A Note on Hagan's Adjusters

Yan Zeng

Version 1.0, last revised on 2012-01-16.

Abstract

We provide some intuition on Hagan's adjusters [1].

1 The algorithm of pricing-via-adjusters

Suppose we have an exotic derivative v to price. Suppose that for "operational reasons", one could not calibrate on its natural hedging instruments h_1, h_2, $\cdots$, h_m, but instead were forced to calibrate on instruments S_1, S_2, $\cdots$, S_n. Let these instruments have market volatilities σ_1, σ_2, $\cdots$, σ_n. Denote the vector $(\sigma_1, \sigma_2, \cdots, \sigma_n)$ by σ. The algorithm of pricing-via-adjusters goes as follows:

Step 1. Compute price of the exotic derivative under the calibrated vol σ: $V^{mod} = v(\sigma)$.

Step 2. Compute prices of the natural hedging instruments under the calibrated vol σ:

$$H^{mod} = (H_1^{mod}, H_2^{mod}, \cdots, H_m^{mod}) = (h_1(\sigma), h_2(\sigma), \cdots, h_m(\sigma)).$$

Step 3. Choose vector $b = (b_1, b_2, \cdots, b_m)^T$ to minimize the vector

$$\frac{\partial V^{mod}}{\partial \sigma} - \frac{\partial H^{mod}}{\partial \sigma} b = \begin{bmatrix} \frac{\partial v(\sigma)}{\partial \sigma_1} \\ \frac{\partial v(\sigma)}{\partial \sigma_2} \\ \vdots \\ \frac{\partial v(\sigma)}{\partial \sigma_n} \end{bmatrix} - \begin{bmatrix} \frac{\partial h_1(\sigma)}{\partial \sigma_1} & \frac{\partial h_2(\sigma)}{\partial \sigma_1} & \cdots & \frac{\partial h_m(\sigma)}{\partial \sigma_1} \\ \frac{\partial h_1(\sigma)}{\partial \sigma_2} & \frac{\partial h_2(\sigma)}{\partial \sigma_2} & \cdots & \frac{\partial h_m(\sigma)}{\partial \sigma_2} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial h_1(\sigma)}{\partial \sigma_n} & \frac{\partial h_2(\sigma)}{\partial \sigma_n} & \cdots & \frac{\partial h_m(\sigma)}{\partial \sigma_n} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Step 4. The adjusted price $V^{adj} = V^{mod} + (H^{mar} - H^{mod}) \cdot b = v(\sigma) + \sum_{k=1}^m (H_k^{mar} - h_k(\sigma)) b_k$, where $H^{mar} = (H_1^{mar}, H_2^{mar}, \cdots, H_m^{mar})$ is the market price of the natural hedging instruments.

2 A simple example

Suppose the exotic derivative is indeed equal to a portfolio of its natural hedging instruments: $v(x) = \sum_{k=1}^m b_k h_k(x)$, where x is any model specification (a point in an abstract configuration space) and $b = (b_1, b_2, \cdots, b_m)^T$ is independent of x. Suppose we don't know b but can directly compute the exotic derivative's price through a formula $v(\cdot)$.

Suppose $x = \sigma^*$ is the correct model specification, but is unavailable due to "operational reasons". Instead, we are given another model specification $\sigma \neq \sigma^*$. The price error is then given by

$$V^* - V^{mod} = H(\sigma^*) \cdot b - H^{mod} \cdot b.$$

But $H(\sigma^*)$ is given directly through market quotes: $H(\sigma^*) = H^{mar}$. By solving the minimization problem in Step 3, we can obtain b, and the correct price for the exotic derivative is consequently

$$V^{mod} + (H^{mar} - H^{mod}) \cdot b.$$

This is the adjusted price V^{adj} in Hagan [1] (2.5b).

3 Heuristics for general case

Suppose the exotic derivative is not necessarily a portfolio of its natural hedging instruments. We can still consider the projection of the derivative's payoff onto the linear space spanned by the payoff functions of hedging instruments. More precisely, if ξ is the payoff of v and η_k is the payoff of h_k $(k = 1, 2, \cdots, m)$, we consider the element in the linear space $\{\eta_1, \eta_2, \cdots, \eta_m\}$ that is "closest" to ξ:

$$\xi \sim \sum_{k=1}^{n} b_k \eta_k.$$

Typically, payoffs ξ, $\eta_1, \cdots, \eta_m$ are random functions parameterized by some model specification, say, volatility σ. For sake of hedging vega risk, the distance of ξ to $\{\eta_1, \eta_2, \cdots, \eta_m\}$ is measured by the distance from $\partial\xi/\partial\sigma$ to $\{\partial\eta_1/\partial\sigma, \cdots, \partial\eta_m/\partial\sigma\}$. That is, we find vector $b = b(\sigma, \omega)$ such that $\sum_{k=1}^{m} b_k \frac{\partial \eta_k}{\partial \sigma}$ represents the element in $\{\partial\eta_1/\partial\sigma, \partial\eta_2/\partial\sigma, \cdots, \partial\eta_m/\partial\sigma\}$ that is closest to $\partial\xi/\partial\sigma$.

If we suppose b is non-random: $b(\sigma, \omega) = b(\sigma)$, we can conclude by risk-neutral pricing that b is the solution of the minimization problem in Step 3. At this moment, b could still be a function of σ, and we have

$$V^* - V^{mod} \sim H^{mar} \cdot b(\sigma^*) - H^{mod} \cdot b(\sigma).$$

Here $\sim$ means "representation that minimizes vega risk". Finally, we assume b is constant, then $V^{adj} = V^{mod} + (H^{mar} - H^{mod}) \cdot b$ gives the price that minimizes vega risk.

References

[1] Patrick Hagan. Adjusters: Turning Good Prices into Great Prices. *Wilmott Magazine* **2**, December 2002, 56-59. http://www.wilmott.com/pdfs/030813_hagan.pdf.

1