# SABR Model

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**Abstract**

Various approximation formulas for SABR model, as well as some general facts on implied volatility.

## Contents



# 1 SABR model

In this model, the forward price and volatility are

$$\begin{cases} dF_t = \alpha_t F_t^{\beta} dW_1(t),\ F_0 = f \\ d\alpha_t = \nu \alpha_t dW_2(t),\ \alpha_0 = \alpha \end{cases}$$

under the forward measure $\mathbb{P}$, where the two processes are correlated by

$$dW_1(t)dW_2(t) = \rho dt.$$

The price of a European call option on $F$ with exercise date $t_{ex}$, settlement date $t_{set}$, and strike $K$ is given by

$$V_{call} = D(t_{set})\mathbb{E}[(F_{t_{ex}} - K)^+]$$

where $\mathbb{E}[\cdot]$ is under the forward measure. Market convention often quotes the price through Black's formula

$$V_{call} = D(t_{set})[f\Phi(d_+) - K\Phi(d_-)],\ d_\pm = \frac{\log(f/K) \pm \frac{1}{2}\sigma_B^2 t_{ex}}{\sigma_B\sqrt{t_{ex}}}.$$

The remaining problem is how to obtain $\sigma_B = \sigma_B(K, f)$.

# 2 Hagan's formula

## 2.1 Formula

Hagan [2] gives the following formula

$$\begin{aligned}\sigma_B(K,f) \quad &\approx \quad \frac{\alpha}{(fK)^{(1-\beta)/2}\left[1+\frac{(1-\beta)^2}{24}\log^2(f/K)+\frac{(1-\beta)^4}{1920}\log^4(f/K)+\cdots\right]}\cdot\frac{z}{x(z)} \\ &\cdot\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right]t_{ex}+\cdots\right\}\end{aligned}$$

where

$$z=\frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log(f/K),\ x(z)=\log\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}$$

For the special case of at-the-money options, options struck at $K=f$, this formula reduces to

$$\sigma_{ATM}=\sigma_B(f,f)\approx\frac{\alpha}{f^{(1-\beta)}}\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}}+\frac{1}{4}\frac{\rho\beta\alpha\nu}{f^{(1-\beta)}}+\frac{2-3\rho^2}{24}\nu^2\right]t_{ex}+\cdots\right\}$$

An alternative representation is through normal volatility:

$$\begin{aligned}\sigma_N(K,f) \quad &\approx \quad \frac{1}{\alpha}\cdot\frac{f^{1-\beta}-K^{1-\beta}}{(1-\beta)(f-K)}\cdot\frac{x(z)}{z} \\ &\cdot\left\{\frac{1+\frac{\beta(2-\beta)}{24}\frac{1-\frac{2-2\beta+\beta^2}{120}\log^2(f/K)}{1+\frac{(1-\beta)^2}{12}\log^2(f/K)}\frac{\alpha^2 t_{ex}}{(fK)^{(1-\beta)}}+\frac{\beta(2-\beta)}{80}\left[(1-\beta)^2+\frac{1}{72}\beta(2-\beta)\right]\frac{\alpha^4 t_{ex}^2}{(fK)^{2-2\beta}}}{1+\frac{\beta\rho}{4}\frac{\alpha\nu t_{ex}}{(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2 t_{ex}}\right\}\end{aligned}$$

For the special case of at-the-money options, this formula reduces to

$$\sigma_N(f,f)\approx\frac{1}{\alpha f^{\beta}}\left\{\frac{1+\frac{\beta(2-\beta)}{24}\frac{\alpha^2 t_{ex}}{f^{2-2\beta}}+\frac{\beta(2-\beta)}{80}\left[(1-\beta)^2+\frac{1}{72}\beta(2-\beta)\right]\frac{\alpha^4 t_{ex}^2}{f^{4-4\beta}}}{1+\frac{\beta\rho}{4}\cdot\frac{\alpha\nu t_{ex}}{f^{1-\beta}}+\frac{2-3\rho^2}{24}\nu^2 t_{ex}}\right\}.$$

In actual implementation, the ATM case is unified under the case of "near ATM": $x(z)/z$ is approximated by $1+\frac{1}{2}\rho z-\frac{1-3\rho^2}{6}z^2$ and

$$\frac{f^{1-\beta}-K^{1-\beta}}{(1-\beta)(f-K)}\approx\frac{1}{(fK)^{(1-\beta)/2}}\cdot\frac{1+\frac{(1-\beta)^2}{24}\log^2(f/K)+\frac{(1-\beta)^2}{1920}\log^4(f/K)}{1+\frac{1}{24}\log^2(f/K)+\frac{1}{1920}\log^4(f/K)}$$

## 2.2 Asymptotics

Assuming $\beta<1$, then $\lim_{K\to\infty}z=-\infty$ and $x(z)\sim-\log(-z)$. So as $K\to\infty$,

$$\sigma_B(K,f)\sim\frac{\nu\log(f/K)}{1+\frac{(1-\beta)^2}{24}\log^2(f/K)+\frac{(1-\beta)^4}{1920}\log^4(f/K)}\cdot\frac{1}{-\log(-z)}\cdot\left(1+\frac{2-3\rho^2}{24}\nu^2 t_{ex}\right).$$

Since $\log(-z)\sim\frac{1-\beta}{2}\log K$,

$$\lim_{K\to\infty}\frac{\nu\log(f/K)}{-\log(-z)}=\frac{2\nu}{1-\beta},$$

and we conclude

$$\lim_{K\to\infty}\sigma(K,f)=0.$$

Assuming $\beta = 1$, Hagan's formula becomes

$$\sigma_B(K,f) \approx \frac{\nu \log(f/K)}{x(z)} \left[1 + \left(\frac{\rho\nu\alpha}{4} + \frac{2-3\rho^2}{24}\nu^2\right) t_{ex}\right],$$

where

$$z = \frac{\nu}{\alpha}\log(f/K),\ x(z) = \log\left\{\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right\}.$$

We still have $\lim_{K\to\infty} z = -\infty$ and $x(z) \sim -\log(-z)$. Hence $\lim_{K\to\infty} \sigma_B(K,f) = \infty$.

In summary,

$$\lim_{K\to\infty} \sigma_B(K,f) = \begin{cases} 0 & \beta < 1 \\ \infty & \beta = 1 \end{cases}$$

We further note when $\beta = 1$, Hagan's formula satisfies

$$\sigma_B(K,f) \sim \frac{\nu \log K}{\log(\log K)} \left[1 + \left(\frac{\rho\nu\alpha}{4} + \frac{2-3\rho^2}{24}\nu^2\right) t_{ex}\right], \text{ as } K \to \infty,$$

This implies

$$\frac{\sigma_B^2(K,f) t_{ex}}{\log(K/f)} \sim \frac{(\nu \log K)^2 t_{ex}}{\log^2(\log K)\cdot \log K} \left[1 + \left(\frac{\rho\nu\alpha}{4} + \frac{2-3\rho^2}{24}\nu^2\right) t_{ex}\right]^2 \overset{K\to\infty}{\longrightarrow} \infty,$$

which contradicts with the moment formula (see Section 5). So Hagan's formula when $\beta = 1$ cannot be an accurate approximation for large strikes.

# 3 Obloj's formula

## 3.1 Formula

Obloj [5] gives the following formula

$$\sigma_B(K,f) \approx I_0(K,f)[1 + I_1(K,f) t_{ex}],$$

where

$$I_0(K,f) = \frac{\nu \log(f/K)}{\log\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)},\ z = \frac{\nu}{\alpha}\frac{f^{1-\beta} - K^{1-\beta}}{1-\beta},$$

$$I_1(K,f) = \frac{(\beta-1)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\nu\alpha\beta}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2$$

For the special case of at-the-money options, the above formula reduces to

$$\sigma_{atm} = \sigma_B(f,f) \approx \frac{\alpha}{f^{1-\beta}}\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2(1-\beta)}} + \frac{1}{4}\frac{\rho\nu\alpha\beta}{f^{1-\beta}} + \frac{2-3\rho^2}{24}\nu^2\right] t_{ex}\right\}.$$

**Remark 1.** *If $\beta = 1$, Obloj's formula agrees with Hagan's formula; if $\nu = 0$, Obloj's formula and Hagan's formula differ by a factor*

$$\frac{1}{1 + \frac{(1-\beta)^2}{24}\log^2(f/K) + \frac{(1-\beta)^4}{1920}\log^4(f/K)}$$

If $\beta = 1$ and $\nu = 0$ hold simultaneously, both formulas give the same constant vol $\sigma_B(K,f) = \alpha$.

### 3.2 Asymptotics

Assuming $\beta < 1$, then $\lim_{K\to\infty} z = -\infty$. Therefore, as $K \to \infty$, $\nu \log(f/K) \sim -\nu \log(K)$ and

$$\begin{aligned}\log\left(\sqrt{1-2\rho z+z^2}+z-\rho\right) &= \log\frac{1-\rho^2}{\sqrt{(z-\rho)^2+(1-\rho^2)}-(z-\rho)} = \log\frac{1-\rho^2}{\sqrt{1+\frac{1-\rho^2}{(z-\rho)^2}}+1} - \log(\rho-z) \\ &\sim -\log(-z)\end{aligned}$$

Therefore

$$\lim_{K\to\infty} I_0(K,f) = \lim_{K\to\infty}\frac{-\nu\log K}{-\log(-z)} = \frac{\nu}{1-\beta}.$$

Consequently

$$\lim_{K\to\infty}\sigma_B(K,f) = \frac{\nu}{1-\beta}\left(1+\frac{2-3\rho^2}{24}\nu^2 t_{ex}\right)$$

Assuming $\beta = 1$, Obloj's formula becomes

$$\sigma_B(K,f) \approx \frac{\nu\log(f/K)}{\log\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)}\left[1+\left(\frac{\rho\nu\alpha}{4}+\frac{2-3\rho^2}{24}\nu^2\right)t_{ex}\right],\ z=\frac{\nu}{\alpha}\log(f/K).$$

We still have $\lim_{K\to\infty} z = -\infty$ and

$$\log\left(\sqrt{1-2\rho z+z^2}+z-\rho\right) \sim -\log(-z) \sim \log(\log K),$$

which implies

$$\lim_{K\to\infty}\sigma_B(K,f) = \infty.$$

In summary

$$\lim_{K\to\infty}\sigma_B(K,f) = \begin{cases}\frac{\nu}{1-\beta}\left(1+\frac{2-3\rho^2}{24}\nu^2 t_{ex}\right) & \beta<1\\ \infty & \beta=1\end{cases}$$

Note in the special case of $\rho = 0$ and $\beta < 1$, the asymptotic limit of $\sigma_B(K,f)$ is $\frac{\nu}{1-\beta}\left(1+\frac{1}{12}\nu^2 t_{ex}\right)$, different from the asymptotic limit $\frac{\nu}{1-\beta}$ of SABR model given by the right tail-wing formula (see Section 5).

# 4 Paulot's formula

Paulot [6] gives the following formula

$$\sigma_B(K,f) = I_0(K,f) + I_1(K,f)t_{ex}$$

where

$$I_0(K,f) = \frac{\nu\log(K/f)}{\log\left(\frac{\sqrt{\alpha^2+2\rho\alpha\nu q+\nu^2q^2}+\rho\alpha+q\nu}{(1+\rho)\alpha}\right)},\ I_1(K,f) = -\nu^2 I_0(K,f)\frac{\tilde{C}+\log\frac{I_0(K,f)\sqrt{Kf}}{\nu}}{\log^2\frac{\sqrt{\alpha^2+2\rho\alpha\nu q+\nu^2q^2}+\rho\alpha+q\nu}{(1+\rho)\alpha}}$$

and

$$\tilde{C} = -\frac{1}{2}\log\frac{\alpha(fK)^\beta V_{\min}}{\nu} - \frac{\rho\beta}{(1-\beta)\sqrt{1-\rho^2}}[G(t_2)-G(t_1)],\ q=\frac{K^{1-\beta}-f^{1-\beta}}{1-\beta},\ V_{\min}=\sqrt{\frac{\alpha^2}{\nu^2}+\frac{2\rho\alpha q}{\nu}+q^2},$$

$$G(t) = \arctan(t) - \frac{a+bX}{\sqrt{(a+bX)^2-(1-\beta)^2R^2}}\arctan\left(\frac{cR+t[a+b(X-R)]}{\sqrt{(a+bX)^2-(1-\beta)^2R^2}}\right)$$

$$t_1 = \sqrt{\frac{R - x_1 + X}{R + x_1 - X}},\ t_2 = \sqrt{\frac{R - x_2 + X}{R + x_2 - X}},\ X = \frac{x_2^2 - x_1^2 + y_2^2 - y_1^2}{2(x_2 - x_1)},\ R = \sqrt{y_1^2 + (x_1 - X)^2},$$

$$x_1 = -\frac{\rho\alpha}{\nu\sqrt{1-\rho^2}},\ y_1 = \frac{\alpha}{\nu},\ x_2 = \frac{q - \rho V_{\min}}{\sqrt{1-\rho^2}},\ y_2 = V_{\min},\ a = f^{1-\beta},\ b = (1-\beta)\sqrt{1-\rho^2},\ c = (1-\beta)\rho.$$

We note the leading term $I_0(K, f)$ in Paulot's formula agrees with that of Obloj's formula, after plugging $q = -\alpha z/\nu$ into the above expression.

# 5 Results on arbitrage-free implied volatility

## 5.1 Model-independent results

Let $x = \log(K/f)$ and $I(x) = \sigma_B(K, f)$. Define $\beta_R$ and $\beta_L$ respectively by

$$\beta_R := \limsup_{x\to\infty} \frac{I^2(x)}{|x|/t_{ex}},\ \beta_L = \limsup_{x\to-\infty} \frac{I^2(x)}{|x|/t_{ex}}.$$

Assuming only the existence of martingale measure $\mathbb{P}$ and $0 < \mathbb{E}[F_{t_{ex}}] < \infty$, Lee [4] shows $0 \le \beta_R, \beta_L \le 2$ and gives the following *moment formula for implied volatility* at large strikes:

$$\frac{1}{2\beta_R} + \frac{\beta_R}{8} - \frac{1}{2} = \sup\{p : \mathbb{E}[F_{t_{ex}}^{1+p}] < \infty\},$$

and the following *moment formula for implied volatility* at small strikes:

$$\frac{1}{2\beta_L} + \frac{\beta_L}{8} - \frac{1}{2} = \sup\{q : \mathbb{E}[F_{t_{ex}}^{-q}] < \infty\}.$$

Equivalently, if we define $\tilde{p} := \sup\{p : \mathbb{E}[F_{t_{ex}}^{1+p}] < \infty\}$ and $\tilde{q} := \sup\{q : \mathbb{E}[F_{t_{ex}}^{-q}] < \infty\}$, $\beta_R$ and $\beta_L$ can be represented as

$$\beta_R = \psi(\tilde{p}),\ \beta_L = \psi(\tilde{q}),$$

where $\psi(x) := 2 - 4(\sqrt{x^2 + x} - x)$.

As an application, for extrapolating the volatility skew with splines, the moment formula raises warnings against spline functions that grow faster than $|x|^{1/2}$, and unless $F_{t_{ex}}$ has finite moments of all orders, against those that grow *slower* than $|x|^{1/2}$.

Benaim et al. [1] extended the moment formula to *tail-wing formula* under additional regularity assumption. Assume that $\mathbb{E}[F_{t_{ex}}^{1+\varepsilon}] < \infty$ for some $\varepsilon > 0$. Let $\varphi(x)$ denote either $1 - \mathbb{P}(\log(F_{t_{ex}}/f) \le x)$ or, if it exists, the density of $\log(F_{t_{ex}}/f)$. If $-\log\varphi$ is *regularly varying with a positive index* (see [1] Definition 5), we have the *right tail-wing formula*

$$\frac{I^2(x)}{x/t_{ex}} \sim \psi(-\log c(x)/x) \sim \psi(-\log\varphi(x)/x - 1),\ \text{as } x \to \infty$$

where $c(x) := \frac{1}{f}\mathbb{E}[(F_{t_{ex}} - K)^+]$.

Assume that $\mathbb{E}[F_{t_{ex}}^{-\varepsilon}] < \infty$ for some $\varepsilon > 0$. Let $\phi(x)$ denote $\mathbb{P}(\log(F_{t_{ex}}/f) \le x)$, or if it exists, the density of $\log(F_{t_{ex}}/f)$. If $-\log\phi(-x)$ is regularly varying with a positive index, we have the *left tail-wing formula*

$$\frac{I^2(-x)}{x/t_{ex}} \sim \psi(-1 - \log p(-x)/x) \sim \psi(-\log\phi(-x)/x),\ \text{as } x \to -\infty$$

where $p(x) := \frac{1}{f}\mathbb{E}[(K - F_{t_{ex}})^+]$.

### 5.2 Results for SABR model

For $\beta = 1$, Benaim et al. [1] shows

$$\limsup_{x\to\infty} \frac{I^2(x)}{x/t_{ex}} = \psi\left(\frac{1}{1-\rho^2} - 1\right)$$

and

$$\limsup_{x\to\infty} \frac{I^2(-x)}{x/t_{ex}} = \psi(0) = 2.$$

For $\beta < 1$, an absorbing boundary condition at 0 leads to the left tail-wing formula

$$\lim_{x\to\infty} \frac{I^2(-x)}{x/t_{ex}} = 2.$$

The right tail-wing formula is only known for the case of $\rho = 0$:

$$\lim_{x\to\infty} I(x) = \frac{\nu}{1-\beta}.$$

Jordan and Tier [3] used WKB or ray method to construct new analytical approximations for vanilla options based on SABR model. Partial differential equation satisfied by the density function

$$p(\hat{x}, \hat{y}, t_{ex}; x, y, t) = \frac{\partial}{\partial \hat{x}} \mathbb{P}(F_{t_{ex}} \le \hat{x}, \alpha_{t_{ex}} \le \hat{y} | F_t = x, \alpha_t = y)$$

and its associated initial and boundary conditions are derived. Exact solution for $p(\hat{x}, \hat{y}, t_{ex}; x, y, t)$ is known for SABR model with $\beta = 0$ ([3] Result 5), which involves integral representation and typically requires numerical integration. For the general case of $\beta \neq 1$, the asymptotic ray solution for $t \to t_{ex}$ is derived ([3] Result 6), and the boundary layer solution for SABR model with $\rho = 0$, $\beta \neq 1$ is also constructed ([3] Result 7). Finally, for the case of $\rho = 0$, $\beta \neq 1$, asymptotic approximations for $t \to t_{ex}$ of call/put prices ([3] Result 9) and delta ([3] Result 10) are derived.

# A Convergence rate of tail integral according to Obloj's formula

For $\beta < 1$, we outline a procedure to estimate the error

$$\left| \int_{\widehat{K}}^{\infty} C(\sigma_B(K, f), K) dK - \int_{\widehat{K}}^{\infty} C(\sigma_*, K) dK \right|$$

for some positive $\widehat{K}$ sufficiently large. Here $\sigma_B(K, f)$ is given by Obloj's formula (Section 3), $\sigma_* = \frac{\nu}{1-\beta}\left(1 + \frac{2-3\rho^2}{24}\nu^2 t_{ex}\right)$ is the asymptotic limit of $\sigma_B(K, f)$ as $K \to \infty$, and $C(\sigma, K)$ is the call price given by Black's formula

$$C(\sigma, K) = f\Phi(d_+) - K\Phi(d_-),\ d_\pm = \frac{\log(f/K)}{\sigma\sqrt{t_{ex}}} \pm \frac{1}{2}\sigma\sqrt{t_{ex}}.$$

Throughout this section, we shall assume the correlation $\rho \in (-1, 1)$, and we shall freely use $x$ or $z$ as an alternative for $K$ when arguing asymptotics, since there's a one-to-one correspondence among them:

$$\begin{cases} x = x(K) = \log(K/f) \\ z = z(K) = \frac{\nu}{\alpha}\frac{f^{1-\beta} - K^{1-\beta}}{1-\beta} = \frac{\nu f^{1-\beta}}{\alpha(1-\beta)}\left[1 - e^{(1-\beta)x}\right] \end{cases}$$

**Basic condition on** $K$: We first summarize a few conditions that $K$ should satisfy in the discussion below. We require $\rho - z > 1$ and $x > 0$, which translate into the condition:

$$\boxed{K > K_0 = \max\left\{ f, \left[\frac{\alpha(1-\beta)}{\nu} \cdot (1-\rho) + f^{1-\beta}\right]^{\frac{1}{1-\beta}} \right\}}$$

**First condition on** $K$: We note (recall $K$ is large enough so that $\log(\rho - z) > 0$)

$$I_0(K,f) \quad = \quad \frac{\nu \log(f/K)}{\log\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)} = \frac{-\nu x}{\log \frac{1+\rho}{\sqrt{1+\frac{1-\rho^2}{(\rho-z)^2}}+1} - \log(\rho - z)} = \frac{\nu x}{\log(\rho - z)} \cdot \frac{1}{1+\theta_1(z)},$$

where

$$0 < \theta_1(z) = -\frac{\log \frac{1+\rho}{\sqrt{1+\frac{1-\rho^2}{(\rho-z)^2}}+1}}{\log(\rho-z)} = \frac{\log\left(1+\sqrt{1+\frac{1-\rho^2}{(\rho-z)^2}}\right) - \log(1+\rho)}{\log(\rho-z)} \sim \frac{\log\frac{2}{1+\rho}}{(1-\beta)\log K}.$$

Note $\theta_1(z)$ is a monotonic increasing function of $z$ and $\lim_{z\to-\infty}\theta_1(z) = 0$. So for any given $\varepsilon_1 > 0$, we can find a unique $z_1$ such that $0 < \theta_1(z) < \varepsilon_1 \Leftrightarrow z < z_1$. Such $z_1$ can be determined by solving the equation $\theta_1(z) = \varepsilon_1$. The corresponding strike $K_1$ can be then given by

$$\boxed{K_1 = \left[f^{1-\beta} - \frac{\alpha(1-\beta)}{\nu} z_1\right]^{1/(1-\beta)}, \text{ where } z_1 \text{ solves } \theta_1(z) = \varepsilon_1}$$

**Remark 2.** *To solve the equation* $\theta_1(z) = \varepsilon_1$*, we need to find a bracketing interval* $[z_l, z_u]$ *such that the solution* $z_1 \in (z_l, z_u)$*. Under the assumption* $\rho - z > 1$*,*

$$0 < \frac{\log 2 - \log(1+\rho)}{\log(\rho - z)} < \theta_1(z) < \frac{\log(1+\sqrt{2-\rho^2}) - \log(1+\rho)}{\log(\rho-z)}$$

*So we can set*

$$z_u = \rho - e^{\frac{\log 2 - \log(1+\rho)}{\varepsilon_1}}, \; z_l = \rho - e^{\frac{\log(1+\sqrt{2-\rho^2})-\log(1+\rho)}{\varepsilon_1}}.$$

**Second condition on** $K$: We further note

$$\frac{\nu x}{\log(\rho-z)} = \frac{\nu x}{\log\left[\rho + \frac{\nu f^{1-\beta}}{\alpha(1-\beta)}(e^{x(1-\beta)}-1)\right]} = \frac{\nu}{1-\beta} \cdot \frac{1}{1+\theta_2(x)},$$

where

$$\theta_2(x) = \frac{\log\left(\frac{\nu}{\alpha}\frac{f^{1-\beta}}{1-\beta}\right) + \log\left[1 + \left(\frac{\rho\alpha}{\nu}\frac{1-\beta}{f^{1-\beta}} - 1\right)e^{-x(1-\beta)}\right]}{(1-\beta)x}.$$

Since $\log(1+y) < y$ for $y > 0$ and $|\log(1-y)| < \frac{y}{1-y}$ for $y \in (0,1)$, the latter of which leads to the relation $|\log(1-y)| < 2y$ for $y \in (0, 0.79)$, we can have $|\log(1+y)| \leq 2|y|$ for $|y| < 0.79$. So by requiring

$$\left|\left(\frac{\rho\alpha}{\nu}\frac{1-\beta}{f^{1-\beta}} - 1\right)e^{-x(1-\beta)}\right| < 0.79$$

we can have

$$|\theta_2(x)| \leq \delta_2(x) := \frac{\left|\log\left(\frac{\nu}{\alpha}\frac{f^{1-\beta}}{1-\beta}\right)\right| + 2\left|\frac{\rho\alpha}{\nu}\frac{1-\beta}{f^{1-\beta}} - 1\right|e^{-x(1-\beta)}}{(1-\beta)x} \sim \frac{\left|\log\left(\frac{\nu}{\alpha}\frac{f^{1-\beta}}{1-\beta}\right)\right|}{(1-\beta)\log K}$$

$\delta_2(x)$ is a monotone decreasing function of $x$ that maps $(0,\infty)$ to $(0,\infty)$. So for any given $\varepsilon_2$, there is a unique solution $x(\varepsilon_2)$ such that $\delta_2(x) < \varepsilon \Leftrightarrow x > x(\varepsilon_2)$. In summary, we have

$$\boxed{K > K_2 = \max\left\{\left(\frac{1}{0.79}\left|\frac{\rho\alpha}{\nu}(1-\beta) - f^{1-\beta}\right|\right)^{\frac{1}{1-\beta}}, f \cdot e^{x(\varepsilon_2)}\right\} \Rightarrow |\theta_2(x)| < \varepsilon_2}$$

where $x(\varepsilon_2)$ is the unique solution of $\delta_2(x) = \varepsilon_2$.

**Remark 3.** *To solve the equation $\delta_2(x) = \varepsilon_2$, we need to find a bracketing interval $[x_l, x_u]$. We can set*

$$x_u = \frac{1}{1-\beta} \cdot \max\left\{1, 2\left|\log\left(\frac{\nu}{\alpha}\frac{f^{1-\beta}}{1-\beta}\right)\right|/\varepsilon_2, 4e^{-1}\left|\frac{\rho\alpha}{\nu}\frac{1-\beta}{f^{1-\beta}} - 1\right|/\varepsilon_2\right\}$$

*and*

$$x_l = \frac{\left|\log\left(\frac{\nu}{\alpha}\frac{f^{1-\beta}}{1-\beta}\right)\right|}{(1-\beta)\varepsilon_2}$$

**Third condition on $K$**: The third condition on $K$ comes from estimating

$$\delta_3(K) := \left|I_1(K,f) - \frac{2-3\rho^2}{24}\nu^2\right| = \frac{(\beta-1)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\nu\alpha\beta}{(fK)^{(1-\beta)/2}} \sim \frac{1}{4}\frac{\rho\nu\alpha\beta}{(fK)^{(1-\beta)/2}}.$$

Note $\delta_3(K)$ is a quadratic function of $\frac{\alpha}{(fK)^{(1-\beta)/2}}$. So the requirement that $\delta_3(K) < \varepsilon_3$ for given positive number $\varepsilon_3$ can be easily translated into the condition

$$\boxed{K > K_3 = \frac{1}{f}\left[\frac{\sqrt{\rho^2\nu^2\beta^2 + \frac{8}{3}(1-\beta)^2\varepsilon_3} + \rho\nu\beta}{8\varepsilon_3} \cdot \alpha\right]^{2/(1-\beta)} \Rightarrow \delta_3(K) < \varepsilon_3}$$

**Estimating convergence rate of implied vol**: We would like to estimate the convergence rate of the implied vol given by Obloj's formula (Section 3) to its asymptotic limit:

$$|\sigma_B(K,f) - \sigma_*| = \left|I_0(K,f)[1 + I_1(K,f)t_{ex}] - \frac{\nu}{1-\beta}\left(1 + \frac{2-3\rho^2}{24}\nu^2 t_{ex}\right)\right|$$

where

$$I_0(K,f) = \frac{\nu}{1-\beta} \cdot \frac{1}{1+\theta_2(x)} \cdot \frac{1}{1+\theta_1(z)},\ I_1(K,f) = \delta_3(K) + \frac{2-3\rho^2}{24}\nu^2.$$

Assuming we have chosen $K$ large enough so that $0 < \theta_1(z) < \varepsilon_1 < 1$, $|\theta_2(x)| < \delta_2(x) < \varepsilon_2 < 1$ and $\delta_3(K) < \varepsilon_3$ ($\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ will be determined later). Then we have

$$\begin{aligned}
|\sigma_B(K,f) - \sigma_*| &\leq \left|I_0(K,f) - \frac{\nu}{1-\beta}\right|[1 + I_1(K,f)t_{ex}] + \frac{\nu}{1-\beta}\delta_3(K)t_{ex} \\
&\leq \frac{\nu}{1-\beta} \cdot \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2}{1-\varepsilon_2}\left[1 + \left(\varepsilon_3 + \frac{2-3\rho^2}{24}\nu^2\right)t_{ex}\right] + \frac{\nu}{1-\beta}t_{ex} \cdot \varepsilon_3 \\
&= \frac{\nu}{1-\beta} \cdot \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2}{1-\varepsilon_2}\left(1 + \frac{2-3\rho^2}{24}\nu^2 t_{ex}\right) + \frac{\nu}{1-\beta}t_{ex}\left[1 + \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2}{1-\varepsilon_2}\right] \cdot \varepsilon_3 \\
&:= err(\varepsilon_1, \varepsilon_2, \varepsilon_3).
\end{aligned}$$

**Estimating convergence rate of tail integral**: We are now ready to estimate the error in tail integral

$$\left|\int_{\widehat{K}}^{\infty} C(\sigma_B(K,f),K)dK - \int_{\widehat{K}}^{\infty} C(\sigma_*,K)dK\right|$$

assuming $\widehat{K}$ is large enough so that $\sup_{K \geq \widehat{K}} |\sigma_B(K,f) - \sigma_*| < \varepsilon$ for some prescribed positive number $\varepsilon$.

By the Lagrange remainder of Taylor's expansion, there exists some $\hat{\sigma}(K) \in [\sigma_* - \varepsilon, \sigma_* + \varepsilon]$ such that

$$\begin{aligned}
&|C(\sigma_B(K,f),K) - C(\sigma_*,K)| \\
\leq\ & \varepsilon f\sqrt{t_{ex}} \cdot \Phi'\left(\frac{-x(K)}{\hat{\sigma}(K)\sqrt{t_{ex}}} + \frac{1}{2}\hat{\sigma}(K)\sqrt{t_{ex}}\right) \\
=\ & \varepsilon K\sqrt{t_{ex}} \cdot \Phi'\left(\frac{x(K)}{\hat{\sigma}(K)\sqrt{t_{ex}}} + \frac{1}{2}\hat{\sigma}(K)\sqrt{t_{ex}}\right).
\end{aligned}$$

For given $x > 0$, the function $\frac{x}{\sigma\sqrt{t_{ex}}} + \frac{1}{2}\sigma\sqrt{t_{ex}}$ as a function of $\sigma$ is monotone decreasing on $(0, \sqrt{2x/t_{ex}})$ and monotone increasing on $(\sqrt{2x/t_{ex}}, \infty)$. Define $\sigma_*^{\pm} = \sigma_* \pm err$ where $err = err(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $K_*^{\pm} = f \cdot e^{\frac{1}{2}(\sigma_*^{\pm})^2 t_{ex}}$. We estimate $\Phi'\left(\frac{x(K)}{\hat\sigma(K)\sqrt{t_{ex}}} + \frac{1}{2}\hat\sigma(K)\sqrt{t_{ex}}\right)$ based on the relationship between $\sqrt{2x(K)/t_{ex}}$ and $[\sigma_*^-, \sigma_*^+]$ (or equivalently, between $K$ and $[K_*^-, K_*^+]$):

$$\Phi'\left(\frac{x(K)}{\hat\sigma(K)\sqrt{t_{ex}}} + \frac{1}{2}\hat\sigma(K)\sqrt{t_{ex}}\right) \le \begin{cases} \Phi'\left(\frac{x(K)}{\sigma_*^+\sqrt{t_{ex}}} + \frac{1}{2}\sigma_*^+\sqrt{t_{ex}}\right), & \sqrt{2x(K)/t_{ex}} \in (\sigma_*^+, \infty) \\ \Phi'\left(\sqrt{2x(K)}\right), & \sqrt{2x(K)/t_{ex}} \in [\sigma_*^-, \sigma_*^+] \\ \Phi'\left(\frac{x(K)}{\sigma_*^-\sqrt{t_{ex}}} + \frac{1}{2}\sigma_*^-\sqrt{t_{ex}}\right), & \sqrt{2x(K)/t_{ex}} \in (0, \sigma_*^-) \end{cases}$$

Note

$$\int_a^b K\Phi'\left(\sqrt{2x(K)}\right) dK = \frac{f}{\sqrt{2\pi}} \cdot (b-a)$$

and

$$\int_a^b K\Phi'\left(\frac{x(K)}{\sigma\sqrt{t_{ex}}} + \frac{1}{2}\sigma\sqrt{t_{ex}}\right) dK = f^2\sigma\sqrt{t_{ex}}e^{\sigma^2 t_{ex}}\left[\Phi\left(\frac{\log(b/f) - \frac{3}{2}\sigma^2 t_{ex}}{\sigma\sqrt{t_{ex}}}\right) - \Phi\left(\frac{\log(a/f) - \frac{3}{2}\sigma^2 t_{ex}}{\sigma\sqrt{t_{ex}}}\right)\right],$$

Then it's easy to see

$$\begin{aligned} & \left|\int_{\widehat{K}}^\infty C(\sigma_B(K,f),K)dK - \int_{\widehat{K}}^\infty C(\sigma_*,K)dK\right| \\ \le\ & \varepsilon\sqrt{t_{ex}}\int_{\widehat{K}}^\infty K\Phi'\left(\frac{x(K)}{\hat\sigma(K)\sqrt{t_{ex}}} + \frac{1}{2}\hat\sigma(K)\sqrt{t_{ex}}\right) dK \\ \le\ & \begin{cases} \varepsilon f^2\sigma_*^+ t_{ex} e^{(\sigma_*^+)^2 t_{ex}}\left[1 - \Phi\left(\frac{\log(\widehat{K}/f) - \frac{3}{2}(\sigma_*^+)^2 t_{ex}}{\sigma_*^+\sqrt{t_{ex}}}\right)\right], & \widehat{K} \in (K_*^+, \infty) \\ \frac{f}{\sqrt{2\pi}}(K_*^+ - \widehat{K}) + \varepsilon f^2\sigma_*^+ t_{ex} e^{(\sigma_*^+)^2 t_{ex}}\left[1 - \Phi\left(\frac{\log(K_*^+/f) - \frac{3}{2}(\sigma_*^+)^2 t_{ex}}{\sigma_*^+\sqrt{t_{ex}}}\right)\right], & \widehat{K} \in [K_*^-, K_*^+] \\ \varepsilon f^2\sigma_*^- t_{ex} e^{(\sigma_*^-)^2 t_{ex}}\left[\Phi\left(\frac{\log(K_*^-/f) - \frac{3}{2}(\sigma_*^-)^2 t_{ex}}{\sigma_*^-\sqrt{t_{ex}}}\right) - \Phi\left(\frac{\log(\widehat{K}/f) - \frac{3}{2}(\sigma_*^-)^2 t_{ex}}{\sigma_*^-\sqrt{t_{ex}}}\right)\right] & \\ + \frac{f}{\sqrt{2\pi}}(K_*^+ - K_*^-) + \varepsilon f^2\sigma_*^+ t_{ex} e^{(\sigma_*^+)^2 t_{ex}}\left[1 - \Phi\left(\frac{\log(K_*^+/f) - \frac{3}{2}(\sigma_*^+)^2 t_{ex}}{\sigma_*^+\sqrt{t_{ex}}}\right)\right], & \widehat{K} \in (0, K_*^-) \end{cases} \end{aligned}$$

**Numerical examples:** We should expect the tail integral to decay slowly, since it's linearly dependent on the product of $O(e^{-\log^2 K})$ and the convergence error of SABR volatility, the latter of which has a convergence rate of the order $O(\log^{-1} K)$ ($\varepsilon_1$ and $\varepsilon_2$).

Example 1. $f = 3.34\%$, $t_{ex} = 10$ (yr), $\alpha = 0.0913$, $\beta = 0.5$, $\nu = 0.2$, $\rho = 0$.

| Obloj | SABR | Hagan | $K_0$ | $K_1/\varepsilon_1$ | $K_2/\varepsilon_2$ | $K_3/\varepsilon_3$ | vol err | $K_*^+$ | tail err |
|---|---|---|---|---|---|---|---|---|---|
| 0.41333 | 0.40 | 0 | 0.16893 | 56.1848/0.2 | 16.8771/0.1 | 0.22573/0.001 | 0.15239 | 0.16546 | 0.001675 |

Example 2. $f = 3.34\%$, $t_{ex} = 10$ (yr), $\alpha = 0.0913$, $\beta = 0.8$, $\nu = 0.2$, $\rho = 0$.

| Obloj | SABR | Hagan | $K_0$ | $K_1/\varepsilon_1$ | $K_2/\varepsilon_2$ | $K_3/\varepsilon_3$ | vol err | $K_*^+$ | tail err |
|---|---|---|---|---|---|---|---|---|---|
| 1.0333 | 1 | 0 | 0.76471 | 476.0425/0.2 | $e^{82.289}/0.1$ | $1.55 \times 10^{-8}/0.001$ | 0.38096 | 736.5453 | 0 |
| 1.0333 | 1 | 0 | 0.76471 | 476.0425/0.2 | $e^{14.317}/0.5$ | $1.55 \times 10^{-8}/0.001$ | 1.6793 | $e^{33.3936}$ | 0 |

Example 3. $f = 3.34\%$, $t_{ex} = 10$ (yr), $\alpha = 0.0913$, $\beta = 0.2$, $\nu = 0.2$, $\rho = 0$.

| Obloj | SABR | Hagan | $K_0$ | $K_1/\varepsilon_1$ | $K_2/\varepsilon_2$ | $K_3/\varepsilon_3$ | vol err | $K_*^+$ | tail err |
|---|---|---|---|---|---|---|---|---|---|
| 0.25833 | 0.25 | 0 | 0.34934 | 21.7927/0.2 | $e^{18.0014}/0.1$ | 4.5697/0.001 | 0.095241 | 0.062404 | 0 |
| 0.25833 | 0.25 | 0 | 0.34934 | 21.7927/0.2 | 2.7898/0.5 | 4.5697/0.001 | 0.41983 | 0.33299 | 0.18223 |

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