Covariance Matrix of Factors in Two-Factor Hull-White Model under Money Market Account Numeraire

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1 Problem description

Short version: we want to find the covariance matrix of all factors at various time points.

Long version: In the two-factor Hull-White short rate model, it is assumed that the short rate r follows the following dynamics under risk-neutral measure with money market account as the numeraire:

$$r(t) = X_1(t) + X_2(t) + \theta(t),\ r(0) = r_0,$$

where the processes $X_1(t)$ and $X_2(t)$ satisfy

$$\begin{cases} dX_1(t) = -\kappa_1 X_1(t)dt + \sigma_1(t)dW_1(t),\ X_1(0) = 0 \\ dX_2(t) = -\kappa_2 X_2(t)dt + \sigma_2(t)dW_2(t),\ X_2(0) = 0 \end{cases}$$

where (W_1, W_2) is a two-dimensional Brownian motion with instantaneous correlation ρ:

$$dW_1(t)dW_2(t) = \rho dt,$$

r_0, κ_1, κ_2 are positive constants, $\sigma_1(t)$ and $\sigma_2(t)$ are positive deterministic functions, and $\rho \in [-1, 1]$. The function $\theta(t)$ is a deterministic function that will be used to fit the initial yield curve.

Define $Z(t) = \int_0^t [X_1(u) + X_2(u)]du$. We are interested in the covariance matrix of the 3-dimensional random vector $(X_1(t), X_2(t), Z(t))$. More generally, for a sequence of increasing time points $t_1 < t_2 < \cdots < t_n$, we are interested in the covariance matrix of the $3n$-dimensional random vector

$$(X_1(t_1), X_2(t_1), Z(t_1), \cdots, X_1(t_n), X_2(t_n), Z(t_n)).$$

This problem arises naturally from the need to price path-dependent interest rate derivatives under two-factor Hull-White model, using Monte-Carlo simulation. We shall first solve this problem in the setting of one-factor Hull-White model, since the notation is easier; then we extend the solution to two-factor case.

2 One-factor case

In the one-factor case, we have $X(t) = e^{-\kappa t}\int_0^t e^{\kappa u}\sigma(u)dW(u)$. Then $X(t)$ is Gaussian with mean and variance

$$E[X(t)] = 0,\ E[X^2(t)] = e^{-2\kappa t}\int_0^t e^{2\kappa u}\sigma(u)du.$$

We further define

$$\begin{cases} h(t) = e^{-\kappa t} \\ H(t) = \int_0^t h(u)du \\ \nu(t) = e^{-2\kappa t}\int_0^t e^{2\kappa u}\sigma^2(u)du \\ \nu^h(t) = h * \nu(t) = \int_0^t e^{-\kappa(t-u)}\nu(u)du \\ \nu^H(t) = H * \nu(t) = \int_0^t H(t-u)\nu(u)du \end{cases}$$

Proposition 2.1. *The pair $(X(t), Z(t))$ is jointly Gaussian with mean 0 and covariance matrix*

$$\Sigma(t) = \begin{pmatrix} \nu(t) & \nu^h(t) \\ \nu^h(t) & 2\nu^H(t) \end{pmatrix}$$

Proof. It's easy to see $X(t)$ and $Z(t)$ are jointly Gaussian and have zero mean. Also, $E[X^2(t)] = \nu(t)$. By integration-by-parts formula, we have

$$X(t)Z(t) = \int_0^t Z(u)dX(u) + \int_0^t X^2(u)du = -\kappa \int_0^t Z(u)X(u)du + \int_0^t X^2(u)du + \text{martingale part.}$$

Define $c_{XZ}(t) = E[X(t)Z(t)]$. Taking expectation on both sides gives

$$c_{XZ}(t) = -\kappa \int_0^t c_{XZ}(u)du + \int_0^t \nu(u)du$$

Solving this integral equation gives

$$c_{XZ}(t) = e^{-\kappa t} \int_0^t e^{\kappa u} \nu(u)du = \nu^h(t).$$

Finally, we note

$$\frac{d}{dt} E[Z^2(t)] = 2E[Z(t)X(t)] = 2c_{XZ}(t) = 2\nu^h(t),$$

so we have

$$E[Z^2(t)] = 2 \int_0^t \nu^h(u)du = 2\nu^H(t).$$

In summary, the covariance matrix of the pair $(X(t), Z(t))$ is

$$\Sigma(t) = \begin{pmatrix} \nu(t) & \nu^h(t) \\ \nu^h(t) & 2\nu^H(t) \end{pmatrix}$$

□

Proposition 2.2. *Suppose $0 \le s \le t$, we have the following relation:*

$$\begin{bmatrix} X(t) \\ Z(t) \end{bmatrix} = \Gamma(t-s) \begin{bmatrix} X(s) \\ Z(s) \end{bmatrix} + \Lambda(s,t) \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix},$$

where $\Gamma(t) = \begin{bmatrix} h(t) & 0 \\ H(t) & 1 \end{bmatrix}$, (w_1, w_2) *is a 2-dimensional standard Gaussian random vector independent of the σ-field $\mathcal{F}_s$, and $\Lambda(s,t)$ is such that*

$$\Lambda(s,t)\Lambda^T(s,t) = \begin{bmatrix} \int_s^t h^2(t-u)\sigma^2(u)du & \int_s^t h(t-u)H(t-u)\sigma^2(u)du \\ \int_s^t h(t-u)H(t-u)\sigma^2(u)du & \int_s^t H^2(t-u)\sigma^2(u)du \end{bmatrix}.$$

Proof. We note

$$X(t) = e^{-\kappa(t-s)}X(s) + e^{-\kappa t} \int_s^t e^{\kappa u}\sigma(u)dW(u) = h(t-s)X(s) + \int_s^t h(t-u)\sigma(u)dW(u).$$

Then $\xi(s,t) := \int_s^t h(t-u)\sigma(u)dW(u)$ is a Gaussian random variable independent of the σ-field $\mathcal{F}_s$, has mean 0, and has variance $\int_s^t h^2(t-u)\sigma^2(u)du$.

We also note

$$\begin{aligned}
Z(t) - Z(s) &= \int_s^t X(u)du = -\frac{1}{\kappa}\int_s^t X(u)e^{\kappa u}de^{-\kappa u} = -\frac{1}{\kappa}\left[X(t) - X(s) - \int_s^t e^{-\kappa u}d(X(u)e^{\kappa u})\right] \\
&= -\frac{1}{\kappa}\left[X(t) - X(s) - \int_s^t \sigma(u)dW(u)\right] \\
&= -\frac{1}{\kappa}\left[(h(t-s)-1)X(s) + \int_s^t h(t-u)\sigma(u)dW(u) - \int_s^t \sigma(u)dW(u)\right] \\
&= H(t-s)X(s) + \int_s^t H(t-u)\sigma(u)dW(u).
\end{aligned}$$

This derivation relies on the assumption that $\kappa \neq 0$. If $\kappa = 0$, we can easily verify the same formula holds. Therefore,

$$Z(t) = H(t-s)X(s) + Z(s) + \eta(s,t),$$

where $\eta(s,t) := \int_s^t H(t-u)\sigma(u)dW(u)$ is a Gaussian random variable independent of the σ-field $\mathcal{F}_s$, has mean 0, and has variance $\int_s^t H^2(t-u)\sigma^2(u)du$.

We finally note

$$E[\xi(s,t)\eta(s,t)] = E\left[\int_s^t h(t-u)\sigma(u)dW(u)\int_s^t H(t-u)\sigma(u)dW(u)\right] = \int_s^t h(t-u)H(t-u)\sigma^2(u)du.$$

So the pair $(\xi(s,t), \eta(s,t))$ is jointly Guassian with mean 0 and covariance matrix

$$\begin{bmatrix} \int_s^t h^2(t-u)\sigma^2(u)du & \int_s^t h(t-u)H(t-u)\sigma^2(u)du \\ \int_s^t h(t-u)H(t-u)\sigma^2(u)du & \int_s^t H^2(t-u)\sigma^2(u)du \end{bmatrix}$$

Writing everything in matrix form, we have

$$\begin{bmatrix} X(t) \\ Z(t) \end{bmatrix} = \begin{bmatrix} h(t-s) & 0 \\ H(t-s) & 1 \end{bmatrix}\begin{bmatrix} X(s) \\ Z(s) \end{bmatrix} + \begin{bmatrix} \xi(s,t) \\ \eta(s,t) \end{bmatrix}$$

Choose a matrix $\Lambda(s,t)$ such that $\Lambda(s,t)\Lambda^T(s,t)$ is equal to the covariance matrix of $(\xi(s,t), \eta(s,t))$ and define

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \Lambda^{-1}(s,t)\begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

Then (w_1, w_2) is a 2-dimensional standard Gaussian random vector independent of $\mathcal{F}_s$. Combining everything together, we conclude

$$\begin{bmatrix} X(t) \\ Z(t) \end{bmatrix} = \begin{bmatrix} h(t-s) & 0 \\ H(t-s) & 1 \end{bmatrix}\begin{bmatrix} X(s) \\ Z(s) \end{bmatrix} + \Lambda(s,t)\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix},$$

where (w_1, w_2) is a 2-dimensional standard Gaussian random vector and $\Lambda(s,t)$ is such that

$$\Lambda(s,t)\Lambda^T(s,t) = \begin{bmatrix} \int_s^t h^2(t-u)\sigma^2(u)du & \int_s^t h(t-u)H(t-u)\sigma^2(u)du \\ \int_s^t h(t-u)H(t-u)\sigma^2(u)du & \int_s^t H^2(t-u)\sigma^2(u)du \end{bmatrix}.$$

□

Corollary 2.1. *Suppose* $0 \le s \le t$*, the covariance matrix of* $(X(s), Z(s))$ *and* $(X(t), Z(t))$ *is*

$$E\left(\begin{bmatrix} X(t) \\ Z(t) \end{bmatrix}\begin{bmatrix} X(s) Z(s) \end{bmatrix}\right) = \Gamma(t-s)\Sigma(s)$$

where $\Sigma(s)$ *is as defined in Proposition 2.1 and* $\Gamma(t)$ *is as defined in Proposition 2.2. More generally, for a sequence of increasing time points* $t_1 < t_2 < \cdots < t_n$*, the covariance matrix of the* $2n$*-dimensional random vector*

$$(X(t_1), Z(t_1), \cdots, X(t_n), Z(t_n))$$

is

$$E\left(\begin{bmatrix}X(t_1)\\ Z(t_1)\\ \vdots\\ X(t_n)\\ Z(t_n)\end{bmatrix}\begin{bmatrix}X(t_1)Z(t_1)\cdots X(t_n)Z(t_n)\end{bmatrix}\right)$$

$$= \begin{bmatrix}\Sigma(t_1) & \Gamma(t_2-t_1)\Sigma(t_1) & \Gamma(t_3-t_1)\Sigma(t_1) & \cdots & \Gamma(t_n-t_1)\Sigma(t_1)\\ \Gamma(t_2-t_1)\Sigma(t_1) & \Sigma(t_2) & \Gamma(t_3-t_2)\Sigma(t_2) & \cdots & \Gamma(t_n-t_2)\Sigma(t_2)\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \Gamma(t_n-t_1)\Sigma(t_1) & \Gamma(t_n-t_2)\Sigma(t_2) & \Gamma(t_n-t_3)\Sigma(t_3) & \cdots & \Sigma(t_n)\end{bmatrix}$$

Proof. Apply the result of Proposition 2.2 and use multiplication of block matrices to make the notation clean. □

3 Two-factor case

To simplify notation, we define for $i, j \in \{1, 2\}$

$$\begin{bmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{bmatrix} = \begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}$$

and

$$\begin{cases} h_i(t) = e^{-\kappa_i t}\\ H_i(t) = \int_0^t h_i(u)du\\ \nu_{ij}(t) = \rho_{ij}\int_0^t e^{-(\kappa_i+\kappa_j)(t-u)}\sigma_i(u)\sigma_j(u)du\\ \nu_{ij}^h(t) = \int_0^t h_i(t-u)\nu_{ij}(u)du\\ \nu_{ij}^H(t) = \int_0^t H_i(t-u)\nu_{ij}(u)du \end{cases}$$

In the two-factor case, we have

$$\begin{cases} X_1(t) = e^{-\kappa_1 t}\int_0^t e^{\kappa_1 u}\sigma_1(u)dW_1(u)\\ X_2(t) = e^{-\kappa_2 t}\int_0^t e^{\kappa_2 u}\sigma_2(u)dW_1(u) \end{cases}$$

So the random vector $(X_1(t), X_2(t))$ is jointly Gaussian, with mean 0 and covariance matrix

$$\begin{bmatrix}\nu_{11}(t) & \nu_{12}(t)\\ \nu_{21}(t) & \nu_{22}(t)\end{bmatrix}$$

Proposition 3.1. *The triple $(X_1(t), X_2(t), Z(t))$ is jointly Gaussian with mean 0 and covariance matrix*

$$\Sigma(t) = \begin{pmatrix}\nu_{11}(t) & \nu_{12}(t) & \nu_{11}^h(t)+\nu_{12}^h(t)\\ \nu_{21}(t) & \nu_{22}(t) & \nu_{21}^h(t)+\nu_{22}^h(t)\\ \nu_{11}^h(t)+\nu_{12}^h(t) & \nu_{21}^h(t)+\nu_{22}^h(t) & 2\left[\nu_{11}^H(t)+\nu_{12}^H(t)+\nu_{21}^H(t)+\nu_{22}^H(t)\right]\end{pmatrix}$$

Proof. It is obvious that the triple is jointly Gaussian with mean 0. We first compute the covariance of $X_1(t)$ and $Z((t)$. By integration-by-parts formula, we have

$$\begin{aligned} X_1(t)Z(t) &= \int_0^t Z(u)dX_1(u) + \int_0^t (X_1^2(u) + X_1(u)X_2(u))du\\ &= -\kappa_1\int_0^t Z(u)X_1(u)du + \int_0^t (X_1^2(u) + X_1(u)X_2(u))du + \text{martingale part.} \end{aligned}$$

Define $c_{X_1Z} = E[X_1(t)Z(t)]$ and take expectation on both sides, we have

$$c_{X_1Z}(t) = -\kappa_1 \int_0^t c_{X_1Z}(u)du + \int_0^t [\nu_{11}(u) + \nu_{12}(u)]du$$

Solving this integral equation gives

$$c_{X_1Z}(t) = e^{-\kappa_1 t} \int_0^t e^{\kappa_1 u}[\nu_{11}(u) + \nu_{12}(u)]du = \nu_{11}^h(t) + \nu_{12}^h(t).$$

Similarly, we have

$$c_{X_2Z}(t) := E[X_2(t)Z(t)] = \nu_{22}^h(t) + \nu_{21}^h(t).$$

To compute the variance of $Z(t)$, we note

$$\begin{aligned} E[Z^2(t)] &= 2E\left[\int_0^t Z(s)(X_1(s) + X_2(s))ds\right] = 2\int_0^t \left[c_{X_1Z}(u) + c_{X_2Z}(u)\right] du \\ &= 2\left[\nu_{11}^H(t) + \nu_{12}^H(t) + \nu_{21}^H(t) + \nu_{22}^H(t)\right] \end{aligned}$$

Combining everything together, we conclude the covariance matrix of $(X_1(t), X_2(t), Z(t))$ is

$$\Sigma(t) = \begin{pmatrix} \nu_{11}(t) & \nu_{12}(t) & \nu_{11}^h(t) + \nu_{12}^h(t) \\ \nu_{21}(t) & \nu_{22}(t) & \nu_{21}^h(t) + \nu_{22}^h(t) \\ \nu_{11}^h(t) + \nu_{12}^h(t) & \nu_{21}^h(t) + \nu_{22}^h(t) & 2\left[\nu_{11}^H(t) + \nu_{12}^H(t) + \nu_{21}^H(t) + \nu_{22}^H(t)\right] \end{pmatrix}$$

□

Proposition 3.2. *Suppose $0 \le s \le t$, we have the following relation*

$$\begin{bmatrix} X_1(t) \\ X_2(t) \\ Z(t) \end{bmatrix} = \Gamma(t-s) \begin{bmatrix} X_1(s) \\ X_2(s) \\ Z(s) \end{bmatrix} + \Lambda(s,t) \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

where $\Gamma(t) = \begin{bmatrix} h_1(t) & 0 & 0 \\ 0 & h_2(t) & 0 \\ H_1(t) & H_2(t) & 1 \end{bmatrix}$, *$(w_1, w_2, w_3)$ is a 3-dimensional standard Gaussian random vector independent of the σ-field $\mathcal{F}_s$, and $\Lambda(s,t)$ is such that $\Lambda(s,t)\Lambda^T(s,t)$ is equal to*

$$\begin{bmatrix} a_{11}(s,t) & a_{12}(s,t) & a_{13}(s,t) \\ a_{21}(s,t) & a_{22}(s,t) & a_{23}(s,t) \\ a_{31}(s,t) & a_{32}(s,t) & a_{33}(s,t) \end{bmatrix}$$

with

$$\begin{cases} a_{11}(s,t) = \int_s^t h_1^2(t-u)\sigma_1^2(u) \\ a_{22}(s,t) = \int_s^t h_2^2(t-u)\sigma_2^2(u) \\ a_{12}(s,t) = a_{21}(s,t) = \rho \int_s^t h_1(t-u)h_2(t-u)\sigma_1(u)\sigma_2(u)du \\ a_{13}(s,t) = a_{31}(s,t) = \int_s^t h_1(t-u)H_1(t-u)\sigma_1^2(u)du + \rho \int_s^t h_1(t-u)H_2(t-u)\sigma_1(u)\sigma_2(u)du \\ a_{23}(s,t) = a_{32}(s,t) = \rho \int_s^t h_2(t-u)H_1(t-u)\sigma_1(u)\sigma_2(u)du + \int_s^t h_2(t-u)H_2(t-u)\sigma_2^2(u)du \\ a_{33}(s,t) = \int_s^t [H_1^2(t-u)\sigma_1^2(u) + 2\rho H_1(t-u)H_2(t-u)\sigma_1(u)\sigma_2(u) + H_2^2(t-u)\sigma_2^2(u)]du \end{cases}$$

Proof. Just like in the proof of Proposition 2.2, for $i = 1, 2$, we have

$$X_i(t) = h_i(t-s)X_i(s) + \int_s^t h_i(t-u)\sigma_i(u)dW_i(u),$$

and $\xi_i(s,t) := \int_s^t h_i(t-u)\sigma_i(u)dW_i(u)$ is a Gaussian random variable independent of the σ-field $\mathcal{F}_s$, has mean 0, and has variance $\int_s^t h_i^2(t-u)\sigma_i^2(u)du$.

Note $Z(t) = \int_0^t X_1(u)du + \int_0^t X_2(u)du$, by the proof of Proposition 2.2, we have

$$Z(t) - Z(s) = \sum_{i=1}^{2} \left[H_i(t-s)X_i(s) + \int_s^t H_i(t-u)\sigma_i(u)dW_i(u) \right].$$

Define $\eta(s,t) = \sum_{i=1}^2 \int_s^t H_i(t-u)\sigma_i(u)dW_i(u)$, then $\eta(s,t)$ is a Gaussian random variable independent of the σ-field $\mathcal{F}_s$, has mean 0, and has variance $\int_s^t [H_1^2(t-u)\sigma_1^2(u) + 2\rho H_1(t-u)H_2(t-u)\sigma_1\sigma_2 + H_2^2(t-u)\sigma_2^2(u)]du$.

We then have the following recurrence relation

$$\begin{bmatrix} X_1(t) \\ X_2(t) \\ Z(t) \end{bmatrix} = \begin{bmatrix} h_1(t-s) & 0 & 0 \\ 0 & h_2(t-s) & 0 \\ H_1(t-s) & H_2(t-s) & 1 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \\ Z(s) \end{bmatrix} + \begin{bmatrix} \xi_1(s,t) \\ \xi_2(s,t) \\ \eta(s,t) \end{bmatrix}$$

The random vector $(\xi_1(s,t), \xi_2(s,t), \eta(s,t))$ is jointly Gaussian with mean 0 and is independent of the σ-field $\mathcal{F}_s$. The covariance matrix of this random vector is given by

$$\begin{cases} E[\xi_1^2(s,t)] = \int_s^t h_1^2(t-u)\sigma_1^2(u) \\ E[\xi_2^2(s,t)] = \int_s^t h_2^2(t-u)\sigma_2^2(u) \\ E[\eta^2(s,t)] = \int_s^t [H_1^2(t-u)\sigma_1^2(u) + 2\rho H_1(t-u)H_2(t-u)\sigma_1(u)\sigma_2(u) + H_2^2(t-u)\sigma_2^2(u)]du \\ E[\xi_1(s,t)\xi_2(s,t)] = \rho \int_s^t h_1(t-u)h_2(t-u)\sigma_1(u)\sigma_2(u)du \\ E[\xi_1(s,t)\eta(s,t)] = \int_s^t h_1(t-u)H_1(t-u)\sigma_1^2(u)du + \rho \int_s^t h_1(t-u)H_2(t-u)\sigma_1(u)\sigma_2(u)du \\ E[\xi_2(s,t)\eta(s,t)] = \rho \int_s^t h_2(t-u)H_1(t-u)\sigma_1(u)\sigma_2(u)du + \int_s^t h_2(t-u)H_2(t-u)\sigma_2^2(u)du \end{cases}$$

Choose a matrix $\Lambda(s,t)$ such that $\Lambda(s,t)\Lambda^T(s,t)$ is equal to the covariance matrix of $(\xi_1(s,t), \xi_2(s,t), \eta(s,t))$. Then define

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \Lambda^{-1}(s,t) \begin{bmatrix} \xi_1(s,t) \\ \xi_2(s,t) \\ \eta(s,t) \end{bmatrix}$$

Then $(\omega_1, \omega_2, \omega_3)$ is a 3-dimensional standard Gaussian random vector independent of the σ-field $\mathcal{F}_s$. $\square$

Corollary 3.1. *Suppose* $0 \le s \le t$*, the covariance matrix of* $(X_1(s), X_2(s), Z(s))$ *and* $(X_1(t), X_2(t), Z(t))$ *is*

$$E\left(\begin{bmatrix} X_1(t) \\ X_2(t) \\ Z(t) \end{bmatrix} [X_1(s), X_2(s), Z(s)] \right) = \Gamma(t-s)\Sigma(s)$$

where $\Sigma(s)$ *is as defined in Proposition 3.1 and* $\Gamma(t)$ *is as defined in Proposition 3.2. More generally, for a sequence of increasing time points* $t_1 < t_2 < \cdots < t_n$*, the covariance matrix of the* $3n$*-dimensional random vector*

$$(X_1(t_1), X_2(t_1), Z(t_1), \cdots, X_1(t_n), X_2(t_n), Z(t_n))$$

is

$$E\left(\begin{bmatrix} X_1(t_1) \\ X_2(t_1) \\ Z(t_1) \\ \vdots \\ X_1(t_n) \\ X_2(t_n) \\ Z(t_n) \end{bmatrix} \left[X_1(t_1) X_2(t_1) Z(t_1) \cdots X_1(t_n) X_2(t_n) Z(t_n) \right] \right)$$

$$= \begin{bmatrix} \Sigma(t_1) & \Gamma(t_2-t_1)\Sigma(t_1) & \Gamma(t_3-t_1)\Sigma(t_1) & \cdots & \Gamma(t_n-t_1)\Sigma(t_1) \\ \Gamma(t_2-t_1)\Sigma(t_1) & \Sigma(t_2) & \Gamma(t_3-t_2)\Sigma(t_2) & \cdots & \Gamma(t_n-t_2)\Sigma(t_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Gamma(t_n-t_1)\Sigma(t_1) & \Gamma(t_n-t_2)\Sigma(t_2) & \Gamma(t_n-t_3)\Sigma(t_3) & \cdots & \Sigma(t_n) \end{bmatrix}$$

Proof. Apply the result of Proposition 3.2 and use multiplication of block matrices to make the notation clean. □

References

[1] D. Brigo and F. Mercurio. *Interest rate models - theory and practice: With smile, inflation and credit.* Second Edition, Springer, 2007.