Elements of LGM Model

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Abstract

In this note, we document the elements of Linear Markov Model (LGM) and its calibration to swaptions.

Contents

1 Elements of one-factor LGM model

In this section, we review the elements of one-factor LGM model and its calibration to swaptions, as presented in Hagan [1] and Piza [4].

1.1 HJM framework

We assume that we have a family of zero-coupon bonds traded in the market. The price at time t of a zero-coupon bond with maturity T ($0 \le t \le T$) will be denoted by $P(t,T)$. We assume the bond price satisfies the following SDE:

$$dP(t,T) = P(t,T)\big[A(t,T)dt + B(t,T)dW_t\big],\ P(T,T) = 1,\ A(T,T) = B(T,T) = 0,$$

where W is a 1-dimensional standard Brownian motion. We assume there is also a strictly positive process N, which will be chosen as the numéraire, that satisfies the following SDE:

$$dN_t = N_t\big(\mu_t^N dt + \sigma_t^N dW_t\big),\ N_0 = 1.$$

By the Fundamental Theorem of Asset Pricing, a necessary and sufficient conditoin for the no arbitrage property (more precisely, no-free-lunch-with-vanishing-risk, NFLVR, for allowable strategies) is that we can find a probability measure Q such that the discounted bond price process

$$\bar{P}(t,T) := \frac{P(t,T)}{N_t}$$

is a Q-local martingale. Itô calculus yields

$$\frac{d\bar{P}(t,T)}{\bar{P}(t,T)} = \big[B(t,T) - \sigma_t^N\big]\left[\frac{A(t,T) - \mu_t^N + (\sigma_t^N)^2 - \sigma_t^N B(t,T)}{B(t,T) - \sigma_t^N}dt + dW_t\right]$$

provided $B(t,T) - \sigma_t^N \neq 0$, $0 \le t \le T$.

If the probability measure Q is defined by (P denotes the original probability measure)

$$\frac{dQ}{dP}\bigg|_{\mathcal{F}_t} = D_t = \exp\left\{\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right\},$$

we necessarily have

$$\frac{A(t,T) - \mu_t^N + (\sigma_t^N)^2 - \sigma_t^N B(t,T)}{B(t,T) - \sigma_t^N} = -\theta_t,$$

which must be independent of T. We are already in the risk-neutral measure (i.e. $P = Q$) if and only if

$$A(t,T) - \mu_t^N + (\sigma_t^N)^2 - \sigma_t^N B(t,T) = 0.$$

1.2 Forward rate model

The results in HJM model can be translated into those in forward rate model. Denote by $f(t,T)$ the forward rate such that $P(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}$. Assume $f(t,T)$ follows the SDE

$$df(t,T) = a(t,T)dt + b(t,T)dW_t.$$

We then have the following relations

$$A(t,T) = f(t,t) - \int_t^T a(t,s)ds + \frac{1}{2}\left(\int_t^T b(t,s)ds\right)^2,\ B(t,T) = -\int_t^T b(t,s)ds$$

and

$$a(t,T) = \frac{\partial B(t,T)}{\partial T} B(t,T) - \frac{\partial A(t,T)}{\partial T}, \ b(t,T) = -\frac{\partial B(t,T)}{\partial T}$$

Then the condition $A(t,T) - \mu_t^N + (\sigma_t^N)^2 - \sigma_t^N B(t,T) = 0$ translates into

$$a(t,T) = \int_t^T b(t,s)ds \cdot b(t,T) + \sigma_t^N b(t,T).$$

1.3 The LGM model

To get the LGM model, we assume that we are already under the risk-neutral measure associated with the numeraire N, where N is specified by the following parameter specification

$$\begin{cases} b(t,T) = H'(T)\alpha_t \\ \sigma_t^N = H(t)\alpha_t \end{cases}$$

Here H and α are two deterministic functions with $H(0) = 0$. This specification gives

$$a(t,T) = H(T)H'(T)\alpha_t^2, \ B(t,T) = -[H(T) - H(t)]\alpha_t.$$

Define $\zeta_t = \int_0^t \alpha_s^2 ds$ and $X_t = \int_0^t \alpha_s dW_s$, we have $f(t,T) = f(0,T) + H'(T)H(T)\zeta_t + H'(T)X_t$. This gives

$$A(t,T) = f(0,t) + H'(t)H(t)\zeta_t + H'(t)X_t - [H(T) - H(t)]H(t)\alpha_t^2$$

and

$$\mu_t^N = f(0,t) + H'(t)H(t)\zeta_t + H'(t)X_t + H^2(t)\alpha_t^2.$$

In summary, the HJM parameter specifications of LGM model are

$$\boxed{\begin{cases} A(t,T) = f(0,t) + H'(t)H(t)\zeta_t + H'(t)X_t - [H(T) - H(t)]H(t)\alpha_t^2 \\ B(t,T) = -[H(T) - H(t)]\alpha_t \\ a(t,T) = H(T)H'(T)\alpha_t^2 \\ b(t,T) = H'(T)\alpha_t \\ \mu_t^N = f(0,t) + H'(t)H(t)\zeta_t + H'(t)X_t + H^2(t)\alpha_t^2 \\ \sigma_t^N = H(t)\alpha_t \end{cases}} \tag{1}$$

where H and α are two deterministic functions with $H(0) = 0$, $\zeta_t = \int_0^t \alpha_s^2 ds$, $X_t = \int_0^t \alpha_s dW_s$, and $f(0,t)$ is given by market quoted yield curve.

Consequently, we have $r_t := f(t,t) = f(0,t) + H'(t)H(t)\zeta_t + H'(t)X_t$,

$$P(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\} = \frac{P(0,T)}{P(0,t)} \exp\left\{-[H(T) - H(t)]X_t - \frac{1}{2}[H^2(T) - H^2(t)]\zeta_t\right\}.$$

and

$$\frac{d\bar{P}(t,T)}{\bar{P}(t,T)} = [B(t,T) - \sigma_t^N]dW_t.$$

The last SDE gives

$$\bar{P}(t,T) = P(0,T)\exp\left\{-H(T)X_t - \frac{1}{2}H^2(T)\zeta_t\right\}.$$

Therefore

$$N_t = \frac{P(t,T)}{\bar{P}(t,T)} = \frac{1}{P(0,t)} \exp\left\{H(t)X_t + \frac{1}{2}H^2(t)\zeta_t\right\}.$$

In summary, we have

$$\begin{cases} f(t,T) = f(0,T) + H'(T)H(T)\zeta_t + H'(T)X_t \\ r_t = f(0,t) + H'(t)H(t)\zeta_t + H'(t)X_t \\ P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left\{-[H(T)-H(t)]X_t - \frac{1}{2}[H^2(T)-H^2(t)]\zeta_t\right\} \\ \bar{P}(t,T) = P(0,T)\exp\left\{-H(T)X_t - \frac{1}{2}H^2(T)\zeta_t\right\} \\ N_t = \frac{1}{P(0,t)}\exp\left\{H(t)X_t + \frac{1}{2}H^2(t)\zeta_t\right\} \end{cases} \tag{2}$$

1.4 Connection with one-factor Hull-White model

Denote by Q the martingale measure associated with money market account numeraire. The one-factor Hull-White model assumes the short rate process r_t follows the following dynamics under Q

$$dr_t = (b_t - \kappa r_t)dt + \sigma_t dW_t^Q,$$

where κ is a constant, b_t and σ_t are deterministic functions of t, and W^Q is a standard Brownian motion under Q.

Define $\theta_t = e^{-\kappa t} r_0 + e^{-\kappa t}\int_0^t e^{\kappa s} b_s ds$ and $X_t^Q = e^{-\kappa t}\int_0^t e^{\kappa s}\sigma_s dW_s^Q$. Then θ_t is a deterministic function of t and X_t^Q is Gaussian process with mean 0 and variance $e^{-2\kappa t}\int_0^t e^{2\kappa s}\sigma_s^2 ds$. In summary, we have

$$r_t = \theta_t + X_t^Q,\ dX_t^Q = -\kappa X_t^Q dt + \sigma_t dW_t^Q,\ X_0^Q = 0,\ E[X_t^Q] = 0,\ E[(X_t^Q)^2] = e^{-2\kappa t}\int_0^t e^{2\kappa s}\sigma_s^2 ds.$$

It's easy to verify that (see Zeng [5])

$$\begin{cases} P(t,T) = P(t,T;X_t^Q) = \frac{P(0,T)}{P(0,t)}\exp\left\{-H^Q(T-t)\left[X_t^Q + \nu^h(t) + \frac{1}{2}\nu(t)H^Q(T-t)\right]\right\} \\ P(0,t) = \exp\left\{-\int_0^t \theta_s ds + \nu_t^{H^Q}\right\} \end{cases} \tag{3}$$

where

$$\begin{cases} h(t) = e^{-\kappa t} \\ H^Q(t) = \int_0^t h(s)ds \\ \nu(t) = e^{-2\kappa t}\int_0^t e^{2\kappa s}\sigma_s^2 ds \\ \nu^h(t) = h * v(t) = \int_0^t e^{-\kappa(t-s)}\nu(s)ds \\ \nu^{H^Q}(t) = H^Q * \nu(t) = \int_0^t H^Q(t-s)\nu(s)ds. \end{cases}$$

We also note that $\frac{d}{dt}\nu^{H^Q}(t) = \nu^h(t)$. The one-to-one correspondence between one-factor LGM model and one-factor Hull-White model is therefore

$$\begin{cases} \alpha_t = e^{\kappa t}\sigma_t \\ \zeta_t = e^{2\kappa t}\nu(t) = \int_0^t \alpha_s^2 ds = \int_0^t e^{2\kappa s}\sigma_s^2 ds \\ H(t) = H^Q(t) = \int_0^t e^{-\kappa s}ds. \end{cases}$$

To verify this relationship, we note

$$\begin{aligned} & -H^Q(T-t)[X_t^Q + \nu^h(t) + \frac{1}{2}\nu(t)H^Q(T-t)] \\ = \quad & -H(T-t)\left[e^{-\kappa t}\int_0^t e^{\kappa s}\sigma_s dW_s^Q + e^{-\kappa t}\int_0^t e^{\kappa s}e^{-2\kappa s}\zeta_s ds + \frac{1}{2}e^{-2\kappa t}\zeta_t H(T-t)\right] \\ = \quad & -[H(T)-H(t)]\left[\int_0^t e^{\kappa s}\sigma_s dW_s^Q + \int_0^t e^{-\kappa s}\zeta_s ds\right] - \frac{1}{2}[H(T)-H(t)]^2\zeta_t \\ = \quad & -[H(T)-H(t)]\left[\int_0^t e^{\kappa s}\sigma_s dW_s^Q - \int_0^t H(s)e^{2\kappa s}\sigma_s^2 ds\right] - \frac{1}{2}[H^2(T)-H^2(t)]\zeta_t. \end{aligned}$$

We shall show $\int_0^t e^{\kappa s}\sigma_s dW_s^Q - \int_0^t H(s)e^{2\kappa s}\sigma_s^2 ds = \int_0^t e^{\kappa s}\sigma_s(dW_s^Q - H(s)e^{\kappa s}\sigma_s ds) = \int_0^t e^{\kappa s}\sigma_s dW_s = X_t$, and thus prove that formula (3) agrees with the zero coupon bond price formula in (2). Indeed, the Radon-Nikodym derivative of Q^N w.r.t. Q is

$$D_t = \frac{N_t}{e^{\int_0^t r_u du}}.$$

So $d\ln D_t = \frac{dN_t}{N_t} + (...)dt$. Since D_t is a martingale under Q, we conclude

$$dD_t = D_t\sigma_t^N dW_t^Q = D_t H(t)\alpha_t dW_t^Q.$$

Girsanov's Theorem (see Appendix A) implies $W_t^Q - \int_0^t H(s)\alpha_s$ is a martingale under Q^N. This proves our claim.

1.5 Pricing formula of swap

Consider a swap with start date t_0, fixed leg pay dates t_1, t_2, $\cdots$, t_n, and fixed rate K. Then the fixed leg makes the payments (assuming notional is one unit of currency)

$$\begin{cases} \tau_i K & \text{paid at } t_i, \text{ for } i = 1, 2, \cdots, n-1 \\ 1 + \tau_n K & \text{paid at } t_n, \end{cases}$$

where τ_i is the day count of $[t_{i-1}, t_i]$ in year fraction. For any $t \le t_0$, these payments have the value

$$V_{fix}(t) = K\sum_{i=1}^n \tau_i P(t, t_i) + P(t, t_n).$$

The swap's floating leg usually has a different frequency than the fixed leg, so let this leg's start and pay dates be

$$t_0 = u_0 < u_1 < \cdots < u_m = t_n.$$

The floating leg pays

$$\begin{cases} \tilde{\tau}_j L_j & \text{paid at } u_j, \text{ for } j = 1, 2, \cdots, m-1 \\ 1 + \tilde{\tau}_m L_m & \text{paid at } u_m = t_n \end{cases}$$

where $\tilde{\tau}_j$ is the day count of $[u_{j-1}, u_j]$ in year fraction and L_j is the Libor or Euribor floating rate for the interval $[u_{j-1}, u_j]$. The rate L_j is set on the fixing date, which is generally two London business days before the interval starts on u_{j-1}. In formula,

$$L_j = \frac{1}{\tilde{\tau}_j}\left[\frac{P(u_{j-1}^{fix}, u_{j-1})}{P(u_{j-1}^{fix}, u_j)} - 1\right] + s_j,$$

where the first part of the formula stands for risk-free floating rate, and the second part s_j stands for a spread for credit risk. The payment of $\tilde{\tau}_j L_j$ at time u_j is equal to a payment of

$$[P(u_{j-1}^{fix}, u_{j-1}) - P(u_{j-1}^{fix}, u_j)] + \tilde{\tau}_j s_j P(u_{j-1}^{fix}, u_j)$$

at time u_j^{fix}, which is further equal to a payment of

$$[P(t, u_{j-1}) - P(t, u_j)] + \tilde{\tau}_j s_j P(t, u_j)$$

at time t. The value of the floating leg is therefore

$$V_{flt}(t) = P(t, t_0) + \sum_{j=1}^m \tilde{\tau}_j s_j P(t, u_j).$$

The value of the receiver swap (receiving the fixed leg, paying the floating leg) is

$$\boxed{V_{rec}(t) = K\sum_{i=1}^{n}\tau_i P(t,t_i) + P(t,t_n) - P(t,t_0) - \sum_{j=1}^{m}\tilde{\tau}_j s_j P(t,u_j)} \tag{4}$$

For $t=0$, we can write the formula in a nicer form

$$V_{rec}(0) = K^{adj}\sum_{i=1}^{n}\tau_i P(0,t_i) + P(0,t_n) - P(0,t_0)$$

where $K^{adj} = K - \frac{\sum_{j=1}^{m}\tilde{\tau}_j s_j P(0,u_j)}{\sum_{i=1}^{n}\tau_i P(0,t_i)}$. This leads to the following pragmatic approximation

$$\boxed{V_{rec}(t) \approx K^{adj}\sum_{i=1}^{n}\tau_i P(t,t_i) + P(t,t_n) - P(t,t_0)} \tag{5}$$

1.6 Pricing formula of swaption

The value of a receiver swaption at time zero is ($t_{ex} \le t_0$ is the option exercise time)

$$V_{rec}^{opt}(0) = N_0 E^{Q_N}\left[\frac{\max\{V_{rec}(t_{ex}),0\}}{N_{t_{ex}}}\right] \approx E^{Q_N}\left[\left(K^{adj}\sum_{i=1}^{n}\tau_i\bar{P}(t_{ex},t_i;X_{t_{ex}}) + \bar{P}(t_{ex},t_n;X_{t_{ex}}) - \bar{P}(t_{ex},t_0;X_{t_{ex}})\right)^+\right]$$

where $X_{t_{ex}} \sim N(0,\zeta_{t_{ex}})$ under the martingale measure Q_N associated with numeraire N. By change of variable $y = x + H(t_0)\zeta_{t_{ex}}$, we have

$$\begin{aligned}
V_{rec}^{opt}(0) \quad &\approx \quad \frac{1}{\sqrt{2\pi\zeta_{t_{ex}}}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\zeta_{t_{ex}}}}\left(K^{adj}\sum_{i=1}^{n}\tau_i P(0,t_i)\exp\left\{-H(t_i)x - \frac{1}{2}H^2(t_i)\zeta_{t_{ex}}\right\}\right. \\
&\quad \left. +P(0,t_n)\exp\left\{-H(t_n)x - \frac{1}{2}H^2(t_n)\zeta_{t_{ex}}\right\} - P(0,t_0)\exp\left\{-H(t_0)x - \frac{1}{2}H^2(t_0)\zeta_{t_{ex}}\right\}\right)^+ dx \\
&= \quad \frac{1}{\sqrt{2\pi\zeta_{t_{ex}}}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\zeta_{t_{ex}}}}\left(K^{adj}\sum_{i=1}^{n}\tau_i D_i\exp\left\{-(H_i-H_0)y - \frac{1}{2}(H_i-H_0)^2\zeta_{t_{ex}}\right\}\right. \\
&\quad \left. +D_n\exp\left\{-(H_n-H_0)y - \frac{1}{2}(H_n-H_0)^2\zeta_{t_{ex}}\right\} - D_0\right)^+ dx
\end{aligned}$$

where $H_i = H(t_i)$, $D_i = P(0,t_i)$ for $i = 0,1,\cdots,n$.

We now assume without loss of generality that H is a strictly increasing function so that $H' > 0$. Then

$$\exp\left\{-[H(T)-H(t)]y - \frac{1}{2}[H(T)-H(t)]^2\zeta_{t_{ex}}\right\}, \; t_{ex} \le t \le T$$

is a monotone decreasing function of y, with limit 0 as $y \to \infty$ and limit ∞ as $y \to -\infty$. So there exists a unique break-even point y^* such that the term inside $(\cdots)^+$ is

$$\begin{cases} <0 & \text{if } y > y^* \\ =0 & \text{if } y = y^* \\ >0 & \text{if } y < y^* \end{cases}$$

Then

$$
\begin{aligned}
&V_{rec}^{opt}(0) \\
\approx\ & \frac{1}{\sqrt{2\pi\zeta_{t_{ex}}}}\int_{-\infty}^{y^*} e^{-\frac{y^2}{2\zeta_{t_{ex}}}}\left(K^{adj}\sum_{i=1}^{n}\tau_i D_i e^{-(H_i-H_0)y-\frac{1}{2}(H_i-H_0)^2\zeta_{t_{ex}}} + D_n e^{-(H_n-H_0)y-\frac{1}{2}(H_n-H_0)^2\zeta_{t_{ex}}} - D_0\right)dx \\
=\ & \boxed{K^{adj}\sum_{i=1}^{n}\tau_i D_i \Phi\left(\frac{y^*+(H_i-H_0)\zeta_{t_{ex}}}{\sqrt{\zeta_{t_{ex}}}}\right) + D_n\Phi\left(\frac{y^*+(H_n-H_0)\zeta_{t_{ex}}}{\sqrt{\zeta_{t_{ex}}}}\right) - D_0\Phi\left(\frac{y^*}{\sqrt{\zeta_{t_{ex}}}}\right)} \qquad (6)
\end{aligned}
$$

where $\Phi(\cdot)$ is the c.d.f. of a standard normal distribution and y^* is the unique solution of

$$
K^{adj}\sum_{i=1}^{n}\tau_i D_i e^{-[H(t_i)-H(t_0)]y^*-\frac{1}{2}[H(t_i)-H(t_0)]^2\zeta_{t_{ex}}} + D_n e^{-[H(t_n)-H(t_0)]y^*-\frac{1}{2}[H(t_n)-H(t_0)]^2\zeta_{t_{ex}}} = D_0.
$$

1.7 Calibration to swaption market

We define the forward swap rate S as

$$
S(t) = \frac{P(t,t_0)-P(t,t_n)}{\sum_{i=1}^{n}\tau_i P(t,t_i)},\ t \le t_0
$$

and the annuity numeraire as

$$
L(t) = \sum_{i=1}^{n}\tau_i P(t,t_i),\ t \le t_0.
$$

Then

$$
V_{rec}(t) \approx K^{adj}\sum_{i=1}^{n}\tau_i P(t,t_i) + P(t,t_n) - P(t,t_0) = (K^{adj}-S(t))L(t)
$$

and the rule of change-of-numeraire gives us

$$
V_{rec}^{opt}(0) = N_0 E^{Q_N}\left[\frac{\max\{V_{rec}(t_{ex}),0\}}{N_{t_{ex}}}\right] = L_0 E^{Q_L}\left[\frac{\max\{V_{rec}(t_{ex}),0\}}{L_{t_{ex}}}\right] \approx L_0 E^{Q_L}[(K^{adj}-S(t_{ex}))^+].
$$

By the pricing formula of zero coupon bond, $S(t)$ is a function of t and X_t. So Ito's formula yields

$$
dS(t) = dS(t,X_t) = \left.\frac{\partial S(t,x)}{\partial x}\right|_{x=X_t}\alpha_t dW_t + (...)dt.
$$

Since $S(t)$ has the form of $\frac{\text{tradable}}{\text{numeraire}}$, it is a martingale under the martingale measure Q_L associated with the annuity numeraire L. Therefore

$$
dS(t) = \left.\frac{\partial S(t,x)}{\partial x}\right|_{x=X_t}\alpha_t dW_t^L,
$$

where W^L is a standard Brownian motion under Q_L.

This is a setup for the equivalent vol techniques of Hagan [3], and we have the following approximation (formula (3.44b), formula (3.44c) of Hagan [1])

$$
\boxed{\sigma_N\sqrt{t_{ex}} \approx \sqrt{\zeta_{t_{ex}}}\frac{S(0)\sum_{i=1}^{n}\tau_i D_i(H_i-H_0)+D_n(H_n-H_0)}{\sum_{i=1}^{n}\tau_i D_i}} \qquad (7)
$$

$$
\boxed{\sigma_B\sqrt{t_{ex}} \approx \sqrt{\zeta_{t_{ex}}}\frac{\log\frac{K^{adj}}{S(0)}}{K^{adj}-S(0)}\frac{S(0)\sum_{i=1}^{n}\tau_i D_i(H_i-H_0)+D_n(H_n-H_0)}{\sum_{i=1}^{n}\tau_i D_i}} \qquad (8)
$$

where σ_N is the implied normal vol and σ_B is the implied Black vol. Here $\frac{\log\frac{K^{adj}}{S(0)}}{K^{adj}-S(0)}$ is interpreted as $\frac{1}{K^{adj}}$ when $S(0)=K^{adj}$.

1.7.1 Calibration to ATM swaption

The ATM calibration function takes as input an ATM vol surface; then for a given expiry, it computes ζ for each tenor by the formula (8)

$$\sigma_B\sqrt{t_{ex}} \approx \sqrt{\zeta_{t_{ex}}}\frac{1}{S(0)}\frac{S(0)\sum_{i=1}^n \tau_i D_i(H_i - H_0) + D_n(H_n - H_0)}{\sum_{i=1}^n \tau_i D_i};$$

finally, it takes a weighted average as the value of ζ at the given expiry. Doing the same thing to each expiry, the function produces the values of ζ at all expiries. That is,

$$\begin{cases}\text{Step1. Input an ATM vol surface with } n \text{ expiries and } m \text{ tenors;}\\ \text{Step2. For } i\text{-th expiry, compute the value of } \zeta \text{ at } i\text{-th expiry } m \text{ times, once for each tenor;}\\ \text{Step3. Take a weighted average of these } m \text{ values of } \zeta\text{, and use this average as the value of } \zeta \text{ at } i\text{-th expiry;}\\ \text{Step4. Go to step2 for } (i+1)\text{-th expiry, until we go through all the } n \text{ expiries.}\end{cases}$$

1.7.2 Calibration to OTM swaption

The OTM calibration function takes as input a volatility band (an array of swaptions); secondly, for a given expiry, it computes the market quoted price of swaption by Black's formula; it then computes ζ by formula (6)

$$V_{rec}^{opt}(0) = K^{adj}\sum_{i=1}^n \tau_i D_i \Phi\left(\frac{y^* + (H_i - H_0)\zeta_{t_{ex}}}{\sqrt{\zeta_{t_{ex}}}}\right) + D_n\Phi\left(\frac{y^* + (H_n - H_0)\zeta_{t_{ex}}}{\sqrt{\zeta_{t_{ex}}}}\right) - D_0\Phi\left(\frac{y^*}{\sqrt{\zeta_{t_{ex}}}}\right),$$

with the initial guess based on

$$\sigma_B\sqrt{t_{ex}} = \sqrt{\zeta_{t_{ex}}}\frac{\log\frac{K^{adj}}{S(0)}\left[1 + \frac{1}{24}(1 - (\log\frac{K^{adj}}{S(0)})^2/120)\sigma_B^2 t_{ex}\right]}{K^{adj} - S(0)}\frac{S(0)\sum_{i=1}^n \alpha_i D_i(H_i - H_0) + D_n(H_n - H_0)}{\sum_{i=1}^n \alpha_i D_i}.$$

That is,

$$\begin{cases}\text{Step1. Input a volatility band with } n \text{ swaptions (} n \text{ different expiries);}\\ \text{Step2. For } i\text{-th swaption, compute the initial guess of } \zeta \text{ at the swaption's expiry;}\\ \text{Step3. Use Black's formula to compute market quoted price of the } i\text{-th swaption;}\\ \text{Step4. Solve equation (6) for } \zeta;\\ \text{Step5. Go to Step2 for } (i+1)\text{-th swaption, until we go through all the } n \text{ swaptions.}\end{cases}$$

1.8 Pricing formula of caplet and calibration to caps market

Consider the Libor rate for time period $[T_s, T_e]$, which is fixed at time $t_f \le T_s$. The market quotes give Black volatilities of caplets for various strikes. Suppose a caplet has strike K and the year fraction of $[T_s, T_e]$ is τ. The market price for this caplet is

$$V_0^{mkt} = P(0, T_e)\tau \text{Bl}(K, F, \sigma_B\sqrt{t_f}, 1)$$

where $F = F(0; T_s, T_e)$ is the forward rate for $[T_s, T_e]$ at time 0, σ_B is the market quoted Black vol, and $\text{Bl}(K, F, v, w)$ is given by

$$\text{Bl}(K, F, v, w) = Fw\Phi(wd_1) - Kw\Phi(wd_2),\ d_1 = \frac{\ln(F/K) + v^2/2}{v},\ d_2 = \frac{\ln(F/K) - v^2/2}{v}$$

The theoretical price of the above caplet based on one-factor LGM model is

$$V_0^{model} = P(0, T_e)\tau \text{Bl}\left(K + \frac{1}{\tau}, F + \frac{1}{\tau}, [H(T_e) - H(T_s)]\sqrt{\zeta_{t_f}}, 1\right)$$

where $H(t) = \int_0^t e^{-\kappa s} ds$ and $\zeta_t = \int_0^t e^{2\kappa s} \sigma_s^2 ds$ ($\sigma_\cdot$ is the volatility parameter in the corresponding one-factor Hull-White model).

Therefore, calibration to caplets in order to obtain ζ_{t_f} requires solving the following equation

$$\boxed{\mathrm{Bl}(K, F, \sigma_B \sqrt{t_f}, 1) = \mathrm{Bl}\left(K + \frac{1}{\tau}, F + \frac{1}{\tau}, [H(T_e) - H(T_s)]\sqrt{\zeta_{t_f}}, 1\right)}$$

We note $\mathrm{Bl}(K, F, v, 1)$ is a monotone increasing function of v with a range of $((F-K)^+, F)$. So the above calibration equation always has a solution.

1.9 Interpolation of LGM model parameter ζ

We assume there is a coupon period $[t_s, t_e]$ and we are given the values of ζ at t_s and t_e: ζ_s and ζ_e, respectively. If the volatility of one-factor Hull-White model (equivalent to one-factor LGM model) is a constant σ over $[t_s, t_e]$, we have for $t \in [t_s, t_e]$

$$\zeta_t = \begin{cases} \zeta_s + \sigma^2 (t - t_s) & \text{if } \kappa = 0 \\ \zeta_s + \sigma^2 \frac{e^{2\kappa t} - e^{2\kappa t_s}}{2\kappa} & \text{if } \kappa \neq 0 \end{cases}$$

By setting t to t_e, we can solve for σ:

$$\sigma = \begin{cases} \frac{\zeta_s (t_e - t) + \zeta_e (t - t_s)}{t_e - t_s} & \text{if } \kappa = 0 \\ \frac{e^{2\kappa(t_e - t_s)} - e^{2\kappa(t - t_s)}}{e^{2\kappa(t_e - t_s)} - 1} \zeta_s + \frac{e^{2\kappa(t - t_s)} - 1}{e^{2\kappa(t_e - t_s)} - 1} \zeta_e & \text{if } \kappa \neq 0 \end{cases}$$

2 Elements of two-factor LGM model

The two-factor LGM model has two state variables, $\mathbf{X}_t = (X_1(t), X_2(t))^T$, where for $k = 1, 2$, $X_k(t) = \int_0^t \alpha_k(s) dW_k(s)$, with W_k a standard Brownian motion, α_k a deterministic function, and $dW_1(t) dW_2(t) = \rho(t) dt$. This is the evolution under the risk neutral measure induced by a numeraire. We choose the numeraire to be

$$\begin{aligned} N_t &= \frac{1}{P(0,t)} \exp\left\{\mathbf{H}(t)\mathbf{X}(t) + \frac{1}{2}\mathbf{H}(t)\zeta(t)\mathbf{H}(t)\right\} \\ &= \frac{1}{P(0,t)} \exp\left\{H_1(t)X_1(t) + H_2(t)X_2(t) + \frac{1}{2}H_1^2(t)\zeta_{11}(t) + H_1(t)H_2(t)\zeta_{12}(t) + \frac{1}{2}H_2^2(t)\zeta_{22}(t)\right\}. \end{aligned}$$

Here $\mathbf{H}(t) = (H_1(t), H_2(t))$ is a deterministic, strictly increasing function of t, and

$$\zeta(t) = \begin{bmatrix} \int_0^t \alpha_1^2(s) ds & \int_0^t \rho(s)\alpha_1(s)\alpha_2(s) ds \\ \int_0^t \rho(s)\alpha_1(s)\alpha_2(s) ds & \int_0^t \alpha_2^2(s) ds \end{bmatrix}.$$

Similar to one-factor LGM model (formual (2)), we have

$$\boxed{\begin{cases} f(t,T) = f(0,T) + \mathbf{H}'(T)\mathbf{X}_t + \mathbf{H}'(T)\zeta(t)\mathbf{H}(T) \\ r_t = f(0,t) + \mathbf{H}'(t)\mathbf{X}_t + \mathbf{H}'(t)\zeta(t)\mathbf{H}(t) \\ P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left\{-[\mathbf{H}(T) - \mathbf{H}(t)]\mathbf{X}_t - \frac{1}{2}\mathbf{H}(T)\zeta(t)\mathbf{H}(T) + \frac{1}{2}\mathbf{H}(t)\zeta(t)\mathbf{H}(t)\right\} \\ \bar{P}(t,T) = P(0,T) \exp\left\{-\mathbf{H}(T)\mathbf{X}_t - \frac{1}{2}\mathbf{H}(T)\zeta(t)\mathbf{H}(T)\right\} \\ N_t = \frac{1}{P(0,t)} \exp\left\{\mathbf{H}(t)\mathbf{X}(t) + \frac{1}{2}\mathbf{H}(t)\zeta(t)\mathbf{H}(t)\right\} \end{cases}} \tag{9}$$

The value of the receiver swap (receiving the fixed leg, paying the floating leg) is

$$\boxed{V_{rec}(t) = K \sum_{i=1}^{n} \tau_i P(t, t_i) + P(t, t_n) - P(t, t_0) - \sum_{j=1}^{m} \tilde{\tau}_j s_j P(t, u_j)} \tag{10}$$

and a pragmatic approximation is

$$\boxed{V_{rec}(t) \approx K^{adj}\sum_{i=1}^{n}\tau_i P(t,t_i) + P(t,t_n) - P(t,t_0)} \tag{11}$$

where $K^{adj} = K - \frac{\sum_{j=1}^{m}\tilde{\tau}_j s_j P(0,u_j)}{\sum_{i=1}^{n}\tau_i P(0,t_i)}$.

2.1 Pricing formula of swaption

2.1.1 Exact formula

By direct integration

$$\begin{aligned}
& V_{rec}^{opt}(0) \\
= \; & E^{Q_N}\left[\left(K^{adj}\sum_{i=1}^{n}\tau_i\bar{P}(t_{ex},t_i) + \bar{P}(t_{ex},t_n) - \bar{P}(t_{ex},t_0)\right)^+\right] \\
= \; & \frac{1}{2\pi|\det\zeta|^{1/2}}\int\int e^{-\frac{1}{2}\mathbf{Y}\zeta^{-1}\mathbf{Y}}\left[K^{adj}\sum_{i=1}^{n}\tau_i D_i e^{-\mathbf{\Delta H}_i\mathbf{Y}-\frac{1}{2}\mathbf{\Delta H}_i\cdot\zeta\mathbf{\Delta H}_i} + D_n e^{-\mathbf{\Delta H}_n\mathbf{Y}-\frac{1}{2}\mathbf{\Delta H}_n\cdot\zeta\mathbf{\Delta H}_n} - D_0\right]^+ dY_1 dY_2
\end{aligned}$$

where $\zeta = \zeta_{t_{ex}}$, $D_i = P(0,t_i)$, and $\mathbf{\Delta H}_i = \mathbf{H}(t_i) - \mathbf{H}(t_0)$, $i = 1,2,\cdots,n$.

By fixing one variable and integrating with respect to the other, we have

$$\begin{aligned}
V_{rec}^{opt}(0) \; = \; & \frac{1}{\sqrt{2\pi\zeta_{22}}}\int\left[K^{adj}\sum_{i=1}^{n}\tau_i D_i e^{-(z+\zeta_{12}\Delta H_i^{(1)}+\zeta_{22}\Delta H_i^{(2)})^2/2\zeta_{22}}\Phi\left(\frac{-\zeta_{12}z+\zeta_{22}y^*+\Delta H_i^{(1)}[\zeta_{11}\zeta_{22}-\zeta_{12}^2]}{\sqrt{\zeta_{22}}\sqrt{\zeta_{11}\zeta_{22}-\zeta_{12}^2}}\right)\right. \\
& +D_n e^{-(z+\zeta_{12}\Delta H_n^{(1)}+\zeta_{22}\Delta H_n^{(2)})^2/2\zeta_{22}}\Phi\left(\frac{-\zeta_{12}z+\zeta_{22}y^*+\Delta H_n^{(1)}[\zeta_{11}\zeta_{22}-\zeta_{12}^2]}{\sqrt{\zeta_{22}}\sqrt{\zeta_{11}\zeta_{22}-\zeta_{12}^2}}\right) \\
& \left. D_0 e^{-z^2/2\zeta_{22}}\Phi\left(\frac{-\zeta_{12}z+\zeta_{22}y^*}{\sqrt{\zeta_{22}}\sqrt{\zeta_{11}\zeta_{22}-\zeta_{12}^2}}\right)\right] dz
\end{aligned} \tag{12}$$

Here $y^* = y^*(z)$ is the numerically determined break-even point, where the vector $\mathbf{Y} = \begin{pmatrix} y^*(z) \\ z \end{pmatrix}$ satisfies

$$K^{adj}\sum_{i=1}^{n}\tau_i D_i e^{-\mathbf{\Delta H}_i\cdot\mathbf{Y}-\frac{1}{2}\mathbf{\Delta H}_i\cdot\zeta\mathbf{\Delta H}_i} + D_n e^{-\mathbf{\Delta H}_n\cdot\mathbf{Y}-\frac{1}{2}\mathbf{\Delta H}_n\cdot\zeta\mathbf{\Delta H}_n} = D_0.$$

In addition,

$$\mathbf{\Delta H}_i = \mathbf{H}(t_i) - \mathbf{H}(t_0) = \begin{pmatrix} \Delta H_i^{(1)} \\ \Delta H_i^{(2)} \end{pmatrix}, \; \zeta_{t_{ex}} = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta 22 \end{pmatrix}.$$

2.1.2 Approximate formula

Same as the one-factor case, the pricing formula of swaption can be written as

$$V_{rec}^{opt}(0) = L_0 E^{Q_L}[(K^{adj} - S(t_{ex}))^+],$$

where

$$L(t) = \sum_{i=1}^{n}\tau_i P(t,t_i)$$

$(t \leq t_0,\ i = 1, \cdots, n)$ is the annuity numeraire, $S(t) = \frac{P(t,t_0)-P(t,t_n)}{\sum_{i=1}^n \tau_i P(t,t_i)}$ is the forward swap rate, and Q_L is the martingale measure associated with annuity numeraire. By the equivalent vol technique, we have (see Hagan [2], formula(3.66a))

$$\boxed{\sigma_N^2 t_{ex} = \mathbf{H}_{tot} \cdot \zeta \mathbf{H}_{tot}} \tag{13}$$

where

$$\mathbf{H}_{tot} = \frac{S(0)\sum_{i=1}^n \tau_i D_i \Delta\mathbf{H}_i + D_n \Delta\mathbf{H}_n}{\sum_{i=1}^n \tau_i D_i}$$

and $\Delta\mathbf{H}_i = \mathbf{H}_i - \mathbf{H}_0$. This allows us to obtain the swaption price by Black's formula with normal vol $\sigma_N = \sqrt{\frac{\mathbf{H}_{tot}\cdot\zeta_{t_{ex}}\mathbf{H}_{tot}}{t_{ex}}}$:

$$\boxed{V_{rec}^{opt}(0) = L_0\left[(K^{adj} - S(0))\Phi(-d_1) + \frac{\sigma_N\sqrt{t_{ex}}}{\sqrt{2\pi}} e^{-d_1^2/2}\right]} \tag{14}$$

where $\Phi(\cdot)$ is the c.d.f. of standard normal distribution and $d_1 = \frac{S(0)-K^{adj}}{\sigma_N\sqrt{t_{ex}}}$.

2.2 Calibration to ATM swaption market

For a given sequence of expiries $t_{ex}^0 < t_{ex}^1 < \cdots < t_{ex}^N$, we shall use the approximate formula (13) to obtain $\zeta(t_{ex}^i)$, $i = 1, \cdots, N$. In the case of one-factor model, formula (13) alone is able to produce $\zeta_{t_{ex}}$; in the case of two-factor model, formula (13) is one equation for two unknowns (assuming α or σ is piecewise constant). So we typically need a sequence of swaptions to deduce $(\zeta_{t_{ex}^i})_{i=1}^N$ by bootstrapping.

More precisely, suppose the tenors are $T_1, T_2, \cdots, T_M$. Formula (13) gives us

$$(\sigma_N^{ij})^2 t_{ex}^i - \mathbf{H}_{tot}^{ij} \cdot \zeta_{t_{ex}^{i-1}} \mathbf{H}_{tot}^{ij} = \mathbf{H}_{tot}^{ij} \cdot (\zeta_{t_{ex}^i} - \zeta_{t_{ex}^{i-1}})\mathbf{H}_{tot}^{ij},$$

where $\mathbf{H}_{tot}^{ij}$ is the quantity similar to the one in formula (13) and corresponds to the $t_{ex}^i \times T^j$ swaption, and σ_N^{ij} is the normal vol for the $t_{ex}^i \times T^j$ swaption. Then $\zeta_{t_{ex}^i}$ is defined via α or σ (recall $\zeta_t = \int_0^t \alpha_s^2 ds = \int_0^t e^{2\kappa s}\sigma_s^2 ds$) such that

$$\sum_{j=1}^M \left\{\mathbf{H}_{tot}^{ij} \cdot (\zeta_{t_{ex}^i} - \zeta_{t_{ex}^{i-1}})\mathbf{H}_{tot}^{ij} - \left[(\sigma_N^{ij})^2 t_{ex}^i - \mathbf{H}_{tot}^{ij}\cdot\zeta_{t_{ex}^{i-1}}\mathbf{H}_{tot}^{ij}\right]\right\}^2 \omega_{ij}^2$$

is minimized (assuming $\zeta_{t_{ex}^{i-1}}$ is already determined). Here $(\omega_{ij})_{0\leq i\leq N, 1\leq j\leq M}$ is a weight matrix. To represent ζ_t in terms of α or σ, we note for $s < t$,

$$\zeta_t - \zeta_s = \begin{cases} \begin{pmatrix} \alpha_1^2(t-s) & \rho\alpha_1\alpha_2(t-s) \\ \rho\alpha_1\alpha_2(t-s) & \alpha_2^2(t-s) \end{pmatrix} & \text{if } \alpha \text{ is constant in } [s,t] \\ \begin{pmatrix} \sigma_1^2\frac{e^{-2\kappa_1 s}-e^{-2\kappa_1 t}}{2\kappa_1} & \rho\sigma_1\sigma_2\frac{e^{-(\kappa_1+\kappa_2)s}-e^{-(\kappa_1+\kappa_2)t}}{\kappa_1+\kappa_2} \\ \rho\sigma_1\sigma_2\frac{e^{-(\kappa_1+\kappa_2)s}-e^{-(\kappa_1+\kappa_2)t}}{\kappa_1+\kappa_2} & \sigma_2^2\frac{e^{-2\kappa_2 s}-e^{-2\kappa_2 t}}{2\kappa_2} \end{pmatrix} & \text{if } \sigma \text{ is constant in } [s,t] \end{cases}$$

Then (recall $\mathbf{H}_{tot} = (H_{tot}^1, H_{tot}^2)$)

$$\begin{aligned} & \mathbf{H}_{tot}\cdot(\zeta_t - \zeta_s)\mathbf{H}_{tot} \\ = \ & (H_{tot}^1)^2(\zeta_t-\zeta_s)_{11} + 2H_{tot}^1H_{tot}^2(\zeta_t-\zeta_s)_{12} + (H_{tot}^2)^2(\zeta_t-\zeta_s)_{22} \\ = \ & \begin{cases} \left[(H_{tot}^1)^2\alpha_1^2 + 2H_{tot}^1H_{tot}^2\rho\alpha_1\alpha_2 + (H_{tot}^2)^2\alpha_2^2\right](t-s) & \text{if } \alpha \text{ is constant in } [s,t] \\ (H_{tot}^1)^2\frac{e^{-2\kappa_1 s}-e^{-2\kappa_1 t}}{2\kappa_1}\sigma_1^2 + 2\rho\Delta H_{tot}^1H_{tot}^2\frac{e^{-(\kappa_1+\kappa_2)s}-e^{-(\kappa_1+\kappa_2)t}}{\kappa_1+\kappa_2}\sigma_1\sigma_2 & \\ +(H_{tot}^2)^2\frac{e^{-2\kappa_2 s}-e^{-2\kappa_2 t}}{2\kappa_2}\sigma_2^2 & \text{if } \sigma \text{ is constant in } [s,t] \end{cases} \\ = \ & \begin{cases} \alpha\cdot\mathbf{A}\alpha & \text{if } \alpha \text{ is constant in } [s,t] \\ \sigma\cdot\mathbf{A}\sigma & \text{if } \sigma \text{ is constant in } [s,t] \end{cases} \end{aligned}$$

where $\alpha = (\alpha^1, \alpha^2)$, $\sigma = (\sigma_1, \sigma_2)$, and

$$\mathbf{A} = \begin{cases} \begin{pmatrix} (H^1_{tot})^2 & \rho H^1_{tot} H^2_{tot} \\ \rho H^1_{tot} H^2_{tot} & (H^2_{tot})^2 \end{pmatrix} (t-s) & \text{if } \alpha \text{ is constant in } [s,t] \\ \begin{pmatrix} (H^1_{tot})^2 \frac{e^{-2\kappa_1 s} - e^{-2\kappa_1 t}}{2\kappa_1} & \rho H^1_{tot} H^2_{tot} \frac{e^{-(\kappa_1+\kappa_2)s} - e^{-(\kappa_1+\kappa_2)t}}{\kappa_1+\kappa_2} \\ \rho H^1_{tot} H^2_{tot} \frac{e^{-(\kappa_1+\kappa_2)s} - e^{-(\kappa_1+\kappa_2)t}}{\kappa_1+\kappa_2} & (H^2_{tot})^2 \frac{e^{-2\kappa_2 s} - e^{-2\kappa_2 t}}{2\kappa_2} \end{pmatrix} & \text{if } \sigma \text{ is constant in } [s,t] \end{cases}$$

Therefore, the calibration problem is reformulated to the following optimization problem:

$$\boxed{\begin{cases} \arg_{\alpha_i} \min \sum_{j=1}^M \left\{ \alpha_i \cdot \mathbf{A}^{ij} \alpha_i - \left[(\sigma_N^{ij})^2 t^i_{ex} - \mathbf{H}^{ij}_{tot} \cdot \zeta_{t^{i-1}_{ex}} \mathbf{H}^{ij}_{tot} \right] \right\}^2 \omega_{ij}^2, \ (i = 1, \cdots, N) & \text{if } \alpha \text{ is piecewise constant} \\ \arg_{\sigma_i} \min \sum_{j=1}^M \left\{ \sigma_i \cdot \mathbf{A}^{ij} \sigma_i - \left[(\sigma_N^{ij})^2 t^i_{ex} - \mathbf{H}^{ij}_{tot} \cdot \zeta_{t^{i-1}_{ex}} \mathbf{H}^{ij}_{tot} \right] \right\}^2 \omega_{ij}^2, \ (i = 1, \cdots, N) & \text{if } \sigma \text{ is piecewise constant} \end{cases}} \quad (15)$$

where for the i-th optimization problem to be solved, the first $(i-1)$ optimization problems must be already solved.

2.3 Calibration to CMS spread option

The calibration of two-factor LGM model to CMS spread option is based on a sequence of swaptions and a sequence of CMS spread options.

More precisely, for a given sequence of expiries $t^0_{ex} < t^1_{ex} < \cdots < t^N_{ex}$ and corresponding sequence of swap maturities $T_1, T_2, \cdots, T_N$, formula (13) gives

$$(\sigma^i_N)^2 t^i_{ex} - \mathbf{H}^i_{tot} \cdot \zeta_{t^{i-1}_{ex}} \mathbf{H}^i_{tot} = \mathbf{H}^i_{tot} \cdot (\zeta_{t^i_{ex}} - \zeta_{t^{i-1}_{ex}}) \mathbf{H}^i_{tot}, \ i = 1, \cdots, N,$$

where σ^i_N is the normal vol for the $t^i_{ex} \times T_i$ swaption and $\mathbf{H}^i_{tot}$ is the quantity similar to the one in formula (13) corresponding to the $t^i_{ex} \times T^i$ swaption.

Assuming $\zeta_{t^1_{ex}}, \cdots, \zeta_{t^{i-1}_{ex}}$ have been given, to find out $\zeta_{t^i_{ex}}$, we repeat what's done in ATM calibration and write $\mathbf{H}^i_{tot} \cdot (\zeta_{t^{i-1}_{ex}} - \zeta_{t^i_{ex}}) \mathbf{H}^i_{tot}$ as

$$\mathbf{H}^i_{tot} \cdot (\zeta_{t^i_{ex}} - \zeta_{t^{i-1}_{ex}}) \mathbf{H}^i_{tot} = \begin{cases} \alpha \cdot \mathbf{A}_i \alpha & \text{if } \alpha \text{ is constant in } [t^{i-1}_{ex}, t^i_{ex}] \\ \sigma \cdot \mathbf{A}_i \sigma & \text{if } \sigma \text{ is constant in } [t^{i-1}_{ex}, t^i_{ex}] \end{cases}$$

We further parameterize $\mathbf{A}_i$ by finding r_1, r_2, ϕ_0 such that

$$\frac{\mathbf{A}_i}{(\sigma^i_N)^2 t^i_{ex} - \mathbf{H}^i_{tot} \cdot \zeta_{t^{i-1}_{ex}} \mathbf{H}^i_{tot}} = \begin{pmatrix} \cos\phi_0 & -\sin\phi_0 \\ \sin\phi_0 & \cos\phi_0 \end{pmatrix} \begin{pmatrix} \frac{1}{r_1^2} & 0 \\ 0 & \frac{1}{r_2^2} \end{pmatrix} \begin{pmatrix} \cos\phi_0 & \sin\phi_0 \\ -\sin\phi_0 & \cos\phi_0 \end{pmatrix}$$

Then the first equation for $\zeta_{t^i_{ex}}$ is

$$1 = \begin{cases} \alpha \cdot \begin{pmatrix} \cos\phi_0 & -\sin\phi_0 \\ \sin\phi_0 & \cos\phi_0 \end{pmatrix} \begin{pmatrix} \frac{1}{r_1^2} & 0 \\ 0 & \frac{1}{r_2^2} \end{pmatrix} \begin{pmatrix} \cos\phi_0 & \sin\phi_0 \\ -\sin\phi_0 & \cos\phi_0 \end{pmatrix} \alpha & \text{if } \alpha \text{ is constant in } [t^{i-1}_{ex}, t^i_{ex}] \\ \sigma \cdot \begin{pmatrix} \cos\phi_0 & -\sin\phi_0 \\ \sin\phi_0 & \cos\phi_0 \end{pmatrix} \begin{pmatrix} \frac{1}{r_1^2} & 0 \\ 0 & \frac{1}{r_2^2} \end{pmatrix} \begin{pmatrix} \cos\phi_0 & \sin\phi_0 \\ -\sin\phi_0 & \cos\phi_0 \end{pmatrix} \sigma & \text{if } \sigma \text{ is constant in } [t^{i-1}_{ex}, t^i_{ex}] \end{cases}$$

For the second equation, consider an ATM CMS spread option with payoff

$$[S_1(t^i_{ex}) - S_2(t^i_{ex}) - K]^+$$

at time t^i_{ex}, where S_1 and S_2 are two swap rates, and $K = E^{Q_{t^i_{ex}}}[S_1(t^i_{ex})] - E^{Q_{t^i_{ex}}}[S_2(t^i_{ex})]$. The normal vol of $S_1(t^i_{ex}) - S_2(t^i_{ex})$ can be quoted directly from market, while Hagan's formula gives the normal spread vol as

$$(\sigma_N^{sprd})^2 t^i_{ex} = (\mathbf{H}^1_{tot} - \mathbf{H}^2_{tot}) \cdot \zeta_{t^i_{ex}} (\mathbf{H}^1_{tot} - \mathbf{H}^2_{tot}),$$

where $\mathbf{H}^1_{tot}$ corresponds to S_1 and $\mathbf{H}^2_{tot}$ corresponds to S_2. Matching σ_N^{sprd} with market quote will give us the second equation for $\zeta_{t^i_{ex}}$.

These two equations combined allow us to solve for $\zeta_{t^i_{ex}}$. More precisely, we use $x = (x_1, x_2)^T$ to stand for either α or σ. On the interval $[t^{i-1}_{ex}, t^i_{ex}]$ we have two equations for two unknowns:

$$\begin{cases} 1 = (x_1, x_2) \cdot \begin{pmatrix} \cos\phi_0 & -\sin\phi_0 \\ \sin\phi_0 & \cos\phi_0 \end{pmatrix} \begin{pmatrix} \frac{1}{r_1^2} & 0 \\ 0 & \frac{1}{r_2^2} \end{pmatrix} \begin{pmatrix} \cos\phi_0 & \sin\phi_0 \\ -\sin\phi_0 & \cos\phi_0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ (\sigma_N^{mkt})^2 t^i_{ex} = (\mathbf{H}^1_{tot} - \mathbf{H}^2_{tot}) \cdot \zeta_{t^i_{ex}} (\mathbf{H}^1_{tot} - \mathbf{H}^2_{tot}) \end{cases} \tag{16}$$

where σ_N^{mkt} is market quote of the normal spread vol and

$$\zeta_{t^i_{ex}} = \zeta_{t^{i-1}_{ex}} + \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \mathbf{B} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}. \tag{17}$$

where

$$\mathbf{B} = \begin{cases} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} (t^i_{ex} - t^{i-1}_{ex}) & \text{if } \alpha \text{ is constant in } [t^{i-1}_{ex}, t^i_{ex}] \\ \begin{pmatrix} \frac{e^{-2\kappa_1 t^{i-1}_{ex}} - e^{-2\kappa_1 t^i_{ex}}}{2\kappa_1} & \rho \frac{e^{-(\kappa_1+\kappa_2) t^{i-1}_{ex}} - e^{-(\kappa_1+\kappa_2) t^i_{ex}}}{\kappa_1+\kappa_2} \\ \rho \frac{e^{-(\kappa_1+\kappa_2) t^{i-1}_{ex}} - e^{-(\kappa_1+\kappa_2) t^i_{ex}}}{\kappa_1+\kappa_2} & \frac{e^{-2\kappa_2 t^{i-1}_{ex}} - e^{-2\kappa_2 t^i_{ex}}}{2\kappa_2} \end{pmatrix} & \text{if } \sigma \text{ is constant in } [t^{i-1}_{ex}, t^i_{ex}] \end{cases}$$

We set the re-parametrization

$$\begin{cases} x_1 = r\cos\phi \\ x_2 = r\sin\phi \end{cases}$$

such that the first equation of system (16) becomes $1 = \frac{r^2\cos^2(\phi-\phi_0)}{r_1^2} + \frac{r^2\sin^2(\phi-\phi_0)}{r_2^2}$, or equivalently,

$$r = \frac{r_1 r_2}{\sqrt{r_2^2\cos^2(\phi-\phi_0) + r_1^2\sin^2(\phi-\phi_0)}}. \tag{18}$$

This allows us to use the following trial-and-error procedure to solve system (16):
Step 1. try a testing value of ϕ;
Step 2. use formula (18) to obtain the value of r;
Step 3. obtain the values of x_1 and x_2 by those of r and ϕ;
Step 4. obtain $\zeta_{t^i_{ex}}$ by formula (17);
Step 5. use the RHS of the second equation of system (16) to obtain theoretical normal spread vol σ_N^{sprd};
Step 6. check the error $|\sigma_N^{sprd} - \sigma_N^{mkt}|$: if sufficently small, stop; otherwise, return to Step 1 with a different value for ϕ.

The above procedure has the advantage of reducing problem's dimensionality. The cost is a complicated nonlinear equation of the unknown ϕ, such that it becomes hard to analyze the effectiveness of global Newton's method:

$$\frac{(\sigma_N^{mkt})^2 t^i_{ex} - (\mathbf{H}^1_{tot} - \mathbf{H}^2_{tot}) \cdot \zeta_{t^{i-1}_{ex}} (\mathbf{H}^1_{tot} - \mathbf{H}^2_{tot})}{r_1^2 r_2^2} = \frac{(\mathbf{H}^1_{tot} - \mathbf{H}^2_{tot}) \cdot \begin{pmatrix} \cos\phi & 0 \\ 0 & \sin\phi \end{pmatrix} \mathbf{B} \begin{pmatrix} \cos\phi & 0 \\ 0 & \sin\phi \end{pmatrix} (\mathbf{H}^1_{tot} - \mathbf{H}^2_{tot})}{r_2^2\cos^2(\phi-\phi_0) + r_1^2\sin^2(\phi-\phi_0)}$$

This equation can actually be simplified to give closed-form solution as follows. We define $\mathbf{h} = \mathbf{H}^1_{tot} - \mathbf{H}^2_{tot} = (h_1, h_2)^T$ and $\mathbf{Q} = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \mathbf{B} \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$. Then the equation becomes

$$\frac{(\sigma_N^{mkt})^2 t^i_{ex} - \mathbf{h} \cdot \zeta_{t^{i-1}_{ex}} \mathbf{h}}{r_1^2 r_2^2} = \frac{\begin{pmatrix} \cos\phi & \sin\phi \end{pmatrix} \mathbf{Q} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}}{r_2^2\cos^2(\phi-\phi_0) + r_1^2\sin^2(\phi-\phi_0)} \tag{19}$$

To solve for ϕ, we set $C = (\sigma_N^{mkt})^2 t_{ex}^i - \mathbf{h}\cdot\zeta_{t_{ex}^{i-1}}\mathbf{h}$, $\theta = \phi - \phi_0$, and

$$\begin{aligned}\hat{\mathbf{Q}} &= \begin{pmatrix}\cos\phi_0 & \sin\phi_0\\ -\sin\phi_0 & \cos\phi_0\end{pmatrix}\mathbf{Q}\begin{pmatrix}\cos\phi_0 & -\sin\phi_0\\ \sin\phi_0 & \cos\phi_0\end{pmatrix}\\ &= \begin{pmatrix} Q_{11}\cos^2\phi_0 + Q_{22}\sin^2\phi_0 + 2Q_{12}\sin\phi_0\cos\phi_0 & (Q_{22}-Q_{11})\sin\phi_0\cos\phi_0 + Q_{12}(\cos^2\phi_0 - \sin^2\phi_0)\\ (Q_{22}-Q_{11})\sin\phi_0\cos\phi_0 + Q_{12}(\cos^2\phi_0 - \sin^2\phi_0) & Q_{11}\sin^2\phi_0 + Q_{22}\cos^2\phi_0 - 2Q_{12}\sin\phi_0\cos\phi_0\end{pmatrix}\end{aligned}$$

Then the equation (19) becomes

$$\sin(2\theta+\gamma) = \frac{C(r_1^2+r_2^2) - r_1^2 r_2^2(\hat{Q}_{11}+\hat{Q}_{22})}{\sqrt{[r_1^2 r_2^2(\hat{Q}_{11}-\hat{Q}_{22}) - C(r_2^2 - r_1^2)]^2 + 4r_1^4 r_2^4 \hat{Q}_{12}^2}} \tag{20}$$

where γ is determined by

$$\begin{cases}\sin\gamma = \frac{r_1^2 r_2^2(\hat{Q}_{11}-\hat{Q}_{22}) - C(r_2^2-r_1^2)}{\sqrt{[r_1^2 r_2^2(\hat{Q}_{11}-\hat{Q}_{22}) - C(r_2^2-r_1^2)]^2 + 4r_1^4 r_2^4\hat{Q}_{12}^2}}\\ \cos\gamma = \frac{2r_1^2 r_2^2 \hat{Q}_{12}}{\sqrt{[r_1^2 r_2^2(\hat{Q}_{11}-\hat{Q}_{22}) - C(r_2^2-r_1^2)]^2 + 4r_1^4 r_2^4\hat{Q}_{12}^2}}\end{cases} \tag{21}$$

A necessary and sufficient for equation (20) to have a solution is

$$\left|C(r_1^2+r_2^2) - r_1^2 r_2^2(\hat{Q}_{11}+\hat{Q}_{22})\right| \le \sqrt{[r_1^2 r_2^2(\hat{Q}_{11}-\hat{Q}_{22}) - C(r_2^2-r_1^2)]^2 + 4r_1^4 r_2^4\hat{Q}_{12}^2},$$

which is equivalent to an inequality for a quadratic polynomial of C. Tedious calculation shows the inequality can be reduced to $C_- \le C \le C_+$, where

$$C_\pm = \frac{r_1^2\hat{Q}_{11} + r_2^2\hat{Q}_{22}}{2} \pm \frac{\sqrt{r_1^4\hat{Q}_{11}^2 + r_2^4\hat{Q}_{22}^2 + 4r_1^2 r_2^2\hat{Q}_{12}^2}}{2}.$$

Therefore, the necessary and sufficient condition for a successful calibration is

$$-\frac{\sqrt{r_1^4\hat{Q}_{11}^2 + r_2^4\hat{Q}_{22}^2 + 4r_1^2 r_2^2\hat{Q}_{12}^2}}{2} \le (\sigma_N^{mkt})^2 t_{ex}^i - \left[\mathbf{h}\cdot\zeta_{t_{ex}^{i-1}}\mathbf{h} + \frac{r_1^2\hat{Q}_{11}+r_2^2\hat{Q}_{22}}{2}\right] \le \frac{\sqrt{r_1^4\hat{Q}_{11}^2 + r_2^4\hat{Q}_{22}^2 + 4r_1^2 r_2^2\hat{Q}_{12}^2}}{2}.$$

Simultaneously, we have discovered the follow procedure to solve system (16) explicitly:
Step 1: obtain the value of $C = (\sigma_N^{mkt})^2 t_{ex}^i - \mathbf{h}\cdot\zeta_{t_{ex}^{i-1}}\mathbf{h}$ by inserting the value of σ_N^{mkt}, and use it to compute the value of γ by equation (21);
Step 2: invert equation (20) to obtain the value of θ;
Step 3: use the value of θ to obtain the value of $\phi = \theta + \phi_0$ and compute the value of r by formula (18);
Step 4: obtain the values of x_1 and x_2: $x_1 = r\cos\phi$, $x_2 = r\sin\phi$;
Step 5: obtain $\zeta_{t_{ex}^i}$ by formula (17).

Remark 1. *From Step 2, we shall have two representative solutions of θ, with the relation*

$$\theta^{(1)} + \theta^{(2)} \mod \pi = \frac{\pi}{2}.$$

This leads to two representative solution pairs of (r,ϕ) with the relation

$$\begin{cases}\phi^{(1)} + \phi^{(2)} \mod \pi = \frac{\pi}{2}\\ r^{(1)} = \frac{r_1 r_2}{\sqrt{r_2^2\cos^2\theta^{(1)} + r_1^2\sin^2\theta^{(1)}}} = \frac{r_1 r_2}{\sqrt{r_2^2\sin^2\theta^{(2)} + r_1^2\cos^2\theta^{(2)}}}\\ r^{(2)} = \frac{r_1 r_2}{\sqrt{r_2^2\cos^2\theta^{(2)} + r_1^2\sin^2\theta^{(2)}}} = \frac{r_1 r_2}{\sqrt{r_2^2\sin^2\theta^{(1)} + r_1^2\cos^2\theta^{(1)}}}\end{cases}$$

A Summary of Girsanov's Theorem for continuous semimartingale

For sake of convenience, we record here a version of Girsanov's Theorem as presented in Revuz and Yor [?]. We will freely use jargons in the theory of continuous semimartingales.

Suppose $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$ is a filtered probability space that satisfies the usual hypotheses. Q is another probability measure such that $Q|_{\mathcal{F}_t}$ is absolutely continuous with respect to $P|_{\mathcal{F}_t}$. We call D_t the Radon-Nikodym derivative of Q with respect to P on $\mathcal{F}_t$. These random variables $(D_t)_{t\geq 0}$ form a $(\mathcal{F}_t, P)$-martingale and can be chosen in such a way that it has cadlag path a.s..

Theorem A.1 (Girsanov's Theorem)**.** *If D is continuous, every continuous $(\mathcal{F}_t, P)$-semimartingale is a continuous $(\mathcal{F}_t, Q)$-semimartingale. More precisely, if M is a continuous $(\mathcal{F}_t, P)$-local martingale, then*

$$\widetilde{M} = M - D^{-1}.\langle M, D\rangle$$

is a continuous $(\mathcal{F}_t, Q)$-local martingale. Moreover, if N is another continuous P-local martingale,

$$\langle \widetilde{M}, \widetilde{N}\rangle = \langle \widetilde{M}, N\rangle = \langle M, N\rangle.$$

To apply Girsanov's Theorem more conveniently, we often use the following results.

Proposition A.1. *If D is a strictly positive continuous local martingale, there exists a unique continuous local martingale L such that*

$$D_t = \exp\left\{L_t - \frac{1}{2}\langle L, L\rangle_t\right\} = \mathcal{E}(L)_t;$$

L is given by the formula

$$L_t = \log D_0 + \int_0^t D_s^{-1} dD_s.$$

Theorem A.2. *If $Q = \mathcal{E}(L)\cdot P$ and M is a continuous P-local martingale, then*

$$\widetilde{M} = M - D^{-1}\cdot\langle M, D\rangle = M - \langle M, L\rangle$$

is a continuous Q-local martingale. Moreover, $P = \mathcal{E}(-L)^{-1}\cdot Q = \mathcal{E}(-\widetilde{L})\cdot Q$.

B Convexity adjustment of Libor rate

We denote by $F(t; t_s, t_e)$ the forward Libor rate over the period $[t_s, t_e]$

$$F(t; t_s, t_e) = \frac{1}{\tau}\left(\frac{P(t, t_s)}{P(t, t_e)} - 1\right)$$

where τ is the year fraction of the period. Denote by t_f the rate's fixing time, which is typically two business days before t_s. For given payment time t_p ($\geq t_s$), we are concerned with computing

$$E_t^{t_p}[F(t_f; t_s, t_e)] = E^{t_p}[F(t_f; t_s, t_e)|\mathcal{F}_t],$$

where $E^{t_p}[\cdot|\mathcal{F}_t]$ is the conditional expectation under the t_p-forward measure.

We focus on computing $E_t^{t_p}\left[\frac{P(t_f,t_s)}{P(t_f,t_e)}\right]$. An $\mathcal{F}_{t_f}$-measurable payment of ξ at time t_p is equivalent to a payment of $P(t_f, t_p)\cdot\xi$ at time t_f; we also would like to price under the martingale measure Q_N associated with the LGM numeraire N. By uniqueness of the arbitrage-free price, we have for any $t \leq t_f$

$$P(t, t_p)E_t^{t_p}\left[\frac{P(t_f, t_s)}{P(t_f, T_e)}\right] = P(t, t_f)E_t^{t_f}\left[P(t_f, t_p)\cdot\frac{P(t_f, t_s)}{P(t_f, T_e)}\right] = N_t E_t^{Q_N}\left[\frac{P(t_f, t_p)}{N_{t_f}}\cdot\frac{P(t_f, t_s)}{P(t_f, T_e)}\right]$$

which gives

$$E_t^{t_p}\left[\frac{P(t_f,t_s)}{P(t_f,t_e)}\right] = E_t^{Q_N}\left[\frac{\bar{P}(t_f,t_p)}{\bar{P}(t,t_p)}\cdot\frac{P(t_f,t_s)}{P(t_f,t_e)}\right]$$

Under Q_N, the state variable X_t is Gaussian with zero mean and variance ζ_t.

By formula (2), we have (note $X_{t_f} - X_t \perp \mathcal{F}_t$ and $X_{t_f} - X_t \sim \mathcal{N}(0, \zeta_{t_f} - \zeta_t)$)

$$\begin{aligned}
& E_t^{Q_N}\left[\frac{\bar{P}(t_f,t_p)}{\bar{P}(t,t_p)}\cdot\frac{P(t_f,t_s)}{P(t_f,t_e)}\right] \\
= \; & E_t^{Q_N}\left[\frac{\bar{P}(t_f,t_p)}{\bar{P}(t,t_p)}\cdot\frac{\bar{P}(t_f,t_s)/\bar{P}(t,t_s)}{\bar{P}(t_f,t_e)/\bar{P}(t,t_e)}\right]\cdot\frac{P(t,t_s)}{P(t,t_e)} \\
= \; & \frac{P(t,t_s)}{P(t,t_e)}\cdot E_t^{Q_N}\left[\exp\left\{-H(t_p)(X_{t_f}-X_t) - \frac{1}{2}H^2(t_p)(\zeta_{t_f}-\zeta_t)\right\}\cdot\right. \\
& \left.\exp\left\{-[H(t_s)-H(t_e)](X_{t_f}-X_t) - \frac{1}{2}[H^2(t_s)-H^2(t_e)](\zeta_{t_f}-\zeta_t)\right\}\right] \\
= \; & \frac{P(t,t_s)}{P(t,t_e)}\cdot\exp\left\{\frac{1}{2}[H(t_p)+H(t_s)-H(t_e)]^2(\zeta_{t_f}-\zeta_t) - \frac{1}{2}[H^2(t_p)+H^2(t_s)-H^2(t_e)](\zeta_{t_f}-\zeta_t)\right\} \\
= \; & \frac{P(t,t_s)}{P(t,t_e)}\exp\left\{[H(t_e)-H(t_p)][H(t_e)-H(t_s)](\zeta_{t_f}-\zeta_t)\right\}
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
E_t^{t_p}[F(t_f;t_s,t_e)] &= \frac{1}{\tau}\left(\frac{P(t,t_s)}{P(t,t_e)}\exp\left\{[H(t_e)-H(t_p)][H(t_e)-H(t_s)](\zeta_{t_f}-\zeta_t)\right\} - 1\right) \\
&= F(t;t_s,t_e) + \frac{1}{\tau}\frac{P(t,t_s)}{P(t,t_e)}\left[\exp\left\{[H(t_e)-H(t_p)][H(t_e)-H(t_s)](\zeta_{t_f}-\zeta_t)\right\} - 1\right]
\end{aligned}$$

and

$$\text{conv. adj.} = \frac{1}{\tau}\frac{P(t,t_s)}{P(t,t_e)}\left[\exp\left\{[H(t_e)-H(t_p)][H(t_e)-H(t_s)](\zeta_{t_f}-\zeta_t)\right\} - 1\right]$$

C Convexity adjustment of CMS rate

Recall the forward swap rate S is defined as

$$S(t) = \frac{P(t,t_0) - P(t,t_n)}{\sum_{i=1}^n \tau_i P(t,t_i)},\ t \le t_0$$

and the annuity numeraire (also called *level numeraire*) L is defined as

$$L(t) = \sum_{i=1}^n \tau_i P(t,t_i),\ t \le t_0.$$

Under the *level measure* Q_L associated with the level numeraire, we have

$$E_t^{t_e}[S(t_f)]$$

D Convexity adjustment of caplet

We denote by $V_{pay}(t)$ and $\bar{V}_{pay}(t)$ the values of a payer's swaption and a numeraire-discounted payer's swaption at time t, respectively. To begin with, we note

$$\begin{aligned} E_t^{t_p}[(F(t_f;t_s,t_e)-K)^+] &= \frac{1}{\tau}E_t^{t_p}\left[\frac{(P(t_f,t_s)-P(t_f,t_e)-K\tau P(t_f,t_e))^+}{P(t_f,t_e)}\right] \\ &= \frac{1}{\tau}E_t^{Q_N}\left[\frac{P(t_f,t_p)/P(t,t_p)}{N_{t_f}/N_t}\cdot\frac{\max\{V_{pay}(t_f;X_{t_f}),0\}}{P(t_f,t_e)}\right] \\ &= \frac{1}{\tau}\frac{1}{\bar{P}(t,t_p)}E_t^{Q_N}\left[\frac{\bar{P}(t_f,t_p)}{\bar{P}(t_f,t_e)}\max\{\bar{V}_{pay}(t_f;X_{t_f}),0\}\right] \end{aligned}$$

Since

$$\begin{aligned} \bar{V}_{pay}(t_f;X_{t_f}) &= \bar{P}(t_f,t_s;X_{t_f})-\bar{P}(t_f,t_e;X_{t_f})-K\tau\bar{P}(t_f,t_e;X_{t_f}) \\ &= P(0,t_s)\exp\{-H(t_s)X_{t_f}-\frac{1}{2}H^2(t_s)\zeta_{t_f}\}-(1+K\tau)P(0,t_e)\exp\{-H(t_e)X_{t_f}-\frac{1}{2}H^2(t_e)\zeta_{t_f}\} \end{aligned}$$

we can find by an argument similar to that of formula (6) a unique $\hat{y}^*=\ln\left[\frac{(1+K\tau)P(0,t_e)}{P(0,t_s)}\right]-\frac{H(t_e)+H(t_s)}{2}\zeta_{t_f}$ such that

$$V_{pay}(t_f;y)\begin{cases} >0 & \text{if } y>\hat{y}^* \\ =0 & \text{if } y=\hat{y}^* \\ <0 & \text{if } y<\hat{y}^* \end{cases}$$

Note $\hat{y}^*=y^*-H(t_s)\zeta_{t_f}$ where y^* is from formula (6). The discrepancy is because that formula (6) applied change-of-variable while we didn't (in order to preserve the neat term $V_{pay}(t_f;X_{t_f})$).

Consequently, we have

$$\begin{aligned} & E_t^{t_p}[(F(t_f;t_s,t_e)-K)^+] \\ =\ & \frac{1}{\tau}\frac{1}{\bar{P}(t,t_p)}E_t^{Q_N}\left[\frac{P(0,t_p)}{P(0,t_e)}\exp\left\{-[H(t_p)-H(t_e)]X_{t_f}-\frac{1}{2}[H^2(t_p)-H^2(t_e)]\zeta_{t_f}\right\}\bar{V}_{pay}(t_f;X_{t_f})1_{\{X_{t_f}>\hat{y}^*\}}\right] \end{aligned}$$

To prepare for the next step computation, we compute $E[e^{-uX}1_{\{X>x_0\}}]$ for $X\sim\mathcal{N}(0,\sigma^2)$:

$$E[e^{-uX}1_{\{X>x_0\}}]=\int_{x_0}^{\infty}\frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}e^{-ux}dx=e^{\frac{1}{2}u^2\sigma^2}\cdot\int_{x_0}^{\infty}\frac{e^{-\frac{(x+u\sigma^2)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}dx=e^{\frac{1}{2}u^2\sigma^2}\Phi\left(-\frac{x_0+u\sigma^2}{\sigma}\right).$$

Then

$$E_t^{Q_N}\left[\exp\left\{-[H(t_p)-H(t_e)]X_{t_f}\right\}\bar{V}_{pay}(t_f;X_{t_f})1_{\{X_{t_f}>\hat{y}^*\}}\right]=\mathrm{I}-\mathrm{II}$$

where

$$\begin{cases} \mathrm{I}=E_t^{Q_N}\left[\exp\left\{-[H(t_p)-H(t_e)]X_{t_f}\right\}P(0,t_s)\exp\{-H(t_s)X_{t_f}-\frac{1}{2}H^2(t_s)\zeta_{t_f}\}1_{\{X_{t_f}>\hat{y}^*\}}\right] \\ \mathrm{II}=E_t^{Q_N}\left[\exp\left\{-[H(t_p)-H(t_e)]X_{t_f}\right\}(1+K\tau)P(0,t_e)\exp\{-H(t_e)X_{t_f}-\frac{1}{2}H^2(t_e)\zeta_{t_f}\}1_{\{X_{t_f}>\hat{y}^*\}}\right] \end{cases}$$

and we have

$$\begin{aligned} \mathrm{I} &= P(0,t_s)\exp\left\{-[H(t_p)-H(t_e)+H(t_s)]X_t-\frac{1}{2}H^2(t_s)\zeta_{t_f}\right\} \\ &\quad\cdot E_t^{Q_N}\left[\exp\left\{-[H(t_p)-H(t_e)+H(t_s)](X_{t_f}-X_t)\right\}1_{\{X_{t_f}-X_t>\hat{y}^*-X_t\}}\right] \\ &= P(0,t_s)\exp\left\{-[H(t_p)-H(t_e)+H(t_s)]X_t-\frac{1}{2}H^2(t_s)\zeta_{t_f}\right\} \\ &\quad\cdot\exp\left\{\frac{1}{2}[H(t_p)-H(t_e)+H(t_s)]^2(\zeta_{t_f}-\zeta_t)\right\}\Phi\left(-\frac{\hat{y}^*-X_t}{\sqrt{\zeta_{t_f}-\zeta_t}}-[H(t_p)-H(t_e)+H(t_s)]\sqrt{\zeta_{t_f}-\zeta_t}\right) \end{aligned}$$

and

$$
\begin{aligned}
\mathbb{I} &= (1+K\tau)P(0,t_e)\exp\left\{-H(t_p)X_t - \frac{1}{2}H^2(t_e)\zeta_{t_f}\right\} E_t^{Q_N}\left[\exp\left\{-H(t_p)(X_{t_f}-X_t)1_{\{X_{t_f}-X_t>\hat{y}^*-X_t\}}\right\}\right] \\
&= (1+K\tau)P(0,t_e)\exp\left\{-H(t_p)X_t - \frac{1}{2}H^2(t_e)\zeta_{t_f}\right\} \\
&\quad \cdot \exp\left\{\frac{1}{2}H^2(t_p)(\zeta_{t_f}-\zeta_t)\right\}\Phi\left(-\frac{\hat{y}^*-X_t}{\sqrt{\zeta_{t_f}-\zeta_t}} - H(t_p)\sqrt{\zeta_{t_f}-\zeta_t}\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& E_t^{t_p}[(F(t_f;t_s,t_e)-K)^+] \\
= & \frac{1}{\tau}\frac{P(t,t_s)}{P(t,t_e)} e^{[H(t_e)-H(t_p)][H(t_e)-H(t_s)](\zeta_{t_f}-\zeta_t)}\Phi\left(-\frac{\hat{y}^*-X_t}{\sqrt{\zeta_{t_f}-\zeta_t}} - [H(t_p)-H(t_e)+H(t_s)]\sqrt{\zeta_{t_f}-\zeta_t}\right) \\
& -\frac{1+K\tau}{\tau}\Phi\left(-\frac{\hat{y}^*-X_t}{\sqrt{\zeta_{t_f}-\zeta_t}} - H(t_p)\sqrt{\zeta_{t_f}-\zeta_t}\right)
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
& E_t^{t_e}[(F(t_f;t_s,t_e)-K)^+] \\
= & \frac{1}{\tau}\frac{P(t,t_s)}{P(t,t_e)}\Phi\left(-\frac{\hat{y}^*-X_t}{\sqrt{\zeta_{t_f}-\zeta_t}} - H(t_s)\sqrt{\zeta_{t_f}-\zeta_t}\right) - \frac{1+K\tau}{\tau}\Phi\left(-\frac{\hat{y}^*-X_t}{\sqrt{\zeta_{t_f}-\zeta_t}} - H(t_e)\sqrt{\zeta_{t_f}-\zeta_t}\right)
\end{aligned}
$$

This gives us the convexity adjustment

$$
\begin{aligned}
& \text{conv. adj.} \\
= & E_t^{t_p}[(F(t_f;t_s,t_e)-K)^+] - E_t^{t_e}[(F(t_f;t_s,t_e)-K)^+] \\
= & \frac{P(t,t_s)}{\tau P(t,t_e)}\left\{e^{[H(t_e)-H(t_p)][H(t_e)-H(t_s)]\Delta\zeta}\Phi\left(-\lambda - [H(t_p)-H(t_e)+H(t_s)]\sqrt{\Delta\zeta}\right) - \Phi\left(-\lambda - H(t_s)\sqrt{\Delta\zeta}\right)\right\} \\
& -\frac{1+K\tau}{\tau}\left[\Phi\left(-\lambda - H(t_p)\sqrt{\Delta\zeta}\right) - \Phi\left(-\lambda - H(t_e)\sqrt{\Delta\zeta}\right)\right]
\end{aligned}
$$

where

$$
\begin{cases}
\Delta\zeta = \zeta_{t_f} - \zeta_t \\
\lambda = \frac{\hat{y}^*-X_t}{\sqrt{\zeta_{t_f}-\zeta_t}}
\end{cases}
$$

E Convexity adjustment of CMS caplet

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