Approximation Formulas for Implied Volatility

Yan Zeng

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Abstract

Summary of various approximation formulas for computing (Black-Scholes) implied volatility.

1 Summary of various formulas

Under the Black-Scholes framework, the value of a European call option on a non-dividend paying stock is

$$C = SN(d_1) - Xe^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},\ d_2 = \frac{\ln(S/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},\ N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du.$$

The stock price, strike price, interest rate, time-to-expiration and volatility are denoted by S, X, r, T, σ, respectively.

For convenience, we define $K = e^{-rT}X$ and say an option is at-the-money (ATM) if $S = K$. We summarize various approximation formulas for implied vol in the following table:

Name	Formula	Applicability
Brenner et al.	$\sigma \approx \sqrt{\frac{2\pi}{T}}\frac{C}{S}$	Exactly ATM
Corrado-Miller	$\sigma \approx \sqrt{\frac{2\pi}{T}}\frac{1}{S+K}\left[C - \frac{S-K}{2} + \sqrt{\left(C - \frac{S-K}{2}\right)^2 - \frac{(S-K)^2}{\pi}}\right]$	Accurate for some cases
Bharadia et al.	$\sigma \approx \sqrt{\frac{2\pi}{T}}\frac{C-(S-K)/2}{S-(S-K)/2}$	Less accurate than Corrado-Miller
Li [1]	$\sigma \approx \frac{2\sqrt{2}}{\sqrt{T}}z - \frac{1}{\sqrt{T}}\sqrt{8z^2 - \frac{6\alpha}{\sqrt{2}z}}$ $\alpha = \frac{C\sqrt{2\pi}}{S},\ z = \cos\left[\frac{1}{3}\cos^{-1}\left(\frac{3\alpha}{\sqrt{32}}\right)\right],$ $8z^2 - \frac{6\alpha}{\sqrt{2}z} > 0$ always, $0 < \frac{3\alpha}{\sqrt{32}} < 1$ if $0 < \frac{C}{S} < 0.7522$	Exactly ATM: consistently more accurate than Brenner et al.
	$\sigma \approx \frac{\tilde{\alpha} + \sqrt{\tilde{\alpha}^2 - \frac{4(\eta-1)^2}{1+\eta}}}{2\sqrt{T}}$, which is equivalent to $\sqrt{\frac{2\pi}{T}}\frac{1}{S+K}\left[C - \frac{S-K}{2} + \sqrt{\left(C - \frac{S-K}{2}\right)^2 - \frac{(S-K)^2}{\pi}\frac{1+K/S}{2}}\right]$ $\eta = \frac{K}{S},\ \tilde{\alpha} = \frac{\sqrt{2\pi}}{1+\eta}\left[\frac{2C}{S} + \eta - 1\right]$	Accurate when $\sigma \ll \sqrt{\frac{\lvert K/S-1\rvert}{T}}$: small volatility, deep in- or out-of-the-money, short time-to-expiration
	$\sigma \approx \frac{2\sqrt{2}}{\sqrt{T}}\tilde{z} - \frac{1}{\sqrt{T}}\sqrt{8\tilde{z}^2 - \frac{6\tilde{\alpha}}{\sqrt{2}\tilde{z}}}$ $\tilde{\alpha} = \frac{\sqrt{2\pi}}{1+\eta}\left[\frac{2C}{S} + \eta - 1\right],\ \tilde{z} = \cos\left[\frac{1}{3}\cos^{-1}\left(\frac{3\tilde{\alpha}}{\sqrt{32}}\right)\right]$ $0 < \frac{3\tilde{\alpha}}{\sqrt{32}} < 1$ if $0 < \frac{C}{S} < 0.88$	Accurate when $\sigma \gg \sqrt{\frac{\lvert K/S-1\rvert}{T}}$: high volatility, nearly ATM, long time-to-expiration

2 Li's formula

Let $\rho = \frac{|K-S|S}{C^2}$. Then approximately, $\sigma \gg \frac{\sqrt{|K/S-1|}}{T}$ is equivalent to $\rho \ll 1$ (by the Brenner-Subrahmanyam formula $\sigma = \sqrt{\frac{2\pi}{T}}\frac{C}{S}$). Li's formula states

$$\boxed{\sigma \approx \begin{cases} \frac{2\sqrt{2}}{\sqrt{T}}\tilde{z} - \frac{1}{\sqrt{T}}\sqrt{8\tilde{z}^2 - \frac{6\tilde{\alpha}}{\sqrt{2}\tilde{z}}} & \text{if } \rho \leq 1.4 \ (\sigma \gg \sqrt{\frac{|K/S-1|}{T}}) \\ \frac{\tilde{\alpha}+\sqrt{\tilde{\alpha}^2 - \frac{4(\eta-1)^2}{1+\eta}}}{2\sqrt{T}} & \text{if } \rho > 1.4 \ (\sigma \ll \sqrt{\frac{|K/S-1|}{T}}) \end{cases}}$$

where

$$\begin{cases} \eta = \frac{K}{S} \\ \tilde{\alpha} = \frac{\sqrt{2\pi}}{1+\eta}\left[\frac{2C}{S} + \eta - 1\right] \\ \tilde{z} = \cos\left[\frac{1}{3}\cos^{-1}\left(\frac{3\tilde{\alpha}}{\sqrt{32}}\right)\right] \end{cases}$$

References

[1] Steven Li. A new formula for computing implied volatility. *Applied Mathematics and Computation*, 170 (2005) 611-625. 1