Numerical Analysis of Calibrating HW1F to Caps Market

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Abstract

Numerical issues of calibrating one-factor Hull-White model to caps market.

Let $\text{Bl}(K, F, v) = F\Phi(d_1) - K\Phi(d_2)$, where $F, K, v > 0$, Φ is the CDF of a standard normal distribution, and

$$d_1 = \frac{\ln\frac{F}{K} + \frac{1}{2}v^2}{v},\ d_2 = d_1 - v = \frac{\ln\frac{F}{K} - \frac{1}{2}v^2}{v}$$

We need to solve for v the equation $V = \text{Bl}(K, F, v)$ where V is the given Black price. In calibration of one-factor Hull-White model to caps market, we have the following calibration equation

$$\text{Bl}(K, F, v_B) = \text{Bl}(K + \delta, F + \delta, v)$$

where strike K, forward rate F, and caplet Black vol v_B are given, δ is the inverse of the year fraction of the caplet's coverage (e.g. $\delta = \frac{1}{0.2555556}$), and we need to invert the implied vol v.

To illustrate the numerical issues involved in this problem, let us suppose the caplet is ATM and we apply the Brenner-Subrahmanyam approximation for call price (see, for example, Wilmott [1] p. 116):

$$\text{Call} \approx 0.4F\sigma_B.$$

Then $v \approx \frac{F}{F+\delta}\sigma_B$. Under the low interest environment, $F << \delta$ makes $\frac{F}{F+\delta}$ very small. In a realistic example where $F = 0.66\%$ and $\delta = \frac{1}{0.2555556}$,

$$\frac{F}{F + \delta} \approx 0.0017$$

So we'll have to shrink v_B 590 times to obtain v – in this example, $v = 0.19\%$ for $v_B = 92.75\%$ (see numerical examples later). This has created some subtlety in numerical root-finding.

1 Newton's method

A typical root finder is the global Newton's method. One problem with this method is that the vega

$$\frac{\partial}{\partial v}\text{Bl}(K, F, v) = F\Phi'(d_1) = \frac{F}{\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{\ln(F/K)}{v} + \frac{1}{2}v\right]^2}$$

could be very small when $F << K$. This phenomenon arises naturally under the low interest rate environment and becomes really prominent when the bracketing interval for root-finding includes small numbers – it's often necessary in practice that we include 0 as an end point (see numerical examples later) and this will practically shrink vega to 0.

2 Brent's method

Brent's method is more robust than Newton's method. But we still need to estimate the lower and upper bounds of the bracketing interval, as well as the tolerance for volatility and price.

2.1 Tolerance for volatility

We assume the tolerance tol_p for price is what we really care and is already given. By the (rather crude) estimate

$$
\begin{aligned}
|\mathrm{Bl}(K,F,v_1)-\mathrm{Bl}(K,F,v_2)| &\leq \left|\int_{v_1}^{v_2}\frac{\partial}{\partial\xi}\mathrm{Bl}(K,F,\xi)d\xi\right| \\
&\leq \frac{F}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\left|\ln\frac{F}{K}\right|+\ln\frac{F}{K}\right)}|v_1-v_2| \\
&= \frac{\min\{F,K\}}{\sqrt{2\pi}}|v_1-v_2|
\end{aligned}
$$

we set the tolerance for volatility as

$$
tol_v = \frac{\sqrt{2\pi}}{\min\{F,K\}}tol_p
$$

2.2 Lower/Upper bound

For ATM deal, we can obtain the analytic solution of $V=\mathrm{Bl}(F,K,v_{atm})$ as

$$
v_{atm}=v_{atm}(F)=-2\Phi^{-1}\left(0.5(1-V/F)\right)
$$

For OTM deal ($F<K$), by

$$
\begin{aligned}
\mathrm{Bl}(F,F,v_{atm}(F)) = V = \mathrm{Bl}(F,K,v_{otm}) < \mathrm{Bl}(F,F,v_{otm}), \\
\mathrm{Bl}(K,K,v_{atm}(K)) = V = \mathrm{Bl}(F,K,v_{otm}) < \mathrm{Bl}(K,K,v_{otm})
\end{aligned}
$$

we conclude $v_{atm}(F)=\max\{v_{atm}(F),v_{atm}(K)\}<v_{otm}$.

For ITM deal ($F>K$), we can similarly conclude $v_{itm}<\min\{v_{atm}(F),v_{atm}(K)\}=v_{atm}(F)$.

In the context of calibrating one-factor Hull-White model to caps market, we note $G(\delta,F,K,v):=\mathrm{Bl}(F+\delta,K+\delta,v)$ is monotone increasing in δ (see Appendix A), so

$$
\mathrm{Bl}(F,K,v_B)=G(0,F,K,v_B)<G(\delta,F,K,v_B)=\mathrm{Bl}(F+\delta,K+\delta,v_B)
$$

This gives an upper bound v_B, regardless the moneyness of the deal. In summary, we have the following bracketing interval for calibrating one-factor Hull-White model to caps market:

$$
\boxed{\begin{cases}
v_{atm}=v_{atm}(F)=-2\Phi^{-1}\left(0.5(1-V/F)\right) \\
v_{otm}\in[v_{atm}(F),v_B] \\
v_{itm}\in[0,\min\{v_{atm}(F),v_B\}]
\end{cases}}
$$

Some numerical example to illustrate the above bounds (price error is prescribed as 10^{-7})

Moneyness	OTM	ITM	OTM	ITM	OTM
F	0.0066367785	0.0168108387	0.0045036260	0.0196151456	0.0051276546
K	0.0102006226	0.0102006226	0.0102006226	0.0102006226	0.0102006226
v_B	0.9275450996	1.1977140233	0.3943233667	1.1801915054	0.5316839845
price V	0.0015146970	0.0098433910	0.0000183185	0.0121844574	0.0001731815
δ	$\frac{1}{0.255556}$	$\frac{1}{0.255556}$	$\frac{1}{0.255556}$	$\frac{1}{0.255556}$	$\frac{1}{0.255556}$
lower bound	0.0009677661	0.0000000000	0.0000119578	0.0000000000	0.0001094491
upper bound	0.9275450996	0.0062209380	0.3943233667	0.0077848794	0.5316839845
implied vol	0.0018931870	0.0037664583	0.0007114381	0.0040884782	0.0009778507
price error	3.258811e-10	5.103501e-12	1.139355e-08	2.610311e-08	1.148679e-08
number of iter.	4	5	24	5	14

Remark 1. *The number of iterations typically seen in experiment is less than 10. For extremely deep OTM deal, the number of iteration could be more than 10, and the largest number we saw is 24.*

2.3 Limit cases

The Black price function $\mathrm{Bl}(F, K, v)$ is a monotone increasing function of $v \in (0, \infty)$, with the limits $\lim_{v \downarrow 0+} \mathrm{Bl}(F, K, v) = (F - K)^+$ and $\lim_{v \uparrow 0+} \mathrm{Bl}(F, K, v) = F$. Numerical experiments show the given Black price can be slightly outside of the range $[(F - K)^+, F]$. For example, in the calibration equation for caplets

$$\mathrm{Bl}(F, K, v_B) = \mathrm{Bl}(F + \delta, K + \delta, v) \ (\delta > 0)$$

the Black price is calculated by the formula on the left side of the equation, which could go outside of the range due to numerical errors.

To deal with the limit cases, we take a prescribed tolerance $\varepsilon < \frac{1}{2} \min\{F, K\}$ such that $(F-K)^+ + \varepsilon < F - \varepsilon$ and let the code take the following actions according to different scenarios:

1) Prices falls outside $[(F - K)^+ - \varepsilon, F + \varepsilon]$: report error.
2) Price falls inside $[(F - K)^+ + \varepsilon, F - \varepsilon]$: call root-finder to find the implied vol.
3) Price falls inside $((F - K)^+ - \varepsilon, (F - K)^+ + \varepsilon)$: set implied vol to 0.
4) Price falls inside $(F - \varepsilon, F + \varepsilon)$: set implied vol by the formula

$$v = -\Phi^{-1}\left(\frac{\varepsilon}{F + K}\right) + \sqrt{\left[\Phi^{-1}\left(\frac{\varepsilon}{F + K}\right)\right]^2 + 2\left|\ln \frac{F}{K}\right|}$$

The last case is based on the following line of reasoning. We want to choose v sufficiently large so that $|\mathrm{Bl}(K, F, v) - F| < \varepsilon$. We note

$$\begin{aligned} |\mathrm{Bl}(K, F, v) - F| &= |F\Phi(d_1) - K\Phi(d_2) - F| \leq F[1 - \Phi(d_1)] + K\Phi(d_2) = F\Phi(-d_1) + K\Phi(d_2) \\ &\leq (F + K)\Phi\left(\frac{|\ln(F/K)|}{v} - \frac{v}{2}\right) \end{aligned}$$

Forcing the last term to be smaller than ε will give the value for v. Note the requirement of $\varepsilon < \frac{1}{2} \min\{F, K\}$ implies $\Phi^{-1}\left(\frac{\varepsilon}{F+K}\right) < 0$, so we always have $v > 0$ in case 4). In practice, we can take

$$\varepsilon = \frac{1}{4} \min\{F, K, 4tol_p\}.$$

A Monotonicity of $G(\delta)$

Define $G(\delta, F, K, v) := \mathrm{Bl}(F + \delta, K + \delta, v)$. We show $G(\delta)$ is monotone increasing in δ. Indeed, define $F(\delta) = F + \delta$, $K(\delta) = K + \delta$, and $d_{1,2}(\delta) = \frac{\ln \frac{F(\delta)}{K(\delta)}}{v} \pm \frac{1}{2}v$. Then

$$\frac{d}{d\delta} G(\delta, F, K, v) = \frac{\partial}{\partial F} \mathrm{Bl}(F(\delta), K(\delta), v) F'(\delta) + \frac{\partial}{\partial K} \mathrm{Bl}(F(\delta), K(\delta), v) K'(\delta)$$

Recall $\frac{\partial}{\partial F} \mathrm{Bl}(K, F, v) = \Phi(d_1)$ (where the key observation is $\Phi'(d_2) = \frac{F}{K}\Phi'(d_1)$) and $\frac{\partial}{\partial K} \mathrm{Bl}(K, F, v) = -\Phi(d_2)$. So

$$\frac{d}{d\delta} G(\delta, F, K, v) = \Phi(d_1(\delta)) - \Phi(d_2(\delta)) > 0.$$

References

[1] Paul Wilmott. *Paul Wilmott on quantitative finance*, 2nd edition, volume 1. John Wiley & Sons, 2006. 1