*. The HJM model provides a mechanism for evolving this curve (a "curve" in the variable T) forward in time (the variable t). The forward rate curve can be deduced from the zero-coupon bond prices, and the zero-coupon bond prices can be deduced from the forward rate curve. Because zero-coupon bond prices are given directly by the HJM model rather than indirectly by the risk-neutral pricing formula, one needs to be careful that the model does not generate prices that admit arbitrage. Hence HJM is more than a model because it provides a necessary and sufficient condition for a model driven by Brownian motion to be free of arbitrage. Every Brownian-motion-driven model must satisfy the HJM no-arbitrage condition,

2.2.1 Forward rates

Simple forward rate: $[1 + \tau L(t,T,T+\tau)] = B(t,T)/B(t,T+\tau)$, which has the intuition "\$1 at time t becomes $1/P(t,T)$ at time T; increased further by the simple forward rate to time $T+\tau$, it should have the same value as $1/P(t,T+\tau)$."

Instantaneous forward rate:

$$f(t,T) = \lim_{\tau \downarrow 0} L(t,T,T+\tau) = \lim_{\tau \downarrow 0} \frac{1}{\tau} \frac{B(t,T) - B(t,T+\tau)}{B(t,T+\tau)} = -\frac{\partial \ln B(t,T)}{\partial T} = \lim_{\tau \downarrow 0} \frac{1}{\tau} \ln \frac{B(t,\tau)}{B(t,T+\tau)}.$$

The last equality gives $f(t,T)$ the interpretation of "the continuous compounding rate of interest that, applied to the 1 invested at time T, would return $\frac{B(t,T)}{B(t,T+\delta)}$ at time $T+\delta$." An important property of $f(t,T)$ is the following

$$B(t,T) = \exp\left\{ -\int_t^T f(t,v)dv \right\},\ 0 \leq t \leq T.$$

Also note its connection with the short rate:

$$R(t) = f(t,t),$$

since $f(t,T) = -\frac{\partial \ln B(t,T)}{\partial T} = \frac{\partial y(t,T)}{\partial T}(T-t) + y(t,T)$ for $B(t,T) = e^{-y(t,T)(T-t)}$. This is the instantaneous rate we can lock in at time t for borrowing at time t.

From the relationship between bond prices and forward rates

$$\begin{cases} f(t,T) = -\frac{\partial \ln B(t,T)}{\partial T} \\ B(t,T) = \exp\left\{ -\int_t^T f(t,v)dv \right\}, \end{cases}$$

it does not appear to matter whether we build a model for forward rates or for bond prices. In fact, the no-arbitrage condition works out to have a simple form when we model forward rates. From a practical point of view, forward rates are a more difficult object to determine from market data because the differentiation of $\ln B(t,T)$ is sensitive to small changes in the bond prices. On the other hand, once we have forward rates, bond prices are easy to determine because the integration is not sensitive to small changes in the forward rates.

2.2.2 Dynamics of forward rates and bond prices

Starting from the objective measure $\mathbb{P}$, we assume the dynamics of the forward rates follow

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t),\ 0 \leq t \leq T,$$

and the dynamics of the zero-coupon bond prices follow

$$dB(t,T)/B(t,T) = \alpha_B(t,T)dt + \sigma_B(t,T)dW(t).$$

Using the fact $B(t,T) = \exp\left\{-\int_t^T f(t,u)du\right\}$, we can derive the following relationship

$$\begin{cases} \alpha_B(t,T) = R(t) - \int_t^T \alpha(t,v)dv + \frac{1}{2}\left(\int_t^T \sigma(t,v)dv\right)^2 \\ \sigma_B(t,T) = -\int_t^T \sigma(t,v)dv. \end{cases}$$

Or equivalently,

$$\begin{cases} \alpha(t,T) = -\frac{\partial}{\partial T}\alpha_B(t,T) + \sigma(t,T)\int_t^T \sigma(t,v)dv \\ \sigma(t,T) = -\frac{\partial}{\partial T}\sigma_B(t,T). \end{cases}$$

2.2.3 No-Arbitrage Condition

By requiring $D(t)B(t,T)$ be a martingale, we can find the existence and form of the risk-neutral measure $\widetilde{\mathbb{P}}$.

Heath-Jarrow-Morton no-arbitrage condition. $\frac{\alpha(t,T)}{\sigma(t,T)} - \int_t^T \sigma(t,v)dv = \frac{\alpha(t,T)}{\sigma(t,T)} + \sigma_B(t,T)$ is independent of T.

Risk-neutral probability. Define $\Theta(t) = \frac{\alpha(t,T)}{\sigma(t,T)} + \sigma_B(t,T)$. The risk-neutral probability $\widetilde{\mathbb{P}}$ is such that $\int_0^t \Theta(u)du + W(t)$ is a Brownian motion under $\widetilde{\mathbb{P}}$. So the density process satisfies the SDE

$$dZ(t)/Z(t) = -\Theta(t)dW(t).$$

2.2.4 HJM Under Risk-Neutral Measure

Under the risk-neutral measure $\widetilde{\mathbb{P}}$, which is defined by the density process $Z(t) = \mathcal{E}\left(-\int_0^t \Theta(u)dW(u)\right)$, the SDE for instantaneous forward rate $f(t,T)$ is

$$df(t,T) = \sigma(t,T)\int_t^T \sigma(t,v)dv + \sigma(t,T)d\widetilde{W}(t).$$

and the SDE for bond price $B(t,T)$ is

$$dB(t,T)/B(t,T) = R(t)dt + \sigma_B(t,T)d\widetilde{W}(t).$$

where $R(t) = f(t,t)$ is the short rate. The relationship between two volatilities is

$$\begin{cases} \sigma_B(t,T) = -\int_t^T \sigma(t,v)dv \\ \sigma(t,T) = -\frac{\partial}{\partial}\sigma_B(t,T). \end{cases}$$

2.2.5 Relation to Affine-Yield Models

We verify the one-factor Hull-White and Cox-Ingersoll-Ross (CIR) models satisfy the HJM no-arbitrage condition. For both these models, the short rate dynamics are of the form

$$dR(t) = \beta(t, R(t))dt + \gamma(t, R(t))d\widetilde{W}(t),$$

where $\widetilde{W}(t)$ is a Brownian motion under a risk-neutral probability measure $\widetilde{\mathbb{P}}$. The zero-coupon bond prices are of the form

$$B(t,T) = e^{-R(t)C(t,T) - A(t,T)},$$

where $C(t,T)$ and $A(t,T)$ are nonrandom functions. Using the equation $f(t,T) = -\frac{\partial}{\partial T}\ln B(t,T)$, we have the forward rate differential

$$df(t,T) = \left[\frac{\partial}{\partial T}C(t,T)\beta(t,R(t)) + R(t)\frac{\partial}{\partial T}C'(t,T) + \frac{\partial}{\partial T}A'(t,T)\right]dt + \frac{\partial}{\partial T}C(t,T)\gamma(t,R(t))d\widetilde{W}(t).$$

This is an HJM model with

$$\sigma(t,T) = \frac{\partial}{\partial T}C(t,T)\gamma(t,R(t)).$$

So the HJM no-arbitrage condition $\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,v)dv$ becomes[1]

$$\begin{aligned} & \frac{\partial}{\partial T}C(t,T)\beta(t,R(t)) + R(t)\frac{\partial}{\partial T}C'(t,T) + \frac{\partial}{\partial T}A'(t,T) \\ = & \left(\frac{\partial}{\partial T}C(t,T)\right)\gamma(t,R(t))\int_t^T \frac{\partial}{\partial v}C(t,v)\gamma(t,R(t))dv \\ = & \left(\frac{\partial}{\partial T}C(t,T)\right)C(t,T)\gamma^2(t,R(t)). \end{aligned}$$

2.2.6 *Implementation of HJM

2.3 Forward LIBOR Model

For practical applicaitons, it would be convenient to build a model where the forward rate had a long-normal distribution. Unfortunately, this is not possible. However, if one instead models the forward LIBOR $L(t,T)$, this problem can be overcome. The model that takes forward LIBOR as its state is often called the *forward LIBOR model*, the *market model*, or the *Brace-Gatarek-Musiela (BGM) model*.

2.3.1 The Problem with Forward Rates

Recall in an arbitrage-free term-structure model, forward rates must evolve according to

$$df(t,T) = \sigma(t,T)\int_t^T \sigma(t,v)dvdt + \sigma(t,T)d\widetilde{W}(t).$$

In order to adapt the Black-Scholes formula for equity options to fixed income markets, and thereby obtain the Black caplet formula, it would be desirable to build a model in which forward rates are log-normal under a risk-neutral measure. To do that, we should set $\sigma(t,T) = \sigma f(t,T)$ in the above SDE for $f(t,T)$, where σ is a positive constant.

Ignoring the random part and assuming T and t are close to each other, we have the ODE for $f(t) \triangleq f(t,t)$

$$f'(t) \approx \sigma^2 f^2(t).$$

The solution is $f(t) = \frac{f(0)}{1-\sigma^2 f(0)t}$, which explodes at time $t = \frac{1}{\sigma^2 f(0)}$. If we take into account the random term, it's even worse because the randomness causes some paths to explode immediately no matter what initial condition is given. This difficulty with continuously compounding forward rates $f(t,T)$ causes us to introduce the forward LIBOR $L(t,T)$.

2.3.2 LIBOR and Forward LIBOR

Let $0 \le t \le T$ and $\delta > 0$ be given. We recall that at time t one can lock in an interest rate for investing over the interval $[T, T+\delta]$ by shorting 1 unit of T-maturity zero-coupon bond and longing $\frac{B(t,T)}{B(t,T+\delta)}$ units of $(T+\delta)$-maturity zero-coupon bond. This position can be created at zero cost at time t, it calls for "investment" of 1 at time T to cover the short position, and it "repays" $\frac{B(t,T)}{B(t,T+\delta)}$ at time $T+\delta$. Using the formula

investment × (1+duration of investiment × interest rate) = repayment,

we can define a *forward LIBOR rate* $L(t,T)$ by

$$1 \times (1 + \delta L(t,T)) = \frac{B(t,T)}{B(t,T+\delta)}.$$

[1]Note we are already under the risk-neutral measure, so $\Theta(t) = \frac{\alpha(t,T)}{\sigma(t,T)} + \sigma_B(t,T) = 0$.

Or equivalently,

$$L(t,T) = \frac{1}{\delta}\frac{B(t,T) - B(t,T+\delta)}{B(t,T+\delta)}.$$

When $t = T$, we call it *spot LIBOR*, or simply *LIBOR*, set at time T. The positive number δ is called the *tenor* of the LIBOR, and it is usually either 0.25 years or 0.5 years.

2.3.3 Pricing a Backset LIBOR Contract

The *backset LIBOR* is defined on one payment date to be the LIBOR set on the previous payment date. Mathematically, let $0 \le t \le T$ and $\delta > 0$ be given. The no-arbitrage price at time t of a contract that pays $L(T,T)$ at time $T + \delta$ is[2]

$$S(t) = \begin{cases} \frac{1}{\delta}[B(t,T) - B(t,T+\delta)] = B(t,T+\delta)L(t,T), & 0 \le t \le T, \\ B(t,T+\delta)L(T,T), & T \le t \le T+\delta. \end{cases}$$

The intuition here is the same as that of a discrete-time model: the payoff $\delta L(T,T) = (1 + \delta L(T,T)) - 1$ at time $T + \delta$ is equivalent to an investment of 1 at time T (which gives the payoff $1 + \delta L(T,T)$ at time $T + \delta$), subtracting a payoff of 1 at time $T + \delta$. The first part has time t price $B(t,T)$ while the second part has time t price $B(t,T+\delta)$.

Pricing under $(T+\delta)$*-forward measure*: Using the $(T+\delta)$-forward measure, the time-t no-arbitrage price of a payoff $\delta L(T,T)$ at time $(T+\delta)$ is

$$B(t,T+\delta)\widetilde{\mathbb{E}}^{T+\delta}[\delta L(T,T)] = B(t,T+\delta)\cdot \delta L(t,T),$$

where we have used the observation that the forward rate process $L(t,T) = \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right)$ $(0 \le t \le T)$ is a martingale under the $(T+\delta)$-forward measure.

2.3.4 Black Caplet Formula

Consider a caplet that pays $(L(T,T) - K)^+$ at time $T + \delta$, where K is some nonnegative constant. Since $L(t,T)$ is a traded asset $B(t,T)$ discounted by the numeraire $B(t,T+\delta)$ of the $(T+\delta)$-forward measure (up to some constant), it is a martingale under the $(T+\delta)$-forward measure. By the martingale representation theorem, $L(t,T)$ has the representation

$$dL(t,T) = \gamma(t,T)L(t,T)d\widetilde{W}^{T+\delta}(t),\ 0 \le t \le T.$$

The forward LIBOR model is constructed so that $\gamma(t,T)$ is nonrandom and forward LIBOR $L(t,T)$ will be log-normal under the $(T+\delta)$-forward measure. This allows us to apply the Black-Scholes formula for equity options: the price of the caplet at time zero is

$$B(0,T+\delta)\widetilde{\mathbb{E}}^{T+\delta}[(L(T,T) - K)^+] = B(0,T+\delta)[L(0,T)N(d_+) - KN(d_-)],$$

where

$$d_\pm = \frac{1}{\sqrt{\int_0^T \gamma^2(t,T)dt}}\left[\log\frac{L(0,T)}{K} \pm \frac{1}{2}\int_0^T \gamma^2(t,T)dt\right].$$

2.3.5 Forward LIBOR and Zero-Coupon Bond Volatilities

Under the HJM framework, zero-coupon bond volatility and forward rate volatility are related by

$$\begin{cases} \sigma_B(t,T) = -\int_t^T \sigma(t,v)dv \\ \sigma(t,T) = -\frac{\partial}{\partial}\sigma_B(t,T). \end{cases}$$

[2] Compare with the no-arbitrage price of FRA in discrete time model. The same logic applies here.

Through explicit calculation under the $(T+\delta)$-forward measure, we can find the forward LIBOR volatility in terms of zero-coupon bond volatility:

$$\gamma(t,T) = \frac{1+\delta L(t,T)}{\delta L(t,T)}[\sigma_B(t,T) - \sigma_B(t,T+\delta)]. \tag{4}$$

2.3.6 A Forward LIBOR Term-Structure Model

The Black caplet formula can be used to calibrate the forward LIBOR model. Suppose at time zero, market data allow us to determine caplet prices for maturity dates $T_j = j\delta$ for $j = 1, \cdots, n$. The Black caplet formula allows us to recover

$$\frac{1}{T_j}\int_0^{T_j} \gamma^2(t,T_j)dt,\ j = 1, \cdots, n.$$

• We choose nonrandom nonnegative functions

$$\gamma(t,T_j), 0 \leq t \leq T_j, j = 1, \cdots, n,$$

so that $\frac{1}{T_j}\int_0^{T_j} \gamma^2(t,T_j)dt$ agrees with the implied volatility obtained from the Black caplet formula ($j = 1, \cdots, n$).

In order to complete the determination of a full term-structure model with bond prices for all maturities T, a discount process, and forward rates, it is necessary to choose $\gamma(t,T)$ for $0 \leq t \leq T$ and $T \in (0, T_n + \delta) \setminus \{T_1, \cdots, T_n\} = (0, T_1) \cup (T_1, T_2) \cup \cdots \cup (T_n, T_n + \delta)$. This can be done, although the model obtained by exercising these choices arbitrarily is not a reliable vehicle for pricing instruments that depend ont these choices. But anyway, the following results show how to go from $\gamma(t,T)$ to the dynamics of bond prices, the discount process, and the forward rates.

Construction of Forward LIBOR Processes. By working backward and using formula (4), we conclude

$$dL(t,T_j) = \gamma(t,T_j)L(t,T_j)\left[-\sum_{i=j+1}^{n} \frac{\delta\gamma(t,T_i)L(t,T_i)}{1+\delta L(t,T_i)}dt + d\widetilde{W}^{T_{n+1}}(t)\right],\ 0 \leq t \leq T_j,\ j = 1, \cdots, n.$$

Thus, to construct the forward LIBOR model, we choose a Brownian motion, which we call $\widetilde{W}^{T_{n+1}}(t)$, $0 \leq t \leq T_{n+1}$, under a probability measure we call $\widetilde{\mathbb{P}}^{T_{n+1}}$. We assume the initial forward LIBORs $L(0,T_j)$, $j = 1, \cdots, n+1$, are known from market data. With these initial conditions, the above equation generates the forward LIBOR processes $L(t,T_j)$ ($0 \leq t \leq T_j$) recursively, starting from $L(t,T_n)$ and backward to $L(t,T_1)$.

Risk-Neutral Measure. We study the $(T+\delta)$-forward measure $\widetilde{\mathbb{P}}^{T+\delta}$. The density process is

$$Z(t) = \mathbb{E}_t\left[\frac{d\mathbb{P}^{T+\delta}}{d\mathbb{P}}\right] = \frac{B(t,T+\delta)/B(0,T+\delta)}{1/D(t)} = \frac{B(t,T+\delta)D(t)}{B(0,T+\delta)}$$

and it satisfies the SDE

$$\frac{dZ(t)}{Z(t)} = \frac{d(D(t)B(t,T+\delta))}{D(t)B(t,T+\delta)} = \sigma_B(t,T+\delta)d\widetilde{W}(t) = -\int_t^{T+\delta}\sigma(t,v)dvd\widetilde{W}(t).$$

So $\widetilde{W}^{T+\delta}(t) = -\int_0^t \sigma_B(u,T+\delta)du + \widetilde{W}(t) = \int_0^t\int_u^{T+\delta}\sigma(u,v)dvdu + \widetilde{W}(t)$ is a Brownian motion under $\widetilde{\mathbb{P}}^{T+\delta}$.

Construction of T_j-Maturity Discounted Bond Prices. Suppose we have determined $L(t,T)$ in the above step, then formula (4) allows us to determine $\sigma_B(t,T_j)$ ($j = 2, \cdots, n$) once we have decided on the initial choice of $\sigma_B(t,T_1)$ for $0 \leq t < T_1$. Then for $j = 1, \cdots, n$, we have

$$d(D(t)B(t,T_j))/(D(t)B(t,T_j)) = \sigma_B(t,T_j)d\widetilde{W}(t) = \sigma_B(t,T_j)[\sigma_B(t,T_n+\delta)dt + d\widetilde{W}^{T_n+\delta}(t)]$$

Note this equation does not determine the discount process $D(t)$ and the bond price $B(t,T_j)$ separately.

References

[1] Steven Shreve. *Stochastic calculus for finance I: The binomial asset pricing model.* Springer-Verlag, New York, 2004. 1

[2] Steven Shreve. *Stochastic calculus for finance II: Continuous-time models.* Springer-Verlag, New York, 2004.

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