Convexity Adjustment: A User's Guide

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Abstract

Elements of convexity adjustment.

Contents

1 Introduction

In this note, we summarize various results on convexity adjustment. The exposition is based on Boenkost and Schmidt [1], [2], Hagan [3], Hull [5], Hunt and Kennedy [4], Lesniewski [6], Pelsser [8], and Piza [9].

Denote by $P(t,T)$ $(0 \le t \le T)$ the time-t value of a zero coupon bond with maturity T. $\tau(S,T)$ is the year fraction between time S and time T $(S < T)$. The *simply-compounded forward interest rate* $F(t;S,T)$ is defined as

$$F(t;S,T) = \frac{1}{\tau(S,T)}\left(\frac{P(t,S)}{P(t,T)} - 1\right).$$

Suppose $T_\alpha < T_{\alpha+1} < \cdots < T_\beta$ is a set of future times such that the LIBOR rate is reset at $T_\alpha, \cdots, T_{\beta-1}$ and is paid at $T_{\alpha+1}, \cdots, T_\beta$ for a floating-rate note. The *forward swap rate* $S_{\alpha,\beta}(t)$ at time t for the set of times $\mathcal{T} = \{T_\alpha, T_{\alpha+1}, \cdots, T_\beta\}$ $(t \le T_\alpha)$ and year fractions $\tau = \{\tau_{\alpha+1}, \cdots, \tau_\beta\}$ $(\tau_i = \tau(T_{i-1}, T_i))$ is defined as

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)}.$$

These two rates often appear as the underlyings in interest rate derivatives, and will serve as the prototype for convexity adjustment.

2 Convexity adjusted interest rates

2.1 LIBOR

The LIBOR rate $L(S,T) = F(S;S,T)$ for the interval $[S,T]$ is given by

$$L(S,T) = \frac{1}{\tau(S,T)}\left(\frac{1}{P(S,T)} - 1\right).$$

Under the forward measure Q_T for which $P(\cdot,T)$ is the numeraire, $F(t;S,T)$ is a martingale and therefore $E^{Q_T}[L(S,T)] = F(0;S,T)$. This leads to the pricing formula of a floater, which resets LIBOR at time S and makes payment at time T.

2.1.1 LIBOR-in-arrears

For LIBOR-in-arrears, we need to evaluate $E^{Q_S}[L(S,T)]$, where Q_S is the forward measure for which $P(\cdot,S)$ is the numeraire. The goal is to express $E^{Q_S}[L(S,T)]$ in terms of the forward rate $F(0;S,T)$ plus some "convexity" adjustment (recall $E^{Q_T}[L(S,T)] = F(0;S,T)$):

$$\begin{aligned}
E^{Q_S}[L(S,T)] &= E^{Q_T}\left[L(S,T)\frac{P(S,S)/P(0,S)}{P(S,T)/P(0,T)}\right] \\
&= E^{Q_T}\left[L(S,T)\cdot(1+\tau(S,T)L(S,T))\cdot\frac{P(0,T)}{P(0,S)}\right] \\
&= E^{Q_T}\left[L(S,T)\cdot\frac{1+\tau(S,T)L(S,T)}{1+\tau(S,T)F(0;S,T)}\right] \\
&= \frac{F(0;S,T)+\tau(S,T)E^{Q_T}[L^2(S,T)]}{1+\tau(S,T)F(0;S,T)}
\end{aligned}$$

Note $E^{Q_T}[L^2(S,T)] = \mathrm{Var}_{Q_T}(L(S,T)) + (E^{Q_T}[L(S,T)])^2$, we conclude

$$\boxed{E^{Q_S}[L(S,T)] = F(0;S,T) + \frac{\tau(S,T)\mathrm{Var}_{Q_T}(L(S,T))}{1+\tau(S,T)F(0;S,T)}} \tag{1}$$

Under the so-called market model which is the model underlying the market valuation for caps, the LIBOR $L(S,T)$ is lognormal under Q_T with volatility σ,

$$L(S,T) = F(S;S,T) = F(0;S,T)\exp\left\{\sigma W_S - \frac{1}{2}\sigma^2 S\right\},$$

where W is a standard Brownian motion. In this case, $\mathrm{Var}_{Q_T}(L(S,T)) = F^2(0;S,T)(e^{\sigma^2 S} - 1)$ and formula (1) becomes

$$E^{Q_S}[L(S,T)] = F(0;S,T)\left[1 + \frac{\tau(S,T)F(0;S,T)(e^{\sigma^2 S} - 1)}{1 + \tau(S,T)F(0;S,T)}\right]. \tag{2}$$

2.1.2 LIBOR paid at arbitrary time under the linear rate model

Suppose the payment is made at an arbitrary time $T' \in [S,T]$. This is the case of Asian floater, where S and T are the starting time and ending time of a coupon period, respectively. Then

$$E^{Q_{T'}}[L(S,T)] = E^{Q_T}\left[\frac{P(S,T')/P(0,T')}{P(S,T)/P(0,T)}L(S,T)\right]$$

The *linear rate model* assumes

$$\frac{P(S,T')}{P(S,T)} = a + b(T')L(S,T),\ \forall T' \in [S,T]$$

which requires $a = 1$ by setting $T' = T$. This is effectively equivalent to assuming

$$L(T',T) = \frac{b(T')}{\tau(T',T)}L(S,T),\ \forall T' \in [S,T].$$

Moreover, the martingale property dictates

$$\frac{P(0,T')}{P(0,T)} = E^{Q_T}\left[\frac{P(S,T')}{P(S,T)}\right] = a + b(T')F(0;S,T).$$

So we have $b(T') = \left(\frac{P(0,T')}{P(0,T)} - 1\right)/F(0;S,T) = \frac{\tau(T',T)F(0;T',T)}{F(0;S,T)}$. In summary, the **linear rate model** assumes

$$\boxed{\frac{L(T',T)}{L(S,T)} = \frac{F(0;T',T)}{F(0;S,T)},\ \forall T' \in [S,T]} \tag{3}$$

which can be summarized in words as

> *The ratio of LIBOR rates over the interval $[T',T]$ and $[S,T]$ is equal to the ratio of time-zero forward rates over the same intervals.*

Note the case of LIBOR-in-arrears, where $T' = S$, satisfies the assumption of linear rate model.

Under the linear rate model assumption, we easily deduce that

$$\boxed{E^{Q_{T'}}[L(S,T)] = F(0;S,T)\left[1 + \frac{1 - P(0,T)/P(0,T')}{F^2(0;S,T)}\mathrm{Var}_{Q_T}(L(S,T))\right],\ \forall T' \in [S,T]} \tag{4}$$

Remark 1. *The original motivation for the linear rate model is probably the consideration that the "natural rate" under T-forward measure Q_T is $L(S,T)$. So one would like to use $L(S,T)$ to approximate $P(S,T')/P(S,T)$, and linear function is obviously the simplest. This idea can be generalized to that of making the Radon-Nikodym derivative a function of the payout rate.*

Remark 2. *For $T' = S$, formula (4) reduces to formula (1).*

Under the market model where $L(S,T)$ is lognormal under Q_T with volatility σ,

$$L(S,T) = F(0;S,T)e^{\sigma W_S - \frac{1}{2}\sigma^2 S}$$

and formula (4) becomes more explicit:

$$E^{Q_{T'}}[L(S,T)] = F(0;S,T)\left[1 + \left(1 - \frac{P(0,T)}{P(0,T')}\right)(e^{\sigma^2 S} - 1)\right].$$

2.2 CMS

From the definition of forward swap rate $S_{\alpha,\beta}(t)$, if we choose the annuity $N_t^{\alpha,\beta} = \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)$ as numeraire and denote by $Q^{\alpha,\beta}$ the associated martingale measure (the "swap measure"), we have by martingale property

$$E^{Q^{\alpha,\beta}}\left[S_{\alpha,\beta}(T_\alpha)\right] = S_{\alpha,\beta}(0).$$

If the payment is to be paid at some time $T' > T_\alpha$, we need to compute under the T'-forward measure $Q_{T'}$

$$E^{Q_{T'}}\left[S_{\alpha,\beta}(T_\alpha)\right] = E^{Q^{\alpha,\beta}}\left[\frac{P(T_\alpha,T')/P(0,T')}{N_{T_\alpha}^{\alpha,\beta}/N_0^{\alpha,\beta}} S_{\alpha,\beta}(T_\alpha)\right] = \frac{N_0^{\alpha,\beta}}{P(0,T')} E^{Q^{\alpha,\beta}}\left[\frac{P(T_\alpha,T')}{N_{T_\alpha}^{\alpha,\beta}} S_{\alpha,\beta}(T_\alpha)\right].$$

The goal is to express $E^{Q_{T'}}\left[S_{\alpha,\beta}(T_\alpha)\right]$ in terms of the time-zero swap rate $S^{\alpha,\beta}(0)$ plus some "convexity" adjustment.

2.2.1 CMS paid at arbitrary time under the linear swap rate model

Under the swap measure $Q^{\alpha,\beta}$ associated with the annuity numeraire $N_t^{\alpha,\beta} = \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)$, the entity most convenient for computation is the swap rate $S_{\alpha,\beta}(T_\alpha)$. Therefore, a natural assumption for the so-called *linear swap rate model* is

$$\frac{P(T_\alpha,T')}{N_{T_\alpha}^{\alpha,\beta}} = a + b(T')S_{\alpha,\beta}(T_\alpha),\ T' \geq T_\alpha.$$

To determine a and b, we first take expectation of both sides under the swap measure and use the martingale property to get

$$\frac{P(0,T')}{N_0^{\alpha,\beta}} = a + b(T')S_{\alpha,\beta}(0).$$

This gives $b(T') = \frac{1}{S_{\alpha,\beta}(0)}\left[\frac{P(0,T')}{N_0^{\alpha,\beta}} - a\right]$. To deduce the second equation for a and b, we note

$$1 \;=\; \frac{\sum_{i=\alpha+1}^{\beta}\tau_i P(T_\alpha,T_i)}{N_{T_\alpha}^{\alpha,\beta}} = \sum_{i=\alpha+1}^{\beta} \tau_i\left[a + b(T_i)S_{\alpha,\beta}(T_\alpha)\right] = a\left(1 - \frac{S_{\alpha,\beta}(T_\alpha)}{S_{\alpha,\beta}(0)}\right)\sum_{i=\alpha+1}^{\beta}\tau_i + \frac{S_{\alpha,\beta}(T_\alpha)}{S_{\alpha,\beta}(0)}.$$

Therefore, we can solve for a: $a = \frac{1}{\sum_{i=\alpha+1}^{\beta}\tau_i}$. In summary, the **linear swap rate model** makes the assumption

$$\begin{cases} \frac{P(T_\alpha,T')}{N_{T_\alpha}^{\alpha,\beta}} = a + b(T')S_{\alpha,\beta}(T_\alpha),\ T' \geq T_\alpha \\ a = \frac{1}{\sum_{i=\alpha+1}^{\beta}\tau_i} \\ b(T') = \frac{1}{S_{\alpha,\beta}(0)}\left[\frac{P(0,T')}{N_0^{\alpha,\beta}} - \frac{1}{\sum_{i=\alpha+1}^{\beta}\tau_i}\right],\ T' \geq T_\alpha \end{cases} \tag{5}$$

and consequently

$$\boxed{E^{Q_{T'}}\left[S_{\alpha,\beta}(T_\alpha)\right] = S_{\alpha,\beta}(0)\left[1 + \frac{1 - \frac{P(0,T_\alpha)-P(0,T_\beta)}{S_{\alpha,\beta}(0)P(0,T')\sum_{i=\alpha+1}^{\beta}\tau_i}}{S^2_{\alpha,\beta}(0)}\mathrm{Var}_N(S_{\alpha,\beta}(T_\alpha))\right]} \tag{6}$$

where $\mathrm{Var}_N(S_{\alpha,\beta}(T_\alpha))$ is the variance of the swap rate $S_{\alpha,\beta}(T_\alpha)$ under the swap measure $Q^{\alpha,\beta}$.

Under the so-called market model for swpation, it's assumed the swap rate $S_{\alpha,\beta}(t)$ satisfies

$$dS_{\alpha,\beta}(t) = \sigma_{\alpha,\beta}S_{\alpha,\beta}(t)dW_t^{\alpha,\beta},\ t \le T_\alpha$$

where $W^{\alpha,\beta}$ is a standard Brownian motion under the swap measure $Q^{\alpha,\beta}$. The variance of the swap rate $S_{\alpha,\beta}(T_\alpha)$ under the swap measure is therefore $S^2_{\alpha,\beta}(0)\left(e^{\sigma^2_{\alpha,\beta}T_\alpha}-1\right)$. Then

$$E^{Q_{T'}}\left[S_{\alpha,\beta}(T_\alpha)\right] = S_{\alpha,\beta}(0)\left[1 + \left(1 - \frac{P(0,T_\alpha)-P(0,T_\beta)}{S_{\alpha,\beta}(0)P(0,T')\sum_{i=\alpha+1}^{\beta}\tau_i}\right)\left(e^{\sigma^2_{\alpha,\beta}T_\alpha}-1\right)\right].$$

Remark 3. *The linear rate model for Libor and CMS can be generalized as follows. Write Y_S for a floating rate which is set at time S. Let N, Q_N denote the natural ("market") numeraire pair associated with Y_S and all we need is*

$$E^{Q_N}[Y_S] = Y_0,$$

where Y_0 is known and a function of the yield curve $P(0,\cdot)$ today.

We are interested in today's price of the rate Y_S to be paid at some time $T' \ge S$,

$$P(0,T')E^{Q_{T'}}[Y_S] = N_0E^{Q_N}\left[\frac{P(S,T')}{N_S}Y_S\right].$$

Assume a linear rate model of the form

$$\boxed{\frac{P(S,T')}{N_S} = a + b(T')Y_S} \tag{7}$$

with some deterministic a, $b(T')$ which have to be determined accordingly to make the model consistent. We then have

$$\boxed{E^{Q_{T'}}[Y_S] = Y_0\left[1 + \frac{b(T')}{Y_0(a+b(T')Y_0)}Var_{Q_N}(Y_S)\right]} \tag{8}$$

If in addition, the distribution of Y_S under Q_N is lognormal with volatility σ_Y: $Y_S = Y_0e^{\sigma_Y W_S - \frac{1}{2}\sigma_Y^2 S}$, then

$$E^{Q_{T'}}[Y_S] = Y_0\left[1 + \frac{b(T')Y_0}{a+b(T')Y_0}(e^{\sigma_Y^2 S}-1)\right].$$

Under the linear rate model for Libor, $Y_S = L(S,T)$ and $N_S = P(S,T)$; under the linear rate mode for CMS, $Y_S = S_{\alpha,\beta}(T_\alpha)$ and $N_S = N^{\alpha,\beta}_{T_\alpha}$.

As a last comment, the linear approximation of linear rate model does seem very crude at first, but can be justified by the following argument. Convexity corrections only become sizeable for large maturities. However, for large maturities the term structure almost moves in parallel. Hence, a change in the level of the long end of the curve is well described by the rate Y. Furthermore, for parallel moves in the curve, the ratio $\frac{P(S,T')}{N_S}$ is closely approximated by a linear function of Y, which is exactly what the linear rate model does. Hence, exactly for long maturities the assumptions of the linear rate model become quite accurate. This leads to a good approximation of the convexity correction for long maturities.

2.2.2 CMS paid at arbitrary time under Hagan's model

As seen in the previous section, the key to the convexity adjustment involving CMS rate, is to express $\frac{P(T_\alpha,T')}{N^{\alpha,\beta}_{T_\alpha}}$ as a function of swap rate $S_{\alpha,\beta}(T_\alpha)$:

$$\frac{P(T_\alpha,T')}{N^{\alpha,\beta}_{T_\alpha}} = G(S_{\alpha,\beta}(T_\alpha)).$$

In this section, we explain several such models proposed by Hagan [3].

Model 1: Standard model. The standard method for computing convexity corrections uses bond math approximations: payments are discounted at a flat rate, and the coverage (day count fraction) for each period is assumed to be $1/q$, where q is the number of periods per year. At any date $t \le T_\alpha$, the annuity is approximated by

$$N^{\alpha,\beta}_t = P(t,T_\alpha)\sum_{i=\alpha+1}^{\beta}\tau_i\frac{P(t,T_i)}{P(t,T_\alpha)} \approx P(t,T_\alpha)\sum_{j=1}^{\beta-\alpha}\frac{1/q}{[1+S_{\alpha,\beta}(t)/q]^j} = \frac{P(t,T_\alpha)}{S_{\alpha,\beta}(t)}\left[1-\frac{1}{(1+S_{\alpha,\beta}(t)/q)^n}\right]$$

Here the forward swap rate $S_{\alpha,\beta}(t)$ is used as the discount rate, since it represents the average rate over the life of the reference swap. In a similar spirit, the zero coupon bond for the pay date T' is approximated as

$$P(t,T') \approx \frac{P(t,T_\alpha)}{(1+S_{\alpha,\beta}(t)/q)^\Delta}$$

where $\Delta = \frac{T'-T_\alpha}{T_{\alpha+1}-T_\alpha}$. Combined, the standard "bond math model" leads to the approximation

$$\boxed{G(S_{\alpha,\beta}(t)) = \frac{P(t,T')}{N^{\alpha,\beta}_t} \approx \frac{S_{\alpha,\beta}(t)}{(1+S_{\alpha,\beta}(t)/q)^\Delta}\cdot\frac{1}{1-\frac{1}{(1+S_{\alpha,\beta}(t)/q)^n}}}$$

Model 2: "Exact yield" model. We can account for the reference swap's schedule and day count exactly by approximating

$$\frac{P(t,T_i)}{P(t,T_\alpha)} \approx \prod_{j=\alpha+1}^{i}\frac{1}{1+\tau_j S_{\alpha,\beta}(t)}$$

and

$$P(t,T') \approx \frac{P(t,T_\alpha)}{(1+\tau_{\alpha+1}S_{\alpha,\beta}(t))^\Delta}$$

where $\Delta = \frac{T'-T_\alpha}{T_{\alpha+1}-T_\alpha}$. Therefore

$$N^{\alpha,\beta}_t \approx P(t,T_\alpha)\sum_{i=\alpha+1}^{\beta}\tau_i\left(\prod_{j=\alpha+1}^{i}\frac{1}{1+\tau_j S_{\alpha,\beta}(t)}\right) = \frac{P(t,T_\alpha)}{S_{\alpha,\beta}(t)}\left(1-\prod_{i=\alpha+1}^{\beta}\frac{1}{1+\tau_i S_{\alpha,\beta}(t)}\right)$$

and

$$\boxed{G(S_{\alpha,\beta}(t)) = \frac{P(t,T')}{N^{\alpha,\beta}_t} \approx \frac{S_{\alpha,\beta}(t)}{(1+\tau_{\alpha+1}S_{\alpha,\beta}(t))^\Delta}\frac{1}{1-\prod_{i=\alpha+1}^{\beta}\frac{1}{1+\tau_i S_{\alpha,\beta}(t)}}}$$

This approximates the yield curve as flat and only allows parallel shifts, but has the schedule right.

Model 3: Parallel shifts. This model takes into account the initial yield curve shape, which can be significant in steep yield curve environments. We still only allow parallel yield curve shifts, so we approximate

$$\frac{P(t,T_i)}{P(t,T_\alpha)} = \frac{P(0,T_i)}{P(0,T_\alpha)}e^{-(T_i-T_\alpha)s}$$

where s is the amount of the parallel shift to be determined. To determine s, note

$$N_t^{\alpha,\beta} = P(t,T_\alpha) \sum_{i=\alpha+1}^{\beta} \tau_i \frac{P(t,T_i)}{P(t,T_\alpha)} = P(t,T_\alpha) \sum_{i=\alpha+1}^{\beta} \tau_i \frac{P(0,T_i)}{P(0,T_\alpha)} e^{-(T_i-T_\alpha)s}$$

and hence

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{N_t^{\alpha,\beta}} = \frac{P(0,T_\alpha) - P(0,T_\beta)e^{-(T_\beta-T_\alpha)s}}{\sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i)e^{-(T_i-T_\alpha)s}}.$$

This equation implicitly determines s as a function of $S_{\alpha,\beta}(t)$. Therefore

$$\frac{N_t^{\alpha,\beta}}{P(t,T_\alpha)} = \sum_{i=\alpha+1}^{\beta} \tau_i \frac{P(0,T_i)}{P(0,T_\alpha)} e^{-(T_i-T_\alpha)s} = \frac{1 - \frac{P(0,T_\beta)}{P(0,T_\alpha)} e^{-(T_\beta-T_\alpha)s}}{S_{\alpha,\beta}(t)}$$

and

$$\boxed{G(S_{\alpha,\beta}(t)) = \frac{P(t,T')/P(t,T_\alpha)}{N_t^{\alpha,\beta}/P(t,T_\alpha)} = \frac{e^{-(T'-T_\alpha)s}}{\left[1 - \frac{P(0,T_\beta)}{P(0,T_\alpha)} e^{-(T_\beta-T_\alpha)s}\right]/S_{\alpha,\beta}(t)} = \frac{S_{\alpha,\beta}(t)e^{-(T'-T_\alpha)s}}{1 - \frac{P(0,T_\beta)}{P(0,T_\alpha)} e^{-(T_\beta-T_\alpha)s}}}$$

where s is determined implicity in terms of $S_{\alpha,\beta}(t)$, by

$$S_{\alpha,\beta}(t) \sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i) e^{-(T_i-T_\alpha)s} = P(0,T_\alpha) - P(0,T_\beta)e^{-(T_\beta-T_\alpha)s}.$$

This model's limitations are that it allows only parallel shifts of the yield curve and it presumes perfect correlation between long and short term rates.

Model 4: Non-parallel shifts. We can allow non-parallel shifts by approximating

$$\frac{P(t,T_i)}{P(t,T_\alpha)} = \frac{P(0,T_i)}{P(0,T_\alpha)} e^{-[h(T_i)-h(T_\alpha)]s}$$

Then similar to Model 3, we have

$$\boxed{G(S_{\alpha,\beta}(t)) = \frac{S_{\alpha,\beta}(t)e^{-[h(T')-h(T_\alpha)]s}}{1 - \frac{P(0,T_\beta)}{P(0,T_\alpha)} e^{-[h(T_\beta)-h(T_\alpha)]s}}}$$

where s is determined implicity in terms of $S_{\alpha,\beta}(t)$, by

$$S_{\alpha,\beta}(t) \sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i) e^{-[h(T_i)-h(T_\alpha)]s} = P(0,T_\alpha) - P(0,T_\beta)e^{-[h(T_\beta)-h(T_\alpha)]s}.$$

To complete the model, we need to select the function $h(\cdot)$ which determines the shape of the non-parallel shift. This is often done by postulating a constant mean reversion

$$h((T) - h(T_\alpha) = \frac{1}{\kappa}\left[1 - e^{-\kappa(T-T_\alpha)}\right].$$

Alternatively, one can choose $h(\cdot)$ by calibrating the vanilla swaptions which have the same start date T_α and varying end dates to their market prices.

In either case, under the assumption $\frac{P(T_\alpha,T')}{N_{T_\alpha}^{\alpha,\beta}} = G(S_{\alpha,\beta}(T_\alpha))$, we have

$$\boxed{E^{Q_{T'}}[S_{\alpha,\beta}(T_\alpha)] = S_{\alpha,\beta}(0) + E^{Q_{\alpha,\beta}}\left\{\left[\frac{G(S_{\alpha,\beta}(T_\alpha))}{G(S_{\alpha,\beta}(0))} - 1\right] S_{\alpha,\beta}(T_\alpha)\right\}} \tag{9}$$

2.3 Hull's approach to convexity adjustment (LIBOR-in-arrears)

This section is based on Hull [5], Chapter 20. We recall the relation between forward LIBOR rate $F(t; S, T)$ and zero coupon bond price $P(t, \cdot)$ is given by

$$\frac{P(t,T)}{P(t,S)} = \frac{1}{1+\tau(S,T)F(t;S,T)}.$$

Write y_t for $F(t; S, T)$ and define $G(y) = \frac{1}{1+\tau(S,T)y}$. Then Taylor expansion gives

$$\frac{P(t,T)}{P(t,S)} = G(y_t) \approx G(y_0) + G'(y_0)(y_t - y_0) + \frac{1}{2}G''(y_0)(y_t - y_0)^2$$

Under the S-forward measure Q_S, $E^{Q_S}\left[\frac{P(t,T)}{P(t,S)}\right] = \frac{P(0,T)}{P(0,S)} = G(y_0)$. So taking expectation of both sides of the Taylor expansion, we have $G(y_0) \approx G(y_0) + G'(y_0)\left(E^{Q_S}[y_t] - y_0\right) + \frac{1}{2}G''(y_0)E^{Q_S}[(y_t - y_0)^2]$. This gives

$$E^{Q_S}[y_t] \approx y_0 - \frac{1}{2}\frac{G''(y_0)}{G'(y_0)}E^{Q_S}[(y_t - y_0)^2].$$

Let $t = S$ and approximate $E^{Q_S}[(y_S - y_0)^2]$ by $\sigma^2 y_0^2 S$ with σ the volatility of y. We then have

$$E^{Q_S}[L(S,T)] \approx F(0;S,T)\left[1 + \frac{\tau(S,T)F(0;S,T)\sigma^2 S}{1+\tau(S,T)F(0;S,T)}\right]$$

This is the first order approximation of convexity adjustment formula (2). Note the approximation of $E^{Q_S}[(y_S - y_0)^2]$ by $\sigma^2 y_0^2 S$ is more or less equivalent to assuming y_S is lognormally distributed.

2.4 Option on interest rates paid at arbitrary time under linear rate model

In this section, we investigate European options on interest rates like LIBOR $L(S, T)$ for period $[S, T]$ or CMS rates $S_{\alpha,\beta}(T_\alpha)$ for tenure structure $\mathcal{T} = \{T_\alpha, T_{\alpha+1}, \cdots, T_\beta\}$. The payment date of the option is an arbitrary time point T' with $T' \geq S$ or $T' \geq T_\alpha$, respectively.

Of particular interest are caps and floors or binaries. For standard caps and floors on LIBOR, we have $T' = T$ and the standard market model postulates a lognormal distribution of $L(S, T)$ under the forward measure Q_T. For standard option on a swap rate $S_{\alpha,\beta}(T_\alpha)$, i.e. swaptions, the market uses a lognormal distribution for $S_{\alpha,\beta}(T_\alpha)$ under the swap measure $Q^{\alpha,\beta}$. However in the general case, i.e. for options on LIBOR or CMS with arbitrary payment date T', a lognormal model would be inconsistent with the market model for standard options. For example, if $L(S, T)$ is lognormal under Q_T, it cannot be lognormal under Q_S in general:

$$E^{Q_T}[f(L(S,T))] = \frac{P(0,S)}{P(0,T)}E^{Q_S}\left[\frac{f(L(S,T))}{1+\tau(S,T)L(S,T)}\right].$$

We follow the general setup of linear rate model. Y_S is a floating interest rate which is set at time S and (N, Q_N) denotes the "market" numeraire pair associated with Y_S. We assume that the distribution of Y_S under Q_N is lognormal with volatility σ_Y,

$$Y_S = Y_0 \exp\left\{\sigma_Y W_S - \frac{1}{2}\sigma_Y^2 S\right\},$$

where W is a standard Brownian motion. For a payment date $T' \geq S$, we further assume a linear rate model of the form (7)

$$\frac{P(S,T')}{N_S} = a + b(T')Y_S.$$

Recall that for the case of $Y_S = L(S, T)$, $N_S = P(S, T)$, and $T' = S$, i.e. LIBOR-in-arrears, the assumption of a linear rate model is trivially satisfied and there is no restriction.

We first consider the valuation of standard options on the rate Y_S but the option payout is at some arbitrary time $T' \geq S$. The value of a call option with strike K is then

$$\begin{aligned} P(0,T')E^{Q_{T'}}\left[(Y_S-K)^+\right] &= N_0 E^{Q_N}\left[(Y_S-K)^+\frac{P(S,T')}{N_S}\right] = N_0 E^{Q_N}\left[(Y_S-K)^+(a+b(T')Y_S)\right] \\ &= P(0,T')\frac{Y_0\Phi(d_1)(a-b(T')K)-aK\Phi(d_2)+b(T')Y_0^2 e^{\sigma_Y^2 S}\Phi(d_1+\sigma_Y\sqrt{S})}{a+b(T')Y_0} \end{aligned}$$

where $\Phi(\cdot)$ is the c.d.f. of the standard normal distribution,

$$d_1 = \frac{\ln\left(\frac{Y_0}{K}\right)+\frac{1}{2}\sigma_Y^2 S}{\sigma_Y\sqrt{S}},\ d_2 = \frac{\ln\left(\frac{Y_0}{K}\right)-\frac{1}{2}\sigma_Y^2 S}{\sigma_Y\sqrt{S}}.$$

It is desirable to be able to use standard valuation formula (i.e. Black's formula) also for options on interest rates which are irregularly paid. For this purpose, we assume Y_S is lognormally distributed under $Q_{T'}$, with adjusted volatility.

Using formula (8), i.e. $E^{Q_{T'}}[Y_S] = Y_0\left[1+\frac{b(T')Y_0}{a+b(T')Y_0}(e^{\sigma_Y^2 S}-1)\right]$, and moment matching, we can derive

$$Y_S \approx E^{Q_{T'}}[Y_S]\exp\left\{\sigma_Y^*\widehat{W}_S - \frac{1}{2}(\sigma_Y^*)^2 S\right\},$$

where $\widehat{W}$ is a standard Brownian motion under $Q_{T'}$ and

$$(\sigma_Y^*)^2 = \sigma_Y^2 + \ln\left[\frac{(a+b(T')Y_0)(a+b(T')Y_0 e^{2\sigma_Y^2 S})}{(a+b(T')Y_0 e^{\sigma_Y^2 S})^2}\right]/S. \tag{10}$$

This will give option price via Black's formula.

With the above lognormal approximation, we can consider an exchange option involving two interest rates, Y_1 and Y_2. We assume Y_1 and Y_2 are set (fixed) at times S_1 and S_2, respectively, with $S_1 \leq S_2$. For example, Y_1 and Y_2 could be LIBOR rates $L(S_1,T_1)$ and $L(S_2,T_2)$ referring to different fixing dates S_1, S_2 (e.g. LIBOR and LIBOR-in-arrears). One could also think of two CMS rates to be set at the same date but with different tenors.

Suppose the option's payoff is $(Y_2-Y_1)^+$, paid at time $T' \geq \max\{S_1,S_2\}$. In view of the above lognormal approximation, we can assume that both interest rates are lognormal under $Q_{T'}$ $(i=1,2)$

$$Y_i = Y_i^0 e^{\sigma_i W_{S_i}^i - \frac{1}{2}\sigma_i^2 S_i},\ Y_i^0 = E^{Q_{T'}}[Y_i],$$

with $E^{Q_{T'}}[Y_i]$ given by formula (4), (6), or (8) and W^i standard Brownian motion under $Q_{T'}$. Suppose the instantaneous correlation between W^1 and W^2 is ρ, the fair price of the exchange option is then given by

$$P(0,T')[Y_2^0 N(b_1) - Y_1^0 N(b_2)],$$

where

$$b_1 = \frac{\ln\left(\frac{Y_2^0}{Y_1^0}\right)+\frac{1}{2}(\sigma_1^2 S_1+\sigma_2^2 S_2-2\sigma_1\sigma_2\rho S_1)}{\sqrt{\sigma_1^2 S_1+\sigma_2^2 S_2-2\sigma_1\sigma_2\rho S_1}},\ b_2 = \frac{\ln\left(\frac{Y_2^0}{Y_1^0}\right)-\frac{1}{2}(\sigma_1^2 S_1+\sigma_2^2 S_2-2\sigma_1\sigma_2\rho S_1)}{\sqrt{\sigma_1^2 S_1+\sigma_2^2 S_2-2\sigma_1\sigma_2\rho S_1}}$$

For details of the computation, see Boenkost and Schmidt [1], Section 4.4, Proposition 7.

Remark 4. *The market standard method of valuing options on convexity adjusted rates is to apply the Black formula using the convexity adjusted rate as the forward rate. But to be conceptually correct, we should also convexity adjust volatility by formula (10). These two adjustments combined is equivalent to assuming the rate is lognormal under the T'-forward measure $Q_{T'}$.*

3 Convexity adjustment with volatility smile: pricing by replication

As shown in Section 2 and Section 2.4, pricing irregular interest cash flows such as LIBOR-in-arrears or CMS requires a convexity correction on the corresponding forward rate. This convexity correction involves the volatility of the underlying rate as traded in the cap/floor or swaption market. In the same spirit, an option on an irregular rate, such as an in-arrears cap or a CMS cap, is often valued by applying Black's formula with a convexity adjusted forward rate and a convexity adjusted volatility. However, to a large extent, this approach ignores the volatility smile, which is quite pronounced in the cap/floor market. This section solves this problem by replicating the irregular interest flow or option with liquidly traded options with different strikes. As an important consequence, one obtains immediately the respective simultaneous delta and vega hedges in terms of liquidly traded options.

3.1 Motivation

Example 1. (LIBOR and caplet paid at arbitrary time) Let $Y = L(S,T)$ be the LIBOR rate for the period $[S,T]$. Suppose our goal is to price an "exotic" payoff $f(L(S,T))$ at time S, or more generally, at time $T' \geq S$. For example, $f(L(S,T)) = \tau(S,T)L(S,T)$ is the payoff of a LIBOR, and $f(L(S,T)) = \tau(S,T)[L(S,T)-K]^+$ is the payoff of a caplet. *How can we use liquid instruments to hedge this exotic payoff?*

In the interest rate derivative market, caplets are quite liquid for (more or less) any strike $K \geq 0$. Therefore, we intend to use caplets as hedging instruments. One difficulty is that by default, a caplet pays in arrears (i.e. at time T).

To hedge the exotic payoff, we first need to "match" the payoff times. Since the exotic payoff is either paid at time S or at time $T' \geq S$, we discount everything to time S. First, the standard caplet paying at time T is equivalent to a payoff of

$$P(S,T)\tau(S,T)[L(S,T)-K]^+ = \frac{\tau(S,T)}{1+\tau(S,T)L(S,T)}[L(S,T)-K]^+$$

at time S. Second, under the linear rate model $\frac{P(S,T')}{P(S,T)} = 1 + \frac{\left(\frac{P(0,T')}{P(0,T)}-1\right)}{F(0;S,T)}L(S,T)$, the exotic payoff $f(L(S,T))$ at time T' is equivalent to a payoff of

$$P(S,T')f(L(S,T)) = \frac{1+\frac{\left(\frac{P(0,T')}{P(0,T)}-1\right)}{F(0;S,T)}L(S,T)}{1+\tau(S,T)L(S,T)}f(L(S,T)) := f_1(L(S,T))$$

at time S. Here $f_1(y) := \frac{1+\frac{\left(\frac{P(0,T')}{P(0,T)}-1\right)}{F(0;S,T)}y}{1+\tau(S,T)y}f(y)$.

In summary, the exotic payoff can be written as $f(L(S,T))$, paid at time S; and the hedging instruments can be generalized to the form of $g(L(S,T))[L(S,T)-K]^+$ $(K \geq 0)$, also paid at time S. Here $g(y) := \frac{\tau(S,T)}{1+\tau(S,T)y}$. We are interested in finding a representation

$$f(y) = C + \int_0^\infty g(y)(y-K)^+ d\mu(K)$$

with some locally finite signed measure μ on $\mathbb{R}^+$ and some constant C. If such a representation exists, the time-zero fair price of the contingent claim $f(L(S,T))$ is given by

$$V_0^{exotic} = C \cdot P(0,S) + \int_0^\infty V_0^{caplet}(K)d\mu(K),$$

where $V_0^{caplet}(K)$ is the time-zero price of a caplet for the period $[S,T]$, with strike K.

Example 2. (CMS swap and CMS caplet paid at arbitrary time) Let $Y = S_{\alpha,\beta}(T_\alpha)$ be the swap rate for the tenure structure $\mathcal{T} = \{T_\alpha, T_{\alpha+1}, \cdots, T_\beta\}$. Suppose our goal is to price an "exotic" payoff $f(S_{\alpha,\beta}(T_\alpha))$ at time T_α. For example, $f(y) = y$ is a cash flow in a CMS swap (but paid in advance), and $f(y) = (y-K)^+$ gives a CMS caplet.

In case the payoff is scheduled from a time $T' \geq T_\alpha$, we can approximately think of a derivative with payoff

$$f(y) = \begin{cases} \frac{y}{(1+y)^{T'-T_\alpha}} & \text{for a CMS swap} \\ \frac{(y-K)^+}{(1+y)^{T'-T_\alpha}} & \text{for a CMS caplet;} \end{cases}$$

at time T_α.

How can we use liquid instruments to hedge this exotic payoff?

In the interest rate derivative market, swaptions are quite liquid for any strike $K \geq 0$. Therefore, we intend to use swaptions as hedging instruments. There are two types of swaptions: the payoff of a *cash-settled payer swaption* at time T_α is

$$\left(\sum_{i=\alpha+1}^{\beta} \frac{\tau(T_{i-1}, T_i)}{[1 + S_{\alpha,\beta}(T_\alpha)]^{T_i - T_\alpha}} \right) (S_{\alpha,\beta}(T_\alpha) - K)^+$$

and the payoff of a *physically-settled payer swaption* at time T_α is

$$\left(\sum_{i=\alpha+1}^{\beta} \tau(T_{i-1}, T_i) P(T_\alpha, T_i) \right) (S_{\alpha,\beta}(T_\alpha) - K)^+.$$

The market does not make a significant difference in pricing a cash- or physically-settled swaption, and both are priced under a lognormal Black model.

Example 3. (CMS swap and CMS caplet paid at arbitrary time under linear rate model) In Example 2, we approximated CMS swap rate and CMS caplet paid at arbitrary time by

$$\begin{cases} \frac{S_{\alpha,\beta}(T_\alpha)}{[1+S_{\alpha,\beta}(T_\alpha)]^{T'-T_\alpha}} & \text{for a CMS swap} \\ \frac{[S_{\alpha,\beta}(T_\alpha)-K]^+}{[1+S_{\alpha,\beta}(T_\alpha)]^{T'-T_\alpha}} & \text{for a CMS caplet.} \end{cases}$$

In the current example, we want to avoid this approximation but we then need the additional assumption of linear rate model. The so-called *linear swap rate model* is

$$\boxed{\begin{cases} \frac{P(T_\alpha, T')}{N^{\alpha,\beta}_{T_\alpha}} = a + b(T') S_{\alpha,\beta}(T_\alpha), \ T' \geq T_\alpha \\ a = \frac{1}{\sum_{i=\alpha+1}^{\beta} \tau_i} \\ b(T') = \frac{1}{S_{\alpha,\beta}(0)} \left[\frac{P(0,T')}{N_0^{\alpha,\beta}} - \frac{1}{\sum_{i=\alpha+1}^{\beta} \tau_i} \right], \ T' \geq T_\alpha \end{cases}}$$

where $N^{\alpha,\beta}_{\cdot}$ is the annuity numeraire $N_t^{\alpha,\beta} = \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)$,

Consequently, an exotic derivative paying $f(S_{\alpha,\beta}(T_\alpha))$ at some time $T' \geq T_\alpha$ is equivalent to a payoff of

$$P(T_\alpha, T') f(S_{\alpha,\beta}(T_\alpha)) = N^{\alpha,\beta}_{T_\alpha} [a + b(T') S_{\alpha,\beta}(T_\alpha)] f(S_{\alpha,\beta}(T_\alpha))$$

at time T_α. Under the swap measure $Q^{\alpha,\beta}$ associated with the annuity numeraire $N^{\alpha,\beta}_{\cdot}$, the price of the contingent claim is given by

$$N_0^{\alpha,\beta} E^{Q^{\alpha,\beta}} \left[(a + b(T') S_{\alpha,\beta}(T_\alpha)) f(S_{\alpha,\beta}(T_\alpha)) \right]$$

The price of physically-settled payer swaption can be written as

$$V_0^{swpt}(K) = N_0^{\alpha,\beta} E^{Q^{\alpha,\beta}} \left[(S_{\alpha,\beta}(T_\alpha) - K)^+ \right].$$

Comparing these two formulas leads us to looking for a representation of the form

$$(a + b(T')Y) f(Y) = C + \int_{[0,\infty)} (Y - K)^+ d\mu(K).$$

3.2 Problem formulation and its solution

Consider a financial underlying with price Y at time T. Suppose there is a liquid market for plain vanilla options on this underlying with all possible strikes K. Somewhat more generally, we suppose that for all K, the time-zero price $V_0^{vanilla}(K)$ of the "plain vanilla" derivative with payoff $g(Y)(Y-K)^+$ at time T is known. Here g is a deterministic function.

Our goal is to price an exotic contingent claim with payoff $f(Y)$ at time T, where f is a deterministic function. The idea is to replicate the exotic payoff $f(Y)$ by a portfolio of traded derivatives $g(Y)(Y-K)^+$ with different strikes K. If this is possible, then the replication is the key to incorporating the volatility smile of the liquid options into the pricing of the exotic derivative.

So we are looking for a representation

$$f(Y) = C + \int_0^\infty g(Y)(Y-K)^+ d\mu(K) \tag{11}$$

with some locally finite signed measure μ on $\mathbb{R}^+$ and some constant C. By risk-neutral pricing, this representation will give the time-zero price of the contingent claim $f(Y)$:

$$\boxed{V_0 = C \cdot P(0,T) + \int_0^\infty V_0^{vanilla}(K) d\mu(K)} \tag{12}$$

Remark 5. *The importance of formula (12) goes beyond the issue of just pricing a derivative with payoff $f(Y)$, since it provides us with an explicit strategy for a simultaneous delta and vega hedge of the derivative $f(Y)$ in terms of liquidly traded products $g(Y)(Y-K)^+$. In practice one would discretize the integral appropriately to get an (approximate) hedging strategy in a finite set of products $g(Y)(Y-K_i)^+$ with different strikes K_i, $i = 1, \cdots$.*

Proposition 3.1. *The exotic payoff $f(Y)$ allows for a replication (11) with some locally finite signed measure μ on $[0,\infty)$ and some constant C if and only if*
(i) $\lim_{x\downarrow 0} f(x) = C$;
(ii) the function $\frac{f-C}{g}$, extended to the domain of definition $[0,\infty)$ by setting $\frac{f(0)-C}{g(0)} = 0$, is a difference of convex functions on $[0,\infty)$.
The measure μ is then generated by the following right continuous generalized dsitrbution function (function of locally bounded variation)

$$d\mu(y) = dD_+\left(\frac{f(y)-C}{g(y)}\right)$$

with D_+ denoting the right derivative and defining $D_+\left(\frac{f-C}{g}\right)(0-) = 0$.

Proof. For necessity, note

$$\frac{f(y)-C}{g(y)} = \int_0^\infty (y-K)^+ d\mu^+(K) - \int_0^\infty (y-K)^+ d\mu^-(K),$$

where μ^+ and μ^- are the positive and negative parts of μ, respectively. For sufficiency, we apply the integral representation of convex functions (Proposition B.1)

$$\frac{f(y)-C}{g(y)} - \frac{f(0)-C}{g(0)} = \int_0^y \mu(K)dK = \mu(y)y - \int_0^y K d\mu(K) = \int_0^y (y-K)d\mu(K) = \int_0^\infty (y-K)^+ d\mu(K)$$

for some function $\mu(\cdot)$ of locally finite variation. □

Remark 6. *By risk-neutral pricing under the T-forward measure, the time-zero price of the contingent claim is also given by the formula*

$$V_0 = P(0,T)E^{Q_T}[f(Y)] = P(0,T)\int_0^\infty f(x)dF_Y(x),$$

where $F_Y(\cdot)$ is the cumulative distribution function of Y under Q_T. Since

$$V_0^{vanilla}(K) = P(0,T)E^{Q_T}[g(Y)(Y-K)^+],$$

we have

$$D_+V_0^{vanilla}(x) = P(0,T)E^{Q_T}[g(Y)D_+(Y-x)^+] = -P(0,T)E^{Q_T}[g(Y)1_{\{x<Y\}}] = -P(0,T)\int_x^\infty g(y)dF_Y(y)$$

and

$$dD_+V_0^{vanialla} = P(0,T)g(x)dF_Y(x).$$

Therefore by integrating by parts twice, we have (define $C = \lim_{x\downarrow 0} f(x)$)

$$\begin{aligned} V_0 &= C\cdot P(0,T) + \int_0^\infty \frac{f(x)-C}{g(x)} dD_+V_0^{vanilla}(x) \\ &= C\cdot P(0,T) - \int_0^\infty D_+V_0^{call}(x)df(x) \\ &= C\cdot P(0,T) + \int_0^\infty V_0^{call}(x)d^2f(x). \end{aligned}$$

This approach is consistent with formula (12).

3.3 Application to motivating examples

3.3.1 LIBOR and caplet paid at arbitrary time under linear rate model

We consider a coupon period $[S,T]$ and denote by τ the day count $\tau(S,T)$ of $[S,T]$ in year fraction. Consider a LIBOR paid at time $T' \geq S$. Under the linear rate model, the payoff is equivalent to a payoff of

$$\frac{1+b(T')L(S,T)}{1+\tau L(S,T)}\tau L(S,T)$$

at time S, where $b(T') = \frac{\left(\frac{P(0,T')}{P(0,T)}-1\right)}{F(0;S,T)}$. Therefore $f(y) = \frac{1+b(T')y}{1+\tau y}\tau y$ and $C = \lim_{x\downarrow 0} f(x) = 0$. The standard caplet paying at time T is equivalent to a payoff of

$$\frac{\tau}{1+\tau L(S,T)}[L(S,T)-K]^+$$

at time S. Therefore, $g(y) = \frac{\tau}{1+\tau y}$. Combined, for LIBOR paid at time $T' \geq S$, we have

$$\frac{f(y)-C}{g(y)} = y[1+b(T')y],$$

which is a convex function. Accordingly,

$$\mu(y) = D_+\left(\frac{f(y)-C}{g(y)}\right) = \begin{cases} 1+2b(T')y & y\geq 0 \\ 0 & y<0 \end{cases}.$$

In view of $d\mu(\{0\}) = \mu(0)-\mu(0-) = 1$, we obtain the formula for the price of a LIBOR expressed in terms of caplet prices $V_0^{caplet}(K)$ with different strikes K:

$$\boxed{\begin{cases} V_0(\tau L(S,T) \text{ paid at time } T' \geq S) = V_0^{caplet}(0) + \int_0^\infty V_0^{caplet}(K)2b(T')dK, \\ b(T') = \frac{\left(\frac{P(0,T')}{P(0,T)}-1\right)}{F(0;S,T)} \end{cases}} \tag{13}$$

Now consider a caplet paid at time $T' \geq S$. Under the linear rate model, the payoff is equivalent to a payoff of

$$\frac{1+b(T')L(S,T)}{1+\tau L(S,T)}\tau[L(S,T)-K]^+$$

at time S. Therefore $f(y) = \frac{1+b(T')y}{1+\tau y}\tau(y-K)^+$ and $C = \lim_{x\downarrow 0} f(y) = 0$. We then have

$$\frac{f(y)-C}{g(y)} = [1+b(T')y](y-K)^+,$$

which is a convex function. Accordingly,

$$\mu(y) = D_+\left(\frac{f(y)-C}{g(y)}\right) = \begin{cases} 1-b(T')K+2b(T')y & y \geq K \\ 0 & y < K \end{cases}.$$

In view of $d\mu(\{K\}) = \mu(K) - \mu(K-) = 1 + b(T')K$, we obtain the formula for the price of a caplet expressed in terms of caplet prices $V_0^{caplet}(K)$ with different strikes K:

$$\boxed{\begin{cases} V_0(\tau[L(S,T)-K]^+ \text{ paid at time } T' \geq S) = V_0^{caplet}(K)(1+b(T')K) + \int_K^\infty V_0^{caplet}(\widetilde{K})2b(T')d\widetilde{K} \\ b(T') = \frac{\left(\frac{P(0,T')}{P(0,T)}-1\right)}{F(0;S,T)} \end{cases}} \tag{14}$$

Remark 7. *In case the cap market quotes no smile, it is easy to verify that formula (13) is reduced to formula (2).*

3.3.2 CMS swap and CMS caplet paid at arbitrary time discounted by swap rate yield

Consider a CMS swap rate $S_{\alpha,\beta}(T_\alpha)$ paid at time $T' \geq T_\alpha$. The payoff is approximated by a payoff of

$$\frac{S_{\alpha,\beta}(T_\alpha)}{(1+S_{\alpha,\beta}(T_\alpha))^{T'-T_\alpha}}$$

at time T_α. Therefore, $f(y) = \frac{y}{(1+y)^{T'-T_\alpha}}$ and $C = \lim_{x\downarrow 0} f(x) = 0$.

The payoff of a *cash-settled payer swaption* at time T_α is

$$\left(\sum_{i=\alpha+1}^{\beta} \frac{\tau(T_{i-1},T_i)}{[1+S_{\alpha,\beta}(T_\alpha)]^{T_i-T_\alpha}}\right)(S_{\alpha,\beta}(T_\alpha)-K)^+$$

So, using cash-settled payer swaption as hedging instruments, we have

$$g(y) = \sum_{i=\alpha+1}^{\beta} \frac{\tau_i}{(1+y)^{T_i-T_\alpha}}$$

We then have

$$\frac{f(y)-C}{g(y)} = \frac{y}{\sum_{i=\alpha+1}^{\beta}\frac{\tau_i}{(1+y)^{T_i-T'}}} = \frac{y}{\text{DV01}(y)},$$

where $\text{DV01}(y) := \sum_{i=\alpha+1}^{\beta}\frac{\tau_i}{(1+y)^{T_i-T'}}$, the so-called present (or dollar) value of one basis point factor. It is kind of hard to determine the convexity of $\frac{f(y)-C}{g(y)}$, or to find the two convex functions whose difference is $\frac{f(y)-C}{g(y)}$. But we can easily see that $\frac{f(y)-C}{g(y)}$ is absolutely continuous and its derivative is of finite variation locally. Therefore, by Proposition B.2, Proposition 3.1 applies. Accordingly,

$$\mu(y) = D_+\left(\frac{f(y)-C}{g(y)}\right) = \begin{cases} \frac{\text{DV01}(y) - y\text{DV01}'(y)}{\text{DV01}^2(y)} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

In view of $d\mu(\{0\}) = \mu(0) - \mu(0-) = \frac{1}{\text{DV01}(0)} = \frac{1}{\sum_{i=\alpha+1}^{\beta}\tau_i}$, we obtain the formula for the price of a CMS swap rate expressed in terms of swaption price $V_0^{swpt}(K)$ with different strikes K:

$$\boxed{\begin{cases} V_0(S_{\alpha,\beta}(T_\alpha) \text{ paid at time } T' \geq T_\alpha) = \frac{V_0^{swpt}(0)}{\sum_{i=\alpha+1}^{\beta}\tau_i} + \int_0^\infty V_0^{swpt}(K)\left(\frac{\text{DV01}(K) - K\text{DV01}'(K)}{\text{DV01}^2(K)}\right)' dK \\ \text{DV01}(y) = \sum_{i=\alpha+1}^{\beta} \frac{\tau_i}{(1+y)^{T_i - T'}} \end{cases}} \tag{15}$$

Now consider a CMS caplet $[S_{\alpha,\beta}(T_\alpha) - K]^+$ paid at time $T' \geq T_\alpha$. The payoff is approximated by a payoff of

$$\frac{[S_{\alpha,\beta}(T_\alpha) - K]^+}{[1 + S_{\alpha,\beta}(T_\alpha)]^{T' - T_\alpha}}$$

at time T_α. Therefore $f(y) = \frac{(y-K)^+}{(1+y)^{T'-T_\alpha}}$ and $C = \lim_{x\downarrow 0} f(x) = 0$.

Like the case of CMS swap rate, by using cash-settled payer swaption as hedging instruments, we have

$$\frac{f(y) - C}{g(y)} = \frac{(y-K)^+}{\sum_{i=\alpha+1}^{\beta} \frac{\tau_i}{(1+y)^{T_i - T'}}} = \frac{(y-K)^+}{\text{DV01}(y)}.$$

Accordingly,

$$\mu(y) = D_+\left(\frac{f(y) - C}{g(y)}\right) = \begin{cases} \frac{\text{DV01}(y) - (y-K)\text{DV01}'(y)}{\text{DV01}^2(y)} & y \geq K \\ 0 & y < K \end{cases}$$

In view of $d\mu(\{K\}) = \mu(K) - \mu(K-) = \frac{1}{\text{DV01}(K)} = \frac{1}{\sum_{i=\alpha+1}^{\beta} \frac{\tau_i}{(1+K)^{T_i - T'}}}$, we obtain the formula for the price of a CMS caplet expressed in terms of swaption price $V_0^{swpt}(K)$ with different strike K:

$$\boxed{\begin{cases} V_0([S_{\alpha,\beta}(T_\alpha) - K]^+ \text{ paid at time } T' \geq T_\alpha) = \frac{V_0^{swpt}(K)}{\text{DV01}(K)} + \int_K^\infty V_0^{swpt}(\widetilde{K}) \cdot h(\widetilde{K}) \cdot d\widetilde{K} \\ \text{DV01}(y) = \sum_{i=\alpha+1}^{\beta} \frac{\tau_i}{(1+y)^{T_i - T'}} \end{cases}} \tag{16}$$

where

$$\begin{aligned} h(y) &= \left(\frac{\text{DV01}(y) - (y-K)\text{DV01}'(y)}{\text{DV01}^2(y)}\right)' \\ &= (y-K)\left[2\frac{(\text{DV01}'(y))^2}{\text{DV01}^3(y)} - \frac{\text{DV01}''(y)}{\text{DV01}^2(y)}\right] - 2\frac{\text{DV01}'(y)}{\text{DV01}^2(y)}. \end{aligned}$$

Formula (16) is widely used by sophisticated practitioners to value CMS caps.

3.3.3 CMS swap and CMS caplet paid at arbitrary time under linear rate model

Consider a CMS swap rate $S_{\alpha,\beta}(T_\alpha)$ paid at time $T' \geq T_\alpha$. As explained in Example 3 of Section 3.1, we are looking for integral representation of the form

$$[a + b(T')S_{\alpha,\beta}(T_\alpha)]S_{\alpha,\beta}(T_\alpha) = C + \int_{[0,\infty)} (S_{\alpha,\beta}(T_\alpha) - K)^+ d\mu(K).$$

Using notation of Proposition 3.1, $f(y) = [a + b(T')y]y$, $g(y) = 1$, and $C = \lim_{x\downarrow 0} f(x) = 0$. Therefore

$$\mu(y) = D_+\left(\frac{f(y) - C}{g(y)}\right) = \begin{cases} a + 2b(T')y & y \geq 0 \\ 0 & y < 0 \end{cases}$$

In view of $d\mu(\{0\}) = \mu(0) - \mu(0-) = a$, we obtain the formula for the price of a CMS swap rate expressed in terms of swaption price $V_0^{swpt}(K)$ with different strikes K:

$$\boxed{\begin{cases} V_0(S_{\alpha,\beta}(T_\alpha) \text{ paid at time } T' \geq T_\alpha) = aV_0^{swpt}(0) + \int_0^\infty V_0^{swpt}(K) \cdot 2b(T')dK \\ a = \frac{1}{\sum_{i=\alpha+1}^{\beta} \tau_i} \\ b(T') = \frac{1}{S_{\alpha,\beta}(0)} \left[\frac{P(0,T')}{N_0^{\alpha,\beta}} - \frac{1}{\sum_{i=\alpha+1}^{\beta} \tau_i} \right], \ T' \geq T_\alpha \end{cases}} \tag{17}$$

Now consider a CMS caplet $[S_{\alpha,\beta}(T_\alpha) - K]^+$ paid at time $T' \geq T_\alpha$. We are looking for integral representation of the form

$$[a + b(T')S_{\alpha,\beta}(T_\alpha)][S_{\alpha,\beta}(T_\alpha) - K]^+ = C + \int_{[0,\infty)} (S_{\alpha,\beta}(T_\alpha) - K)^+ d\mu(K).$$

Using notation of Proposition 3.1, $f(y) = [a+b(T')y](y-K)^+$, $g(y) = 1$, and $C = \lim_{x\downarrow 0} f(x) = 0$. Therefore

$$\mu(y) = D_+\left(\frac{f(y) - C}{g(y)}\right) = \begin{cases} [a - b(T')K] + 2b(T')y & y \geq K \\ 0 & y < K \end{cases}$$

In view of $d\mu(\{K\}) = \mu(K) - \mu(K-) = a + b(T')K$, we obtain the formula for the price of a CMS caplet expressed in terms of swaption price $V_0^{swpt}(K)$ with different strikes K:

$$\boxed{\begin{cases} V_0([S_{\alpha,\beta}(T_\alpha) - K]^+ \text{ paid at time } T' \geq T_\alpha) = [a + b(T')K]V_0^{swpt}(K) + \int_K^\infty V_0^{swpt}(\widetilde{K}) \cdot 2b(T')d\widetilde{K} \\ a = \frac{1}{\sum_{i=\alpha+1}^{\beta} \tau_i} \\ b(T') = \frac{1}{S_{\alpha,\beta}(0)} \left[\frac{P(0,T')}{N_0^{\alpha,\beta}} - \frac{1}{\sum_{i=\alpha+1}^{\beta} \tau_i} \right], \ T' \geq T_\alpha \end{cases}} \tag{18}$$

3.3.4 CMS caplet, floorlet, and floater paid at arbitrary time under Hagan's model

In Section 2.2.2, we explained several models for the ratio

$$P(t,T')/N_t^{\alpha,\beta} = G(S_{\alpha,\beta}(t))$$

and

$$P(0,T')/N_0^{\alpha,\beta} = G(S_{\alpha,\beta}(0)),$$

where $G(x)$ could be

$$G(x) = \begin{cases} \frac{x}{(1+x/q)^\Delta} \cdot \frac{1}{1-\frac{1}{(1+x/q)^n}} & \text{Model 1} \\ \frac{x}{(1+\tau_{\alpha+1}x)^\Delta} \cdot \frac{1}{1-\prod_{i=\alpha+1}^{\beta} \frac{1}{1+\tau_i x}} & \text{Model 2} \\ \frac{xe^{-(T'-T_\alpha)s}}{1-\frac{P(0,T_\beta)}{P(0,T_\alpha)}e^{-(T_\beta-T_\alpha)s}} & \text{Model 3} \\ \frac{xe^{-[h(T')-h(T_\alpha)]s}}{1-\frac{P(0,T_\beta)}{P(0,T_\alpha)}e^{-[h(T_\beta)-h(T_\alpha)]s}} & \text{Mdoel 4} \end{cases}$$

with s determined implicitly in terms of x by

$$x\sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i)e^{-(T_i-T_\alpha)s} = P(0,T_\alpha) - P(0,T_\beta)e^{-(T_\beta-T_\alpha)s}$$

for Model 3, and s determined implicity in terms of x by

$$x\sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i)e^{-[h(T_i)-h(T_\alpha)]s} = P(0,T_\alpha) - P(0,T_\beta)e^{-[h(T_\beta)-h(T_\alpha)]s}.$$

for Model 4.

We first consider **CMS caplet**. The payoff of $[S_{\alpha,\beta}(T_\alpha)-K]^+$ at time $T' \geq T_\alpha$ is equivalent to a payoff of

$$P(T_\alpha, T')[S_{\alpha,\beta}(T_\alpha)-K]^+.$$

at time T_α. By risk-neutral pricing,

$$\begin{aligned}
& V_0([S_{\alpha,\beta}(T_\alpha)-K]^+ \text{ paid at time } T' \geq T_\alpha) \\
= \; & N_0^{\alpha,\beta} E^{Q^{\alpha,\beta}}\left[\frac{P(T_\alpha,T')[S_{\alpha,\beta}(T_\alpha)-K]^+}{N_{T_\alpha}^{\alpha,\beta}}\right] \\
= \; & P(0,T') E^{Q^{\alpha,\beta}}\left[\frac{P(T_\alpha,T')/N_{T_\alpha}^{\alpha,\beta}}{P(0,T')/N_0^{\alpha,\beta}}[S_{\alpha,\beta}(T_\alpha)-K]^+\right] \\
= \; & P(0,T') E^{Q^{\alpha,\beta}}\left\{[S_{\alpha,\beta}(T_\alpha)-K]^+\right\} + P(0,T') E^{Q^{\alpha,\beta}}\left[\left(\frac{P(T_\alpha,T')/N_{T_\alpha}^{\alpha,\beta}}{P(0,T')/N_0^{\alpha,\beta}}-1\right)[S_{\alpha,\beta}(T_\alpha)-K]^+\right]
\end{aligned}$$

where we observe that $\left(\frac{P(t,T')}{N_t^{\alpha,\beta}}\right)_{0\leq t\leq T'}$ is a martingale under swap measure $Q^{\alpha,\beta}$ and

$$E^{Q^{\alpha,\beta}}\left[\frac{P(T_\alpha,T')}{N_{T_\alpha}^{\alpha,\beta}}\right] = \frac{P(0,T')}{N_0^{\alpha,\beta}}$$

The first term is exactly the price of a European swaption with notional $P(0,T')/N_0^{\alpha,\beta}$, regardless of how the swap rate $S_{\alpha,\beta}(t)$ is modeled. The second term is the "convexity correction". Since $S_{\alpha,\beta}(t)$ is a martingale and $\frac{P(T_\alpha,T')/N_{T_\alpha}^{\alpha,\beta}}{P(0,T')/N_0^{\alpha,\beta}}-1$ is zero on average, this term goes to zero linearly with the variance of the swap rate $S_{\alpha,\beta}(t)$ and is much smaller than the first term.

Under a general rate model, we assume $P(t,T')/N_t^{\alpha,\beta} = G(S_t^{\alpha,\beta})$. Then for $f(x) := [G(x)/G(S_{\alpha,\beta}(0)) - 1](x-K)$, we have the property

$$f'(K)(x-K)^+ + \int_K^\infty (x-\zeta)^+ f''(\zeta)d\zeta = \begin{cases} f(x) & \text{for } x \geq K \\ 0 & \text{for } x < K \end{cases}$$

So

$$[G(x)/G(S_{\alpha,\beta}(0)) - 1](x-K)^+ = f'(K)(x-K)^+ + \int_K^\infty (x-\zeta)^+ f''(\zeta)d\zeta$$

and

$$\begin{aligned}
& E^{Q^{\alpha,\beta}}\left[\left(\frac{P(T_\alpha,T')/N_{T_\alpha}^{\alpha,\beta}}{P(0,T')/N_0^{\alpha,\beta}}-1\right)[S_{\alpha,\beta}(T_\alpha)-K]^+\right] \\
= \; & E^{Q^{\alpha,\beta}}\left\{[G(S_{T_\alpha}^{\alpha,\beta})/G(S_{\alpha,\beta}(0)) - 1](S_{T_\alpha}^{\alpha,\beta}-K)^+\right\} \\
= \; & f'(K)E^{Q^{\alpha,\beta}}[(S_{T_\alpha}^{\alpha,\beta}-K)^+] + \int_K^\infty E^{Q^{\alpha,\beta}}[(S_{\alpha,\beta}(T_\alpha)-\zeta)^+]f''(\zeta)d\zeta
\end{aligned}$$

Therefore, the convexity correction is equal to

$$\boxed{cc = \frac{P(0,T')}{N_0^{\alpha,\beta}}\left[f'(K)C(K) + \int_K^\infty C(\zeta)f''(\zeta)d\zeta\right]}$$

and the time zero price of a CMS caplet paid at time $T' \geq T_\alpha$ is

$$\boxed{V_0([S_{\alpha,\beta}(T_\alpha)-K]^+ \text{ paid at time } T' \geq T_\alpha) = \frac{P(0,T')}{N_0^{\alpha,\beta}}\left\{[1+f'(K)]C(K) + \int_K^\infty C(\zeta)f''(\zeta)d\zeta\right\}}$$

Here $C(K) = N_0^{\alpha,\beta} E^{Q^{\alpha,\beta}}[(S_{\alpha,\beta}(T_\alpha) - K)^+]$ is the time zero value of a vanilla payer swaption with strike K.

Repeating the above argument shows that the value of a **CMS floorlet** is

$$\boxed{V_0([K - S_{\alpha,\beta}(T_\alpha)]^+ \text{ paid at time } T' \geq T_\alpha) = \frac{P(0,T')}{N_0^{\alpha,\beta}} \left\{ [1 + f'(K)]P(K) - \int_{-\infty}^{K} P(\zeta) f''(\zeta) d\zeta \right\}}$$

where $f(x)$ is the same as before and $P(K) = N_0^{\alpha,\beta} E^{Q^{\alpha,\beta}}[(K - S_{\alpha,\beta}(T_\alpha))^+]$ is the value of the receiver swaption with strike K.

By the call-put parity and by setting $K = S_{\alpha,\beta}(0)$, we have $C(S_{\alpha,\beta}(0)) - P(S_{\alpha,\beta}(0)) = 0$. Hence the value of a **CMS floater** is

$$\boxed{\begin{aligned} & V_0(S_{\alpha,\beta}(T_\alpha) \text{ paid at time } T' \geq T_\alpha) \\ =& P(0,T') S_{\alpha,\beta}(0) + \frac{P(0,T')}{N_0^{\alpha,\beta}} \left[\int_{S_{\alpha,\beta}(0)}^{\infty} C(\zeta) f''_{atm}(\zeta) d\zeta + \int_{-\infty}^{S_{\alpha,\beta}(0)} P(\zeta) f''_{atm}(\zeta) d\zeta \right] \end{aligned}}$$

where

$$f_{atm}(x) = [G(x)/G(S_{\alpha,\beta}(0)) - 1](x - S_{\alpha,\beta}(0))$$

As a last word, by call-put parity, we can price an in-the-money caplet or floorlet as a swaplet plus an out-of-the-money floorlet or caplet.

Analytical formulas: The function $G(x)$ is smooth and slowly varying, regardless of the model used to obtain it. Since the probable swap rates $S_{\alpha,\beta}(\cdot)$ are heavily concentrated around around $S_{\alpha,\beta}(0)$, it makes sense to expand $G(x)$ as

$$G(x) \approx G(S_{\alpha,\beta}(0)) + G'(S_{\alpha,\beta}(0))(x - S_{\alpha,\beta}(0)) + \cdots$$

If we take the expansion to the linear term, $f(x)$ becomes a quadratic function

$$f(x) \approx \frac{G'(S_{\alpha,\beta}(0))}{G(S_{\alpha,\beta}(0))}(x - S_{\alpha,\beta}(0))(x - K).$$

Then

$$f'(K) = \frac{G'(S_{\alpha,\beta}(0))}{G(S_{\alpha,\beta}(0))}\left[K - S_{\alpha,\beta}(0)\right], \; f''(\zeta) = 2\frac{G'(S_{\alpha,\beta}(0))}{G(S_{\alpha,\beta}(0))}.$$

Consequently, we have (note $G(S_{\alpha,\beta}(0)) = \frac{P(0,T')}{N_0^{\alpha,\beta}}$)

$$\boxed{\begin{aligned} & V_0([S_{\alpha,\beta}(T_\alpha) - K]^+ \text{ paid at time } T' \geq T_\alpha) \\ =& \frac{P(0,T')}{N_0^{\alpha,\beta}} C(K) + G'(S_{\alpha,\beta}(0)) \left[(K - S_{\alpha,\beta}(0)) C(K) + 2 \int_K^{\infty} C(\zeta) d\zeta \right] \end{aligned}}$$

$$\boxed{\begin{aligned} & V_0([K - S_{\alpha,\beta}(T_\alpha)]^+ \text{ paid at time } T' \geq T_\alpha) \\ =& \frac{P(0,T')}{N_0^{\alpha,\beta}} P(K) + G'(S_{\alpha,\beta}(0)) \left[(K - S_{\alpha,\beta}(0)) P(K) - 2 \int_{-\infty}^{K} P(\zeta) d\zeta \right] \end{aligned}}$$

and

$$\boxed{V_0(S_{\alpha,\beta}(T_\alpha) \text{ paid at time } T' \geq T_\alpha) = P(0,T') S_{\alpha,\beta}(0) + 2G'(S_{\alpha,\beta}(0)) \left[\int_{S_{\alpha,\beta}(0)}^{\infty} C(\zeta) d\zeta + \int_{-\infty}^{S_{\alpha,\beta}(0)} P(\zeta) d\zeta \right]}$$

We note [1]

$$\int_K^\infty C(\zeta)d\zeta = N_0^{\alpha,\beta}\int_K^\infty E^{Q^{\alpha,\beta}}[(S_{\alpha,\beta}(T_\alpha)-\zeta)^+]d\zeta = \frac{1}{2}N_0^{\alpha,\beta}E^{Q^{\alpha,\beta}}\left\{([S_{\alpha,\beta}(T_\alpha)-K]^+)^2\right\}$$

Therefore, an alternative form of the formulas is

$$\begin{aligned} & V_0([S_{\alpha,\beta}(T_\alpha)-K]^+ \text{ paid at time } T' \geq T_\alpha) \\ = & \boxed{\frac{P(0,T')}{N_0^{\alpha,\beta}}C(K) + G'(S_{\alpha,\beta}(0))N_0^{\alpha,\beta}E^{Q^{\alpha,\beta}}\left\{[S_{\alpha,\beta}(T_\alpha)-S_{\alpha,\beta}(0)]\,[S_{\alpha,\beta}(T_\alpha)-K]^+\right\}} \end{aligned}$$

for the CMS caplets,

$$\begin{aligned} & V_0([K-S_{\alpha,\beta}(T_\alpha)]^+ \text{ paid at time } T' \geq T_\alpha) \\ = & \boxed{\frac{P(0,T')}{N_0^{\alpha,\beta}}P(K) - G'(S_{\alpha,\beta}(0))N_0^{\alpha,\beta}E^{Q^{\alpha,\beta}}\left\{[S_{\alpha,\beta}(0)-S_{\alpha,\beta}(T_\alpha)]\,[K-S_{\alpha,\beta}(T_\alpha)]^+\right\}} \end{aligned}$$

for CMS floorlets, and

$$\begin{aligned} & V_0(S_{\alpha,\beta}(T_\alpha) \text{ paid at time } T' \geq T_\alpha) \\ = & \boxed{P(0,T')S_{\alpha,\beta}(0) + G'(S_{\alpha,\beta}(0))N_0^{\alpha,\beta}E^{Q^{\alpha,\beta}}\left\{[S_{\alpha,\beta}(T_\alpha)-S_{\alpha,\beta}(0)]^2\right\}} \end{aligned}$$

for floater.

To finish the calculation, one needs an explicit model for the swap rate $S_{\alpha,\beta}(t)$. There are two simple models one can use. The first model is Black's model, which assumes that the swap rate $S_{\alpha,\beta}(T_\alpha)$ is log normal with a volatility σ_B:

$$S_{\alpha,\beta}(T_\alpha) = S_{\alpha,\beta}(0)e^{\sigma_B W_{T_\alpha} - \frac{1}{2}\sigma_B^2 T_\alpha}.$$

With this model, one obtains (see Appendix D)

$$V_0^{floater} = P(0,T')S_{\alpha,\beta}(0) + G'(S_{\alpha,\beta}(0))N_0^{\alpha,\beta}(S_{\alpha,\beta}(0))^2\left(e^{\sigma_B^2 T_\alpha}-1\right)$$

for CMS floaters,

$$\begin{aligned} & V_0^{caplet} \\ = & \frac{P(0,T')}{N_0^{\alpha,\beta}}C(K) + G'(S_{\alpha,\beta}(0))N_0^{\alpha,\beta}\Big[(S_{\alpha,\beta}(0))^2 e^{\sigma_B^2 T_\alpha}\Phi(d_{3/2}) - S_{\alpha,\beta}(0)(S_{\alpha,\beta}(0)+K)\Phi(d_{1/2}) \\ & + S_{\alpha,\beta}(0)K\Phi(d_{-1/2})\Big] \end{aligned}$$

for CMS caplets, and

$$\begin{aligned} & V_0^{floorlet} \\ = & \frac{P(0,T')}{N_0^{\alpha,\beta}}P(K) - G'(S_{\alpha,\beta}(0))N_0^{\alpha,\beta}\Big[(S_{\alpha,\beta}(0))^2 e^{\sigma_B^2 T_\alpha}\Phi(-d_{3/2}) - S_{\alpha,\beta}(0)(S_{\alpha,\beta}(0)+K)\Phi(-d_{1/2}) \\ & + S_{\alpha,\beta}(0)K\Phi(-d_{-1/2})\Big] \end{aligned}$$

for CMS floorlets. Here

$$d_\lambda = \frac{\ln\left(\frac{S_{\alpha,\beta}(0)}{K}\right) + \lambda\sigma_B^2 T_\alpha}{\sigma_B\sqrt{T_\alpha}}.$$

[1] The interpretation of $\int_K^\infty C(\zeta)d\zeta = \frac{1}{2}N_0^{\alpha,\beta}E^{Q^{\alpha,\beta}}\left\{([S_{\alpha,\beta}(T_\alpha)-K]^+)^2\right\}$ is that the value of the right side is obtained through the value of the left side: $C(K)$'s are directly given by market quotes for various K's, and we don't know the distribution of $S_{\alpha,\beta}(T_\alpha)$ without further assumptions.

The second model is the normal, or absolute model, which assumes that the swap rate follows

$$dS_{\alpha,\beta}(t) = \sigma_N dW_t.$$

This yields

$$V_0^{floater} = P(0,T')S_{\alpha,\beta}(0) + G'(S_{\alpha,\beta}(0))N_0^{\alpha,\beta}\sigma_N^2 T_\alpha$$

for the CMS floaters,

$$V_0^{caplet} = \frac{P(0,T')}{N_0^{\alpha,\beta}}C(K) + G'(S_{\alpha,\beta}(0))N_0^{\alpha,\beta}\sigma_N^2 T_\alpha \Phi\left(\frac{S_{\alpha,\beta}(0)-K}{\sigma_N\sqrt{T_\alpha}}\right)$$

for CMS caplets, and

$$V_0^{floorlet} = \frac{P(0,T')}{N_0^{\alpha,\beta}}P(K) - G'(S_{\alpha,\beta}(0))N_0^{\alpha,\beta}\sigma_N^2 T_\alpha \Phi\left(\frac{K-S_{\alpha,\beta}(0)}{\sigma_N\sqrt{T_\alpha}}\right)$$

for CMS floorlets. We can obtain the normal vol σ_N by noting that if σ_B is the lognormal volatility for a swaption with forward rate $S_{\alpha,\beta}(0)$ and strike K, then the normal volatility σ_N of this swaption is

$$\sigma_N = \sigma_B \frac{S_{\alpha,\beta}(0)-K}{\ln\left(\frac{S_{\alpha,\beta}(0)}{K}\right)} \frac{1}{1+\frac{1}{24}\sigma_B^2 T_\alpha + \frac{1}{5760}\sigma_B^4 T_\alpha^2}$$

Near the money ($(S_{\alpha,\beta}(0)-K)/K$ is less than 20% or so), we can replace this formula with

$$\sigma_N = \sigma_B\sqrt{S_{\alpha,\beta}(0)K} \cdot \frac{1+\frac{1}{24}\ln^2\left(\frac{S_{\alpha,\beta}(0)}{K}\right)+\frac{1}{1920}\ln^4\left(\frac{S_{\alpha,\beta}(0)}{K}\right)}{1+\frac{1}{24}\sigma_B^2 T_\alpha + \frac{1}{5760}\sigma_B^4 T_\alpha^2}$$

N.B. The key concern with Black's model is that it does not address the smiles and/or skews seen in the marketplace. This can be partially mitigated by using the correct volatilities. For CMS floaters, the volatility σ_{atm} for at-the-money swpations should be used, since the expected value includes high and low strike swaption equally. For out-of-the-money caplets and floorlets, the volatility σ_K for strike K should be used, since the swap rate $S_{\alpha,\beta}(T_\alpha)$ near K provide the largest contribution to the expected value. For in-the-money options, the largest contributions come from swap rates $S_{\alpha,\beta}(T_\alpha)$ near the mean value $S_{\alpha,\beta}(0)$. Accordingly, call-put parity should be used to evaluate in-the-money caplets and floorlets as a CMS swap payment plus an out-of-the-money floorlet or caplet.

3.3.5 CMS digitals paid at arbitrary time under Hagan's model

To compute the time-zero value of a CMS digital call/put, we note

$$\begin{aligned}
& V_0(1_{\{S_{\alpha,\beta}(T_\alpha)>K\}} \text{ paid at time } T' \geq T_\alpha) \\
= \; & P(0,T')E^{Q_{T'}}\left[1_{\{S_{\alpha,\beta}(T_\alpha)>K\}}\right] \\
= \; & -\frac{d}{dK}P(0,T')E^{Q_{T'}}\left[(S_{\alpha,\beta}(T_\alpha)-K)^+\right] \\
= \; & -\frac{d}{dK}V_0([S_{\alpha,\beta}(T_\alpha)-K]^+ \text{ paid at time } T' \geq T_\alpha)
\end{aligned}$$

Similarly,

$$V_0(1_{\{S_{\alpha,\beta}(T_\alpha)<K\}} \text{ paid at time } T' \geq T_\alpha) = \frac{d}{dK}V_0([K-S_{\alpha,\beta}(T_\alpha)]^+ \text{ paid at time } T' \geq T_\alpha).$$

Therefore, from the integral representation of CMS caplet/floorlet price, we have

$$\boxed{\begin{aligned}
& V_0(1_{\{S_{\alpha,\beta}(T_\alpha)>K\}} \text{ paid at time } T' \geq T_\alpha) \\
= & -\frac{P(0,T')}{N_0^{\alpha,\beta}}\frac{d}{dK}C(K) + G'(S_{\alpha,\beta}(0))\left[(S_{\alpha,\beta}(0)-K)\frac{d}{dK}C(K)+C(K)\right]
\end{aligned}}$$

$$\boxed{\begin{aligned}&V_0(1_{\{S_{\alpha,\beta}(T_\alpha)<K\}}\text{ paid at time }T'\geq T_\alpha)\\ =&\frac{P(0,T')}{N_0^{\alpha,\beta}}\frac{d}{dK}P(K)-G'(S_{\alpha,\beta}(0))\left[(S_{\alpha,\beta}(0)-K)\frac{d}{dK}P(K)+P(K)\right]\end{aligned}}$$

Under the lognormal model,

$$C(K)=N_0^{\alpha,\beta}\left[S_{\alpha,\beta}(0)\Phi(d_1)-K\Phi(d_2)\right]$$

where $d_1=d_1(\sigma_K)=\frac{\ln\left(\frac{S_{\alpha,\beta}(0)}{K}\right)+\frac{1}{2}\sigma_K^2T_\alpha}{\sigma_K\sqrt{T_\alpha}}$ and $d_2=d_2(\sigma_K)=d_1(\sigma_K)-\sigma_K\sqrt{T_\alpha}$. Since

$$\begin{aligned}&\frac{d}{dK}\left[S_{\alpha,\beta}(0)\Phi(d_1)-K\Phi(d_2)\right]=\frac{\partial}{\partial K}\left[S_{\alpha,\beta}(0)\Phi(d_1)-K\Phi(d_2)\right]+\frac{\partial}{\partial\sigma}\left[S_{\alpha,\beta}(0)\Phi(d_1)-K\Phi(d_2)\right]\frac{d\sigma_K}{dK}\\ =&-\frac{1}{K\sigma_K\sqrt{T_\alpha}}\left[S_{\alpha,\beta}(0)\Phi'(d_1)-K\Phi'(d_2)\right]-\Phi(d_2)+S_{\alpha,\beta}(0)\Phi'(d_1)\sqrt{T_\alpha}\frac{d\sigma_K}{dK}\end{aligned}$$

and

$$S_{\alpha,\beta}(0)\Phi'(d_1)=S_{\alpha,\beta}(0)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{(d_2+\sigma_K\sqrt{T_\alpha})^2}{2}}=S_{\alpha,\beta}(0)e^{-d_2\sigma_K\sqrt{T_\alpha}-\frac{1}{2}\sigma_K^2T_\alpha}\Phi'(d_2)=K\Phi'(d_2),$$

we conclude $\frac{d}{dK}C(K)=N_0^{\alpha,\beta}\left[-\Phi(d_2)+\Phi'(d_2)K\frac{d\sigma_K}{dK}\sqrt{T_\alpha}\right]$. Therefore, the value of a **digital call** under lognormal model is

$$\begin{aligned}&V_0(1_{\{S_{\alpha,\beta}(T_\alpha)>K\}}\text{ paid at time }T'\geq T_\alpha)\\ =&P(0,T')\left[\Phi(d_2)-\Phi'(d_2)K\frac{d\sigma_K}{dK}\sqrt{T_\alpha}\right]+G'(S_{\alpha,\beta}(0))N_0^{\alpha,\beta}S_{\alpha,\beta}(0)\left\{\Phi(d_1)-\Phi(d_2)\right.\\ &\left.-[\Phi'(d_1)-\Phi'(d_2)]K\frac{d\sigma_K}{dK}\sqrt{T_\alpha}\right\}\end{aligned}$$

By the call-put parity, the value of a **digital put** under lognormal model is

$$\begin{aligned}&V_0(1_{\{S_{\alpha,\beta}(T_\alpha)<K\}}\text{ paid at time }T'\geq T_\alpha)\\ =&P(0,T')-V_0(1_{\{S_{\alpha,\beta}(T_\alpha)>K\}}\text{ paid at time }T'\geq T_\alpha)\\ =&P(0,T')\left[\Phi(-d_2)+\Phi'(d_2)K\frac{d\sigma_K}{dK}\sqrt{T_\alpha}\right]+G'(S_{\alpha,\beta}(0))N_0^{\alpha,\beta}S_{\alpha,\beta}(0)\left\{\Phi(d_2)-\Phi(d_1)\right.\\ &\left.-[\Phi'(d_2)-\Phi'(d_1)]K\frac{d\sigma_K}{dK}\sqrt{T_\alpha}\right\}\end{aligned}$$

Under the normal model,

$$C(K)=N_0^{\alpha,\beta}\left[(S_{\alpha,\beta}(0)-K)\Phi(d_1)+\Phi'(d_1)\sigma\sqrt{T_\alpha}\right]$$

where $d_1=\frac{S_{\alpha,\beta}(0)-K}{\sigma\sqrt{T_\alpha}}$ and σ is the normal vol. Since

$$\begin{aligned}&\frac{d}{dK}\left[(S_{\alpha,\beta}(0)-K)\Phi(d_1)+\Phi'(d_1)\sigma\sqrt{T_\alpha}\right]\\ =&\frac{\partial}{\partial K}\left[(S_{\alpha,\beta}(0)-K)\Phi(d_1)+\Phi'(d_1)\sigma\sqrt{T_\alpha}\right]+\frac{\partial}{\partial\sigma}\left[(S_{\alpha,\beta}(0)-K)\Phi(d_1)+\Phi'(d_1)\sigma\sqrt{T_\alpha}\right]\frac{d\sigma}{dK}\\ =&-\Phi(d_1)+\Phi'(d_1)\sqrt{T_\alpha}\frac{d\sigma}{dK},\end{aligned}$$

we conclude $\frac{d}{dK}C(K)=N_0^{\alpha,\beta}\left[-\Phi(d_1)+\Phi'(d_1)\sqrt{T_\alpha}\frac{d\sigma}{dK}\right]$. Therefore, the value of a **digital call** under normal model is

$$\begin{aligned}&V_0(1_{\{S_{\alpha,\beta}(T_\alpha)>K\}}\text{ paid at time }T'\geq T_\alpha)\\ =&P(0,T')\left[\Phi(d_1)-\Phi'(d_1)\sqrt{T_\alpha}\frac{d\sigma}{dK}\right]+G'(S_{\alpha,\beta}(0))N_0^{\alpha,\beta}\Phi'(d_1)\sqrt{T_\alpha}\left[(S_{\alpha,\beta}(0)-K)\frac{d\sigma}{dK}+\sigma\right]\end{aligned}$$

By the call-put parity, the value of a **digital put** under normal model is

$$\begin{aligned} & V_0(1_{\{S_{\alpha,\beta}(T_\alpha)<K\}} \text{ paid at time } T' \geq T_\alpha) \\ = \; & P(0,T')\left[\Phi(-d_1)+\Phi'(d_1)\sqrt{T_\alpha}\frac{d\sigma}{dK}\right] + G'(S_{\alpha,\beta}(0))N_0^{\alpha,\beta}\Phi'(d_1)\sqrt{T_\alpha}\left[(K-S_{\alpha,\beta}(0))\frac{d\sigma}{dK}-\sigma\right] \end{aligned}$$

3.3.6 CMS fracl paid at arbitrary time under linear rate model

A floating CMS range accrual pays out

$$\left[\text{leverage}\cdot S_1(t_0^1)+\text{spread}\right]1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}+\text{fixRate}$$

at time $t_p \geq t_0^2 \geq t_0^1$, where $S_1(\cdot)$ is a swap rate with schedule $t_0^1 < t_1^1 < \cdots < t_{n_1}^1$ and corresponding coverage $\tau_1^1 < \cdots < \tau_{n_1}^1$, and $S_2(\cdot)$ is a swap rate with schedule $t_0^2 < t_1^2 < \cdots < t_{n_2}^2$ and corresponding coverage $\tau_1^2 < \cdots < \tau_{n_2}^2$. The time-zero value of the payoff is

$$\begin{aligned} V_0 \;=\; & P(0,t_p)\left(\text{leverage}\cdot E^{Q_{t_p}}\left[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}S_1(t_0^1)\right]+\text{spread}\cdot E^{Q_{t_p}}\left[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}\right]\right) \\ & +P(0,t_p)\cdot\text{fixRate} \end{aligned}$$

Denote by L_1 and L_2 the annuities corresponding to S_1 and S_2, respectively. Then by formula (5) from the linear rate model

$$\begin{aligned} E^{Q_{t_p}}\left[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}S_1(t_0^1)\right] &= E^{L_2}\left[\frac{P(t_0^2,t_p)/P(0,t_p)}{L_2(t_0^2)/L_2(0)}\cdot 1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}S_1(t_0^1)\right] \\ &= E^{L_2}\left[(1+a+bS_2(t_0^2))\cdot 1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}S_1(t_0^1)\right] \end{aligned}$$

where $a = \frac{L_2(0)}{P(0,t_p)\sum_{i=1}^{n_2}\tau_i^2}-1$, $b=\frac{1}{S_2(0)}\left[1-\frac{L_2(0)}{P(0,t_p)\sum_{i=1}^{n_2}\tau_i^2}\right]$. Assume $E^{L_2}[S_1(t_0^1)|S_2(t_0^2)]$ is a linear function of $S_2(t_0^2)$:

$$E^{L_2}[S_1(t_0^1)|S_2(t_0^2)]-E^{L_2}[S_1(t_0^1)] \approx C\left(S_2(t_0^2)-E^{L_2}[S_2(t_0^2)]\right) = C[S_2(t_0^2)-S_2(0)]$$

By requiring $E^{L_2}\left[E^{L_2}[S_1(t_0^1)|S_2(t_0^2)]\cdot S_2(t_0^2)\right] = E^{L_2}\left[S_1(t_0^1)S_2(t_0^2)\right]$, we obtain

$$C = corr^{L_2}(S_1(t_0^1),S_2(t_0^2))\sqrt{\frac{var^{L_2}S_1(t_0^1)}{var^{L_2}S_2(t_0^2)}}$$

Consequently, we have

$$\begin{aligned} & E^{Q_{t_p}}\left[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}S_1(t_0^1)\right] \\ = \; & E^{L_2}\left[(1+a+bS_2(t_0^2))\cdot 1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}\left(E^{L_2}[S_1(t_0^1)]+C\left[S_2(t_0^2)-S_2(0)\right]\right)\right] \end{aligned}$$

To compute $E^{L_2}[S_1(t_0^1)]$, we apply the linear rate model $\frac{L_2(t)}{L_1(t)} = a_1 + b_1 S_1(t)$

$$\begin{aligned} E^{L_2}[S_1(t_0^1)] &= E^{L_1}\left[\frac{L_2(t_0^1)/L_2(0)}{L_1(t_0^1)/L_1(0)}S_1(t_0^1)\right] \\ &= E^{L_1}\left[(a_1+b_1S_1(t_0^1))S_1(t_0^1)\right] \\ &= a_1S_1(0)+b_1S_1^2(0)+b_1var^{L_1}S_1(t_0^1) \end{aligned}$$

where $a_1 = \frac{L_1(0)\sum_{i=1}^{n_2}\tau_i^2}{L_2(0)\sum_{j=1}^{n_1}\tau_j^1}$ and $b_1 = \frac{1}{S_1(0)}\left[1-\frac{L_1(0)\sum_{i=1}^{n_2}\tau_i^2}{L_2(0)\sum_{j=1}^{n_1}\tau_j^1}\right]$ (see Appendix A). Define

$$E_0 = E^{Q_{t_p}}\left[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}\right],$$

then

$$V_0 = P(0,t_p)\left(\text{leverage}\cdot E^{Q_{t_p}}\left[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}S_1(t_0^1)\right]+\text{spread}\cdot E_0\right)+P(0,t_p)\cdot\text{fixRate}$$

Define

$$\begin{cases}\widetilde{E}_0 = E^{L_2}\left[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}\right]\\ \widetilde{E}_1 = E^{L_2}\left[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}S_2(t_0^2)\right]\\ \widetilde{E}_2 = E^{L_2}\left[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}S_2^2(t_0^2)\right]\\ \widetilde{E}_0cc = a\widetilde{E}_0+b\widetilde{E}_1 = E^{L_2}\left[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}(a+bS_2(t_0^2))\right]\\ \widetilde{E}_1cc = a\widetilde{E}_1+b\widetilde{E}_2 = E^{L_2}\left[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}S_2(t_0^2)(a+bS_2(t_0^2))\right]\\ \widetilde{EA}_1 = E^{L_1}[(a_1+b_1S_1(t_0^1))S_1(t_0^1)] = E^{L_2}[S_1(t_0^1)]\end{cases}$$

We can write $E^{Q_{t_p}}\left[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}S_1(t_0^1)\right]$ as

$$\begin{aligned}
& E^{Q_{t_p}}\left[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}S_1(t_0^1)\right]\\
= \;& E^{L_2}\left[(1+a+bS_2(t_0^2))\cdot 1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}\left(E^{L_2}[S_1(t_0^1)]+C\left[S_2(t_0^2)-S_2(0)\right]\right)\right]\\
= \;& E^{L_2}\left[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}\left(E^{L_2}[S_1(t_0^1)]+C\left[S_2(t_0^2)-S_2(0)\right]\right)\right]\\
& +E^{L_2}\left[(a+bS_2(t_0^2))\cdot 1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}\right]\left(E^{L_2}[S_1(t_0^1)]-C\cdot S_2(0)\right)\\
& +E^{L_2}\left[(a+bS_2(t_0^2))\cdot 1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}C\cdot S_2(t_0^2)\right]\\
= \;& [\widetilde{EA}_1 - C\cdot S_2(0)]\cdot\widetilde{E}_0 + C\cdot\widetilde{E}_1 + [\widetilde{EA}_1 - C\cdot S_2(0)]\cdot\widetilde{E}_0cc + C\cdot\widetilde{E}_1cc\\
= \;& \widetilde{E}+\widetilde{cc}
\end{aligned}$$

where $\widetilde{E} = [\widetilde{EA}_1 - C\cdot S_2(0)]\cdot\widetilde{E}_0 + C\cdot\widetilde{E}_1$ and $\widetilde{cc} = [\widetilde{EA}_1 - C\cdot S_2(0)]\cdot\widetilde{E}_0cc + C\cdot\widetilde{E}_1cc$. Therefore

$$\boxed{V_0 = P(0,t_p)\left[\text{leverage}\cdot(\widetilde{E}+\widetilde{cc})+\text{spread}\cdot E_0\right]+P(0,t_p)\cdot\text{fixRate}} \tag{19}$$

The following table shows how to compute the various quantities in the above derivation

Formula	Black Model ($S>0$ always) $S_1(t_0^1) = S_1(0)e^{\sigma_B^1 W_{t_0^1}-\frac{1}{2}(\sigma_B^1)^2t_0^1}$, $S_2(t_0^2) = S_2(0)e^{\sigma_B^2 W_{t_0^2}-\frac{1}{2}(\sigma_B^2)^2t_0^2}$
$E_0 = E^{Q_{t_p}}\left[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}\right]$	replication (Appendix C)
$\widetilde{E}_0 = E^{L_2}[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}]$	$\Phi\left(\frac{\ln\frac{R_{\max}}{S_2(0)}}{\sigma_B^2\sqrt{t_0^2}}+\frac{1}{2}\sigma_B^2\sqrt{t_0^2}\right)-\Phi\left(\frac{\ln\frac{R_{\min}}{S_2(0)}}{\sigma_B^2\sqrt{t_0^2}}+\frac{1}{2}\sigma_B^2\sqrt{t_0^2}\right)$
$\widetilde{E}_1 = E^{L_2}[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}S_2(t_0^2)]$	$S_2(0)\left[\Phi\left(\frac{\ln\frac{R_{\max}}{S_2(0)}}{\sigma_B^2\sqrt{t_0^2}}-\frac{1}{2}\sigma_B^2\sqrt{t_0^2}\right)-\Phi\left(\frac{\ln\frac{R_{\min}}{S_2(0)}}{\sigma_B^2\sqrt{t_0^2}}-\frac{1}{2}\sigma_B^2\sqrt{t_0^2}\right)\right]$
$\widetilde{E}_2 = E^{L_2}[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}S_2^2(t_0^2)]$	$S_2^2(0)e^{(\sigma_B^2)^2t_0^2}\left[\Phi\left(\frac{\ln\frac{R_{\max}}{S_2(0)}}{\sigma_B^2\sqrt{t_0^2}}-\frac{3}{2}\sigma_B^2\sqrt{t_0^2}\right)-\Phi\left(\frac{\ln\frac{R_{\min}}{S_2(0)}}{\sigma_B^2\sqrt{t_0^2}}-\frac{3}{2}\sigma_B^2\sqrt{t_0^2}\right)\right]$
$\widetilde{E}_0cc = a\widetilde{E}_0+b\widetilde{E}_1$	
$\widetilde{E}_1cc = a\widetilde{E}_1+b\widetilde{E}_2$	
$\widetilde{EA}_1 = E^{L_1}[(a_1+b_1S_1(t_0^1))S_1(t_0^1)]$	$a_1S_1(0)+b_1S_1^2(0)e^{(\sigma_B^1)^2t_0^1}$
$\widetilde{E} = [\widetilde{EA}_1 - C\cdot S_2(0)]\cdot\widetilde{E}_0 + C\cdot\widetilde{E}_1$	
$\widetilde{cc} = [\widetilde{EA}_1 - C\cdot S_2(0)]\cdot\widetilde{E}_0cc + C\cdot\widetilde{E}_1cc$	

By noting $\Phi''(x) = -x\Phi'(x)$, it's easy to derive $\int_{-\infty}^{y} x\Phi'(x)dx = -\Phi'(y)$ and $\int_{-\infty}^{y} x^2\Phi'(x)dx = \Phi(y) - y\Phi'(y)$. Therefore we have

Formula	Normal Model (S could be negative) $S_1(t_0^1) = S_1(0) + \sigma_N^1 W_{t_0^1},\ S_2(t_0^1) = S_2(0) + \sigma_N^2 W_{t_0^2}$
$E_0 = E^{Q_{t_p}}\left[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}\right]$	replication (Appendix C)
$\widetilde{E}_0 = E^{L_2}[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}]$	$\Phi\left(\frac{R_{\max}-S_2(0)}{\sigma_N^2\sqrt{t_0^2}}\right) - \Phi\left(\frac{R_{\min}-S_2(0)}{\sigma_N^2\sqrt{t_0^2}}\right)$
$\widetilde{E}_1 = E^{L_2}[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}S_2(t_0^2)]$	$\sigma_N^2\sqrt{t_0^2}\left[\Phi'\left(\frac{R_{\min}-S_2(0)}{\sigma_N^2\sqrt{t_0^2}}\right) - \Phi'\left(\frac{R_{\max}-S_2(0)}{\sigma_N^2\sqrt{t_0^2}}\right)\right]$ $+S_2(0)\left[\Phi\left(\frac{R_{\max}-S_2(0)}{\sigma_N^2\sqrt{t_0^2}}\right) - \Phi\left(\frac{R_{\min}-S_2(0)}{\sigma_N^2\sqrt{t_0^2}}\right)\right]$
$\widetilde{E}_2 = E^{L_2}[1_{\{R_{\min}<S_2(t_0^2)<R_{\max}\}}S_2^2(t_0^2)]$	$(\sigma_N^2)^2 t_0^2\left[\Phi(y) - y\Phi'(y)\right]\Big\vert_{(R_{\min}-S_2(0))/(\sigma_N^2 t_0^2)}^{(R_{\max}-S_2(0))/(\sigma_N^2 t_0^2)}$ $+2S_2(0)\widetilde{E}_1 - S_2^2(0)\widetilde{E}_0$
$\widetilde{E}_0 cc = a\widetilde{E}_0 + b\widetilde{E}_1$	
$\widetilde{E}_1 cc = a\widetilde{E}_1 + b\widetilde{E}_2$	
$\widetilde{EA}_1 = E^{L_1}[(a_1 + b_1 S_1(t_0^1))S_1(t_0^1)]$	$a_1 S_1(0) + b_1 S_1^2(0) + b_1(\sigma_N^1)^2 t_0^1$
$\widetilde{E} = [\widetilde{EA}_1 - C\cdot S_2(0)]\cdot\widetilde{E}_0 + C\cdot\widetilde{E}_1$	
$\widetilde{cc} = [\widetilde{EA}_1 - C\cdot S_2(0)]\cdot\widetilde{E}_0 cc + C\cdot\widetilde{E}_1 cc$	

4 Comparison of various convexity adjusted prices

4.1 LIBOR-in-arrears swaps

We compare pricing formulas of LIBOR-in-arrears swaps under replication (formula (13)), convexity correction with adjusted forward rate (formula (2)), and convexity correction with adjusted forward rate (formula (2)) and adjusted volatility (formula (10))

$$
\begin{aligned}
& V_0(\tau(S,T)L(S,T) \text{ paid at time } S) \\
= & \begin{cases} V_0^{caplet}(0) + 2\tau(S,T)\int_0^\infty V_0^{caplet}(K)dK & \text{replication} \\ \tau(S,T)P(0,S)F(0;S,T)\left[1 + \frac{\tau(S,T)F(0;S,T)(\exp\{\sigma_{atm}^2 S\}-1)}{1+\tau(S,T)F(0;S,T)}\right] & \text{adj Fwd} \\ \tau(S,T)P(0,S)F(0;S,T)\left[1 + \frac{\tau(S,T)F(0;S,T)(\exp\{(\sigma^*)^2 S\}-1)}{1+\tau(S,T)F(0;S,T)}\right] & \text{adj Fwd \& Vol} \end{cases}
\end{aligned}
$$

where σ_{atm} is the at-the-money volatility of the caplet for the period $[S, T]$ and

$$
\begin{aligned}
(\sigma^*)^2 &= (\sigma_{atm})^2 + \ln\left\{\frac{[a + b(S)F(0;S,T)][a + b(S)F(0;S,T)e^{2\sigma_{atm}^2 S}]}{[a + b(S)F(0;S,T)e^{\sigma_{atm}^2 S}]^2}\right\}/S \\
&= (\sigma_{atm})^2 + \ln\left\{\frac{[1 + \tau(S,T)F(0;S,T)][1 + \tau(S,T)F(0;S,T)e^{2\sigma_{atm}^2 S}]}{[1 + \tau(S,T)F(0;S,T)e^{\sigma_{atm}^2 S}]^2}\right\}/S
\end{aligned}
$$

The case of "adj Fwd & Vol" is equivalent to assuming $L(S,T)$ is lognormal under Q_S, while the case of "adj Fwd" is kind of halfway from "lognormal under Q_T" to "lognormal under Q_S".

Numerical experiments show results by these formulas are quite close and the two naive approaches (adjusted forward rates without and with additional adjusted volatilities) yield results whose difference to the correct one from replication are negligible in practice.

4.2 In-arrears caplets

We compare pricing formulas of in-arrears caplets under replication (formula (14)), Black formula with adjusted forward rate (formula (2)), and Black's formula with adjusted forward rate (formula (2)) and adjusted volatility (formula (10))

$$
\begin{aligned}
& V_0(\tau(S,T)[L(S,T)-K]^+ \text{ paid at time } S) \\
= & \begin{cases} V_0^{caplet}(K)[1+\tau(S,T)K] + 2\tau(S,T)\int_K^\infty V_0^{caplet}(\widetilde{K})d\widetilde{K} & \text{replication} \\ \tau(S,T)P(0,S)\left[F_{adj}(\sigma_{strike})\Phi(d_1) - K\Phi(d_2)\right] & \text{Black formula with adj Fwd} \\ \tau(S,T)P(0,S)\left[F_{adj}(\sigma^*)\Phi(d_1^*) - K\Phi(d_2^*)\right] & \text{Black formula with adj Fwd \& Vol} \end{cases}
\end{aligned}
$$

where $F_{adj}(\sigma) = F(0;S,T)\left[1 + \frac{\tau(S,T)F(0;S,T)(\exp\{\sigma^2 S\}-1)}{1+\tau(S,T)F(0;S,T)}\right]$ is the convexity adjusted forward rate, σ_{strike} is the volatility taken from the smile according to the strike rate of the caplet, $\Phi(\cdot)$ is the c.d.f. of a standard normal distribution,

$$d_1 = \frac{\ln(F_{adj}(\sigma_{strike})/K) + \sigma_{strike}^2 S/2}{\sigma_{strike}\sqrt{S}},\ d_2 = d_1 - \sigma_{strike}\sqrt{S},$$

and

$$(\sigma^*)^2 = (\sigma_{strike})^2 + \ln\left\{\frac{[1+\tau(S,T)F(0;S,T)][1+\tau(S,T)F(0;S,T)e^{2\sigma_{strike}^2 S}]}{[1+\tau(S,T)F(0;S,T)e^{\sigma_{strike}^2 S}]^2}\right\}/S$$

$$d_1^* = \frac{\ln(F_{adj}(\sigma^*)/K) + (\sigma^*)^2 S/2}{\sigma^*\sqrt{S}},\ d_2^* = d_1^* - \sigma^*\sqrt{S}.$$

The case of "Black formula with adj Fwd & Vol" is the market standard method of valuing options on convexity adjusted rates: assume the underlying rate is lognormal under Q_S and apply the Black formula.

Numerical experiments show results by these formulas are quite close and the two naive approaches (adjusted forward rates without and with additional adjusted volatilities) yield results whose difference to the correct one from replication are negligible in practice.

4.3 CMS swaps and CMS caps

Formula (16) and formula (18) give prices by replication. For prices in terms of adjusted swap rate, the adjusted swap rate is given by

$$E^{Q_{T'}}[S_{\alpha,\beta}(T_\alpha)] = S_{\alpha,\beta}(0)\left[1 + \left(1 - \frac{P(0,T_\alpha) - P(0,T_\beta)}{S_{\alpha,\beta}(0)P(0,T')\sum_{i=\alpha+1}^{\beta}\tau_i}\right)\left(e^{\sigma_{atm}^2 T_\alpha} - 1\right)\right].$$

For price in terms of adjusted forward swap rate and adjusted volatility, the adjusted volatility is given by

$$(\sigma^*)^2 = (\sigma_{atm})^2 + \ln\left\{\frac{[a + b(T')S_{\alpha,\beta}(0)]\left[a + b(T')S_{\alpha,\beta}(0)e^{2(\sigma_{atm})^2 T_\alpha}\right]}{[a + b(T')S_{\alpha,\beta}(0)e^{\sigma_{atm}^2 T_\alpha}]^2}\right\}/T_\alpha.$$

Again, price in terms of adjusted forward rate and adjusted volatility is obtained by assuming the underlying rate is lognormal under $Q_{T'}$ and applying Black formula.

The naive approaches, although taking into account the smile by taking the caplet volatility from the swaption smile, show a significant mispricing relative to the correct valuations based on replication. Comparing the results of the two replication formulas, it turns out that the replication based on the idea of cash-settled swaptions consistently leads to slightly higher CMS caplet prices.

5 Variable interest rates in foreign currency

We are interested in the price of a domestic interest rate to be paid in foreign currency units at some time $T > 0$.

5.1 Multi-currency change of numeraire theorem

Suppose we have a domestic economy d and a foreign economy f, together with the exchange rate X that expresses the value of one unit foreign currency in terms of domestic currency. Let N^d denote a numeraire process with associated martingale measure Q^d for the domestic economy, and N^f a numeraire process with associated martingale measure Q^f for the foreign economy.

The domestic value today of a domestic payoff Z_T^d to be paid in foreign units and at time T is by the general theory

$$X_0 N_0^f E^{Q^f}\left[\frac{Z_T^d}{N_T^f}\right].$$

On the other hand, the same payoff translated back into domestic currency with the exchange rate at time T should trade at the same domestic price, therefore

$$X_0 N_0^f E^{Q^f}\left[\frac{Z_T^d}{N_T^f}\right] = N_0^d E^{Q^d}\left[\frac{Z_T^d X_T}{N_T^d}\right].$$

This gives us the multi-currency change of numeraire theorem

$$\boxed{\left.\frac{dQ^d}{dQ^f}\right|_{\mathcal{F}_T} = \frac{N_T^d/N_0^d}{N_T^f/N_0^f}\cdot\frac{X_0}{X_T},\ \left.\frac{dQ^f}{dQ^d}\right|_{\mathcal{F}_T} = \frac{N_T^f/N_0^f}{N_T^d/N_0^d}\cdot\frac{X_T}{X_0}.} \tag{20}$$

5.2 Quanto adjustments

We consider a domestic variable interest rate Y_S^d set at time S and paid at time $T' \geq S$ in foreign currency units. Let $P^f(\cdot, T)$ denote the time-T maturity foreign zero coupon bond and Q_T^f the associated foreign T-forward measure. The time-zero value of Y_S^d in foreign currency is

$$P^f(0,T')E^{Q^f_{T'}}[Y_S^d] = P^f(0,T')E^{Q^d}\left[Y_S^d \frac{P^f(S,T')/P^f(0,T')}{N_S^d/N_0^d}\cdot\frac{X_S}{X_0}\right] = \frac{N_0^d}{X_0}E^{Q^d}\left[Y_S^d \frac{X_S P^f(S,T')}{P^d(S,T')}\frac{P^d(S,T')}{N_S^d}\right]$$

By definition the ratio $\frac{P^d(S,T')}{N_S^d}$ is a Q^d-martingale in the time variable $S \leq T'$. The expression $\frac{X_S P^f(S,T')}{P^d(S,T')}$ is the time-S forward foreign exchange rate for delivery at time $T' \geq S$.

Assumption 1: We assume the domestic numeraire N^d is the natural numeraire associated with Y_S^d such that $Y_0^d = E^{Q^d}[Y_S^d]$, and

$$Y_S^d = Y_0^d e^{\sigma_Y W_S - \frac{1}{2}\sigma_Y^2 S}$$

where W is a standard Brownian motion under Q^d.

Assumption 2: We assume the linear rate model for $\frac{P^d(S,T')}{N_S^d}$:

$$\frac{P^d(S,T')}{N_S^d} = a + b(T')Y_S^d,\ T' \geq S.$$

Assumption 3: The forward foreign exchange rate is lognormally distributed:

$$\frac{X_S P^f(S,T')}{P^d(S,T')} = X_0^* e^{\sigma_{fx} W_S^{fx} - \frac{1}{2}\sigma_{fx}^2 S}$$

where W^{fx} is a standard Brownian motion under Q^d and $X_0^* = E^{Q^d}\left[\frac{X_S P^f(S,T')}{P^d(S,T')}\right]$. Here σ_{fx} is identified with the implied volatility of a foreign exchange rate option with maturity S, which is a crucial simplification as long as the payment date T' is not close to S.

To calculate X_0^*, we note

$$\frac{X_0 P^f(0,T')}{N_0^d} = E^{Q^d}\left[\frac{X_S P^f(S,T')}{N_S^d}\right] = E^{Q^d}\left[\frac{X_S P^f(S,T')}{P^d(S,T')}\frac{P^d(S,T')}{N_S^d}\right] = X_0^*[a + b(T')Y_0^d e^{\rho\sigma_{fx}\sigma_Y S}],$$

with ρ as correlation between the driving Brownian motions W^{fx} and W. Therefore

$$X_0^* = \frac{X_0 P^f(0,T')}{N_0^d[a + b(T')Y_0^d e^{\rho\sigma_{fx}\sigma_Y S}]}.$$

Through tedious calculation, we can conclude

$$\boxed{E^{Q^f_{T'}}[Y_S^d] = Y_0^d \frac{e^{\rho\sigma_{fx}\sigma_Y S}[a + b(T')Y_0^d e^{\rho\sigma_{fx}\sigma_Y S + \sigma_Y^2 S}]}{a + b(T')Y_0^d e^{\rho\sigma_{fx}\sigma_Y S}}}$$

5.3 Quantoed options on interest rates

Under Assumption 1-3, the call option on the domestic variable rate Y_S^d set at time S and paid at time $T' \geq S$ in foreign currency units is given by

$$\begin{aligned}
& P^f(0,T')E^{Q^f_{T'}}[(Y_S^d - K)^+] \\
= \quad & \frac{P^f(0,T')}{a + b(T')Y_0^d e^{\rho\sigma_{fx}\sigma_Y S}}\Big[Y_0^d e^{\rho\sigma_{fx}\sigma_Y S}\Phi(d_1 + \rho\sigma_{fx}\sqrt{S})(a - b(T')K) + \\
& b(T')(Y_0^d)^2 e^{(\sigma_Y^2 + 2\rho\sigma_{fx}\sigma_Y)S}\Phi(d_1 + (\rho\sigma_{fx} + \sigma_Y)\sqrt{S}) - aK\Phi(d_2 + \rho\sigma_{fx}\sqrt{S})\Big]
\end{aligned}$$

with

$$d_1 = \frac{\ln\left(\frac{Y_0^d}{K}\right) + \frac{1}{2}\sigma_Y^2 S}{\sigma_Y\sqrt{S}}, \; d_2 = \frac{\ln\left(\frac{Y_0^d}{K}\right) - \frac{1}{2}\sigma_Y^2 S}{\sigma_Y\sqrt{S}}.$$

The corresponding put formula is

$$\begin{aligned}
& P^f(0,T')E^{Q^f_{T'}}[(Y_S^d - K)^+] \\
= \quad & \frac{P^f(0,T')}{a + b(T')Y_0^d e^{\rho\sigma_{fx}\sigma_Y S}}\Big[-Y_0^d e^{\rho\sigma_{fx}\sigma_Y S}\Phi(-d_1 - \rho\sigma_{fx}\sqrt{S})(a - b(T')K) - \\
& b(T')(Y_0^d)^2 e^{(\sigma_Y^2 + 2\rho\sigma_{fx}\sigma_Y)S}\Phi(-d_1 - (\rho\sigma_{fx} + \sigma_Y)\sqrt{S}) + aK\Phi(-d_2 - \rho\sigma_{fx}\sqrt{S})\Big]
\end{aligned}$$

Finally, the formula for a digital is

$$P^f(0,T')E^{Q^f_{T'}}[1_{\{Y_S^d > K\}}] = \frac{P^f(0,T')}{a + b(T')Y_0^d e^{\rho\sigma_{fx}\sigma_Y S}}\left[b(T')Y_0^d e^{\rho\sigma_{fx}\sigma_Y S}\Phi(d_1 + \rho\sigma_{fx}\sqrt{S}) + a\Phi(d_2 + \rho\sigma_{fx}\sqrt{S})\right].$$

Remark 8. *In the special case of Y^d being the domestic LIBOR rate $L^d(S,T)$ for the period of $[S,T]$ and $T' = T$, we have $N^d(\cdot) = P^d(\cdot,T)$ and $a = 1$, $b \equiv 0$ in Assumption 2. The price of a call option on $L^d(S,T)$ paid at time T in foreign currency units is therefore*

$$\begin{aligned}
P^f(0,T)E^{Q^f_T}[(L^d(S,T) - K)^+] &= P^f(0,T)[Y_0^d e^{\rho\sigma_{fx}\sigma_Y S}\Phi(d_1 + \rho\sigma_{fx}\sqrt{S}) - K\Phi(d_2 + \rho\sigma_{fx}\sqrt{S})] \\
&= P^f(0,T)[\widetilde{Y}_0^d\Phi(\widetilde{d}_1) - K\Phi(\widetilde{d}_2)]
\end{aligned}$$

where

$$d_1 = \frac{\ln\left(\frac{\widetilde{Y}_0^d}{K}\right) + \frac{1}{2}\sigma_Y^2 S}{\sigma_Y\sqrt{S}}, \; d_2 = \frac{\ln\left(\frac{\widetilde{Y}_0^d}{K}\right) - \frac{1}{2}\sigma_Y^2 S}{\sigma_Y\sqrt{S}}.$$

This is the market standard method of valuing options on convexity corrected rates: apply the Black formula using the convexity corrected rate as the forward rate.

A Approximation of ratio of annuities under linear rate model

In pricing floating rate CMS range accrual, we need to approximate the ratio of two annuities by swap rate. More formally, suppose $t_0^1 < t_1^1 < \cdots < t_{n_1}^1$ and $t_0^2 < t_1^2 < \cdots < t_{n_2}^2$ are two schedules with their respective day counting conventions $\{\tau_1^1, \tau_2^1, \cdots, \tau_{n_1}^1\}$ and $\{\tau_1^2, \tau_2^2, \cdots, \tau_{n_2}^2\}$. For $k = 1, 2$, annuity $L_k(t)$ is defined as

$$L_k(t) = \sum_{i=1}^{n_k} \tau_i^k P(t, t_i^k),\ t \leq t_0^k$$

and forward swap rate $S_k(t)$ is defined as

$$S_k(t) = \frac{P(t, t_0^k) - P(t, t_{n_k}^k)}{\sum_{i=1}^{n_k} \tau_i^k P(t, t_i^k)} = \frac{P(t, t_0^k) - P(t, t_{n_k}^k)}{L_k(t)},\ t \leq t_0^k.$$

We want to find out the linear approximation $\frac{L_2(t)}{L_1(t)} = \alpha + \beta S_1(t)$. Indeed, by formula (5)

$$\begin{aligned} \frac{L_2(t)}{L_1(t)} &= \sum_{i=1}^{n_2} \frac{\tau_i^2 P(t, t_i^2)}{L_1(t)} = \sum_{i=1}^{n_2} \tau_i^2 \left[a + b(t_i^2) S_1(t)\right] \\ &= \frac{\sum_{i=1}^{n_2} \tau_i^2}{\sum_{j=1}^{n_1} \tau_j^1} + \sum_{i=1}^{n_2} \tau_i^2 \cdot \frac{1}{S_1(0)} \left[\frac{P(0, t_i^2)}{L_1(0)} - \frac{1}{\sum_{j=1}^{n_1} \tau_j^1}\right] \cdot S_1(t) \end{aligned}$$

Therefore, the linear swap rate model implies

$$\boxed{\begin{cases} \frac{L_2(t)}{L_1(t)} = \alpha + \beta S_1(t) \\ \alpha = \frac{\sum_{i=1}^{n_2} \tau_i^2}{\sum_{j=1}^{n_1} \tau_j^1} \\ \beta = \frac{1}{S_1(0)} \cdot \left[\frac{L_2(0)}{L_1(0)} - \frac{\sum_{i=1}^{n_2} \tau_i^2}{\sum_{j=1}^{n_1} \tau_j^1}\right] \end{cases}} \tag{21}$$

B Integral representation of convex functions

This section is based on Niculescu and Persson [7], Section 1.6.

It is well known that the differentiation and the integration are operations inverse to each other. A consequence of this fact is the existence of a certain duality between the class of convex functions on an open interval and the class of nondecreasing functions on that interval.

Given a nondecreasing function $\varphi : I \to \mathbb{R}$ and a point $c \in I$, we can attach to them a new function f, given by

$$f(x) = \int_c^x \varphi(t)dt.$$

It is easy to verify that f is a convex function and f is differentiable at each point of continuity of φ with $f' = \varphi$ at such points.

On the other hand, the subdifferential allows us to state the following generalization of the fundamental formula of integral calculus:

Proposition B.1. *Let $f : I \to \mathbb{R}$ be a continuous convex function and let $\varphi : I \to \mathbb{R}$ be a function such that $\varphi(x) \in \partial f(x)$ for every $x \in intI$. Then for every $a < b$ in I we have*

$$f(b) - f(a) = \int_a^b \varphi(t)dt.$$

Proof. Clearly, we may restrict ourselves to the case where $[a, b] \subset \text{int}I$. If $a = t_0 < t_1 < \cdots < t_n = b$ is a partition of $[a, b]$, then

$$f'_-(t_{k-1}) \leq f'_+(t_{k-1}) \leq \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}} \leq f'_-(t_k) \leq f'_+(t_k)$$

for every k. As $f(b)-f(a)=\sum_{k=1}^{n}[f(t_k)-f(t_{k-1})]$, a moment's reflection shows that

$$f(b)-f(a)=\int_a^b f'_-(t)dt=\int_a^b f'_+(t)dt.$$

As $f'_- \le \varphi \le f'_+$, this forces the equality in the statement. $\square$

A property very useful when joint with Proposition B.1 is the following

Proposition B.2. *If $f: I \to \mathbb{R}$ is absolutely continuous and its derivative φ is of finite variation locally, then f can be written as the difference of two convex functions.*

Proof. By Fundamental Theorem of Calculus for absolutely continuous functions, we have

$$f(b)-f(a)=\int_a^b \varphi(t)dt.$$

By properties of functions of finite variation, φ can be written as the difference of two monotone increasing functions: $\varphi=\varphi^+-\varphi^-$. Then

$$f(x)=f(a)+\int_a^x \varphi^+(t)dt-\int_a^x \varphi^-(t)dt.$$

This proves the claim. $\square$

C Pricing of "trapezoidal" payoff by replication

The payoff of a "trapezoidal" payoff is $1_{\{R_{\min}<Y<R_{\max}\}}(Y\cdot \text{leverage}+\text{spread})$. It can be decomposed as

$$\begin{aligned}
& 1_{\{R_{\min}<Y<R_{\max}\}}(Y\cdot \text{leverage}+\text{spread}) \\
= \; & \text{leverage}\cdot[(Y-R_{\min})^+-(Y-R_{\max})^+]+1_{\{Y>R_{\min}\}}(R_{\min}\cdot \text{leverage}+\text{spread}) \\
& -1_{\{Y>R_{\max}\}}(R_{\max}\cdot \text{leverage}+\text{spread})
\end{aligned}$$

Therefore, we have the following pricing formula via replication

$$\begin{aligned}
V_0 \;=\; & \text{leverage}\cdot[V_0^{caplet}(R_{\min})-V_0^{caplet}(R_{\max})]+V_0^{digital_call}(R_{\min})(R_{\min}\cdot \text{leverage}+\text{spread}) \\
& -V_0^{digital_call}(R_{\max})(R_{\max}\cdot \text{leverage}+\text{spread})
\end{aligned}$$

D Computation of $\int_{K_0}^{\infty} C(K)dK$ and $\int_0^{K_0} P(K)dK$

We recall $C(K)=F\Phi(d_{1/2}(K))-K\Phi(d_{-1/2}(K))$ and $P(K)=-F\Phi(-d_{1/2}(K))+K\Phi(-d_{-1/2}(K))$, where $d_\lambda(K)=\frac{\ln(F/K)}{\sigma\sqrt{T}}+\lambda\sigma\sqrt{T}$. We shall verify that

$$\boxed{\int_K^{\infty} C(x)dx=\frac{1}{2}F^2e^{\sigma^2T}\Phi(d_{3/2}(K))-FK\Phi(d_{1/2}(K))+\frac{1}{2}K^2\Phi(d_{-1/2}(K))}$$

and

$$\boxed{\int_0^{K} P(x)dx=\frac{1}{2}F^2e^{\sigma^2T}\Phi(-d_{3/2}(K))-FK\Phi(-d_{1/2}(K))+\frac{1}{2}K^2\Phi(-d_{-1/2}(K))}$$

We first note by $\Phi'(d_{-1/2}(K))=\frac{F}{K}\Phi'(d_{1/2}(K))$,

$$\frac{d}{dK}C(K)=F\Phi'(d_{1/2}(K))\left(-\frac{1}{K\sigma\sqrt{T}}\right)-\Phi(d_{-1/2}(K))-K\Phi'(d_{-1/2}(K))\cdot\left(-\frac{1}{K\sigma\sqrt{T}}\right)=-\Phi(d_{-1/2}(K)).$$

So by integration-by-parts formula, we have

$$\begin{aligned}\int_K^\infty C(x)dx &= xC(x)|_K^\infty - \int_K^\infty x\frac{d}{dx}C(x)dx = -KC(K) + \int_K^\infty x\Phi(d_{-1/2}(x))dx \\ &= -KC(K) + \frac{x^2}{2}\Phi(d_{-1/2}(x))\Big|_K^\infty - \int_K^\infty \frac{x^2}{2}\Phi'(d_{-1/2}(x))\left(-\frac{1}{x\sigma\sqrt{T}}\right)dx \\ &= -KC(K) - \frac{K^2}{2}\Phi(d_{-1/2}(K)) + \frac{1}{2\sigma\sqrt{T}}\int_K^\infty x\Phi'(d_{-1/2}(x))dx \\ &= -KF\Phi(d_{1/2}(K)) + \frac{K^2}{2}\Phi(d_{-1/2}(K)) + \frac{1}{2\sigma\sqrt{T}}\int_K^\infty x\Phi'(d_{-1/2}(x))dx\end{aligned}$$

By the change-of-variable $y = \ln\frac{x}{F}$, we have

$$\int_K^\infty x\Phi'(d_{-1/2}(x))dx = \int_{\ln\frac{K}{F}}^\infty F^2e^{2y}\frac{e^{-\frac{1}{2}\left(\frac{-y}{\sigma\sqrt{T}}-\frac{1}{2}\sigma\sqrt{T}\right)^2}}{\sqrt{2\pi}}dy = \sigma\sqrt{T}\cdot F^2e^{\sigma^2T}\Phi(d_{3/2}(K))$$

Combined, we have obtained

$$\int_K^\infty C(x)dx = \frac{1}{2}F^2e^{\sigma^2T}\Phi(d_{3/2}(K)) - FK\Phi(d_{1/2}(K)) + \frac{1}{2}K^2\Phi(d_{-1/2}(K))$$

By the put-call parity $C(K) - P(K) = F - K$ and

$$\int_0^\infty C(x)dx = \lim_{K\downarrow 0}\int_K^\infty C(x)dx = \frac{1}{2}F^2e^{\sigma^2T},$$

we conclude

$$\begin{aligned}\int_0^K P(x)dx &= \int_0^K[C(x)-F+x]dx = \int_0^\infty C(x)dx - \int_K^\infty C(x)dx - FK + \frac{K^2}{2} \\ &= \frac{1}{2}F^2e^{\sigma^2T} - FK + \frac{1}{2}K^2 - \frac{1}{2}F^2e^{\sigma^2T}\Phi(d_{3/2}(K)) + FK\Phi(d_{1/2}(K)) - \frac{1}{2}K^2\Phi(d_{-1/2}(K)) \\ &= \frac{1}{2}F^2e^{\sigma^2T}\Phi(-d_{3/2}(K)) - FK\Phi(-d_{1/2}(K)) + \frac{1}{2}K^2\Phi(-d_{-1/2}(K))\end{aligned}$$

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