Elements of Hull-White Model

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Abstract

In this note, we summarize the elements of Hull-White model.

Contents

1 One-factor Hull-White model

1.1 Dynamics of short rate r_t and state variable X_t under risk-neutral measure

Under the assumption of one-factor Hull-White model, the short rate process under the risk-neutral measure Q (the martingale measure associated with money market account numeraire) follows the dynamics

$$dr_t = (b_t - \kappa r_t)dt + \sigma_t dW_t,$$

where κ is a constant, b_t and σ_t are deterministic functions of t, and W is a standard Brownian motion.

Solving the SDE gives $r_t = e^{-\kappa t} r_0 + e^{-\kappa t}\int_0^t e^{\kappa s} b_s ds + e^{-\kappa t}\int_0^t e^{\kappa s}\sigma_s dW_s$. Setting $\theta_t = e^{-\kappa t} r_0 + e^{-\kappa t}\int_0^t e^{\kappa s} b_s ds$ and $X_t = e^{-\kappa t}\int_0^t e^{\kappa s}\sigma_s dW_s$. Then θ_t is a deterministic function of t and X_t is Gaussian process with mean 0 and variance $e^{-2\kappa t}\int_0^t e^{2\kappa s}\sigma_s^2 ds$. In summary, we have

$$\boxed{r_t = \theta_t + X_t,\ dX_t = -\kappa X_t dt + \sigma_t dW_t,\ X_0 = 0,\ E^Q[X_t] = 0,\ E^Q[X_t^2] = e^{-2\kappa t}\int_0^t e^{2\kappa s}\sigma_s^2 ds} \quad (1)$$

1.2 Pricing formula of zero coupon bond

1.2.1 Formula

Denote by $P(t,T)$ the time-t price of a zero-coupon bond with maturity T, we have (note $P(t,T)$ is a function of the state variable X_t)

$$\boxed{\begin{cases} P(t,T) = P(t,T;X_t) = \frac{P(0,T)}{P(0,t)}\exp\left\{-H(T-t)\left[X_t + \nu^h(t) + \frac{1}{2}\nu(t)H(T-t)\right]\right\} \\ P(0,t) = \exp\left\{-\int_0^t \theta_s ds + \nu_t^H\right\} \end{cases}} \quad (2)$$

where

$$\begin{cases} h(t) = e^{-\kappa t} \\ H(t) = \int_0^t h(s)ds \\ \nu(t) = e^{-2\kappa t}\int_0^t e^{2\kappa s}\sigma_s^2 ds \\ \nu^h(t) = h * v(t) = \int_0^t e^{-\kappa(t-s)}\nu(s)ds \\ \nu^H(t) = H * \nu(t) = \int_0^t H(t-s)\nu(s)ds. \end{cases}$$

We also note that $\frac{d}{dt}\nu^H(t) = \nu^h(t)$.

1.2.2 Derivation

For a derivation of formula (2), define $\alpha_t = \sigma_t e^{\kappa t}$. It's easy to see $X_t = h(t)\int_0^t \alpha_s dW_s$. For the convenience of later computation, we note for $u > t$,

$$X_u = e^{-\kappa u}\int_0^t \alpha_s dW_s + e^{-\kappa u}\int_t^u \alpha_s dW_s = h(u-t)X_t + h(u)\int_t^u \alpha_s dW_s.$$

Therefore, risk neutral pricing formula yields

$$P(t,T;X_t = x) = E^Q\left[e^{-\int_t^T r_u du}\middle|\mathcal{F}_t, X_t = x\right] = e^{-\int_t^T \theta_u du - H(T-t)x}E^Q\left[e^{-\int_t^T h(u)(\int_t^u \alpha_s dW_s)du}\right]$$

Define $\eta = \int_t^T h(u)(\int_t^u \alpha_s dW_s)du$. Then η is a Gaussian random variable with 0 mean and

$$\begin{aligned} \eta^2 &= \left[H(u)\int_t^u \alpha_s dW_s\Big|_{u=t}^{u=T} - \int_t^T H(u)\alpha_u dW_u\right]^2 \\ &= H^2(T)\left(\int_t^T \alpha_s dW_s\right)^2 - 2H(T)\int_t^T \alpha_s dW_s\int_t^T H(u)\alpha_u dW_u + \left(\int_t^T H(u)\alpha_u dW_u\right)^2. \end{aligned}$$

Therefore the variance of η is equal to

$$E^Q[\eta^2] = H^2(T)\int_t^T \alpha_s^2 ds - 2H(T)\int_t^T H(s)\alpha_s^2 ds + \int_t^T H^2(s)\alpha_s^2 ds$$

and

$$\begin{aligned} P(t,T;X_t=x) &= e^{-\int_t^T \theta_u du - H(T-t)x} E^Q[e^{-\eta}] = e^{-\int_t^T \theta_u du - H(T-t)x} \exp\left\{\frac{1}{2}Var(\eta)\right\} \\ &= \exp\left\{-\int_t^T \theta_u du - H(T-t)x + \frac{1}{2}\left[H^2(T)\int_t^T \alpha_s^2 ds + \int_t^T H^2(s)\alpha_s^2 ds\right] - H(T)\int_t^T H(s)\alpha_s^2 ds\right\}. \end{aligned}$$

As particular cases, we have

$$\begin{cases} P(0,T) = \exp\left\{-\int_0^T \theta_u du + \frac{1}{2}\left[H^2(T)\int_0^T \alpha_s^2 ds + \int_0^T H^2(s)\alpha_s^2 ds\right] - H(T)\int_0^T H(s)\alpha_s^2 ds\right\} \\ P(0,t) = \exp\left\{-\int_0^t \theta_u du + \frac{1}{2}\left[H^2(t)\int_0^t \alpha_s^2 ds + \int_0^t H^2(s)\alpha_s^2 ds\right] - H(t)\int_0^t H(s)\alpha_s^2 ds\right\}. \end{cases}$$

Therefore, we have

$$\begin{aligned} \frac{P(0,T)}{P(0,t)P(t,T;X_t=x)} &= \exp\left\{H(T-t)x + \frac{1}{2}\left[H^2(T)\int_0^t \alpha_s^2 ds + \int_0^t H^2(s)\alpha_s^2 ds\right] - H(T)\int_0^t H(s)\alpha_s^2 ds\right\} \\ &\quad \cdot \exp\left\{-\frac{1}{2}\left[H^2(t)\int_0^t \alpha_s^2 ds + \int_0^t H^2(s)\alpha_s^2 ds\right] + H(t)\int_0^t H(s)\alpha_s^2 ds\right\} \\ &= \exp\left\{H(T-t)x + \frac{1}{2}[H^2(T) - H^2(t)]\int_0^t \alpha_s^2 ds - [H(T)-H(t)]\int_0^t H(s)\alpha_s^2 ds\right\}. \end{aligned}$$

That is,

$$P(t,T;X_t=x) = \frac{P(0,T)}{P(0,t)}\exp\left\{-H(T-t)x - \frac{1}{2}[H^2(T)-H^2(t)]\int_0^t \alpha_s^2 ds + [H(T)-H(t)]\int_0^t H(s)\alpha_s^2 ds\right\}$$

Note $[H(T)-H(t)]e^{\kappa t} = \frac{e^{-\kappa t}-e^{-\kappa T}}{\kappa}e^{\kappa t} = H(T-t)$, we get

$$\begin{aligned} & P(t,T;X_t=x) \\ = & \frac{P(0,T)}{P(0,t)}\exp\left\{-H(T-t)\left[x + \frac{H(T)+H(t)}{2}e^{-\kappa t}\int_0^t \alpha_s^2 ds - e^{-\kappa t}\int_0^t H(s)\alpha_s^2 ds\right]\right\} \\ = & \frac{P(0,T)}{P(0,t)}\exp\left\{-H(T-t)\left[x + H(t)e^{-\kappa t}\int_0^t \alpha_s^2 ds + \frac{H(T)-H(t)}{2}e^{-\kappa t}\int_0^t \alpha_s^2 ds - e^{-\kappa t}\int_0^t H(s)\alpha_s^2 ds\right]\right\} \\ = & \frac{P(0,T)}{P(0,t)}\exp\left\{-H(T-t)\left[x + H(t)e^{-\kappa t}\int_0^t \alpha_s^2 ds + \frac{1}{2}H(T-t)\nu(t) - e^{-\kappa t}\int_0^t H(s)\alpha_s^2 ds\right]\right\}. \end{aligned}$$

Note

$$\begin{aligned} H(t)e^{-\kappa t}\int_0^t \alpha_s^2 ds - e^{-\kappa t}\int_0^t H(s)\alpha_s^2 ds &= h(t)\left[H(t)\int_0^t \alpha_s^2 ds - H(s)\int_0^s \alpha_u^2 du\Big|_0^t + \int_0^t h(s)\left(\int_0^s \alpha_u^2 du\right) ds\right] \\ &= h(t)\int_0^t e^{\kappa s}\nu(s) ds = \nu^h(t), \end{aligned}$$

We have obtained

$$P(t,T;X_t=x) = \frac{P(0,T)}{P(0,t)}\exp\left\{-H(T-t)\left[x + \nu^h(t) + \frac{1}{2}\nu(t)H(T-t)\right]\right\},$$

which gives formula (2).

1.3 Joint density of $(\int_0^t X_s ds, X_t)$ and value of $E^Q\left[e^{-\int_0^t X_s ds}\middle| X_t\right]$ under risk-neutral measure

To price a European contingent claim with payoff $f(X_T)$ at terminal time T, we typically need to evaluate

$$V_0 = E^Q\left[e^{-\int_0^T r_s ds} f(X_T)\right] = e^{-\int_0^T \theta_s ds} E^Q\left[e^{-\int_0^T X_s ds} f(X_T)\right] = e^{-\int_0^T \theta_s ds} E^Q\left[E^Q\left[e^{-\int_0^T X_s ds}\middle| X_T\right] f(X_T)\right].$$

This demands the knowledge of the joint density of $(\int_0^t X_s ds, X_t)$ or the value of $E^Q\left[e^{-\int_0^t X_s ds}\middle| X_t\right]$. We derive these two quantities in this section.

It's easy to see $Z_t := \int_0^t X_s ds$ and X_t are jointly Gaussian. In order to know their joint density, it's sufficient to know their respective mean and variance, as well as their covariance. In this regard, we note $E[X_t] = E[Z_t] = 0$. Define

$$v_X(t) = \sqrt{E[X_t^2]},\ v_Z(t) = \sqrt{E[Z_t^2]},\ \rho_{XZ}(t) = \frac{E[X_t Z_t]}{v_X(t) v_Z(t)},\ c_{XZ}(t) = E[X_t Z_t].$$

Then $v_X^2(t) = E[X_t^2] = e^{-2\kappa t}\int_0^t e^{2\kappa s}\sigma_s^2 ds = \nu(t)$, and by integration-by-parts formula, we have

$$X_t Z_t = \int_0^t Z_s dX_s + \int_0^t X_s^2 ds = -\kappa \int_0^t X_s Z_s ds + \int_0^t X_s^2 ds + \text{mart. part}.$$

Taking expectation on both sides gives

$$c_{XZ}(t) = -\kappa \int_0^t c_{XZ}(s) ds + \int_0^t \nu(s) ds.$$

Solving this integral equation gives

$$c_{XZ}(t) = e^{-\kappa t}\int_0^t e^{\kappa s}\nu(s) ds = \nu^h(t).$$

Finally, note $\frac{d}{dt}E\left[Z_t^2\right] = 2E[Z_t X_t] = 2c_{XZ}(t) = 2\nu^h(t)$, we have $v_Z^2(t) = E[Z_t^2] = 2\int_0^t \nu^h(s) ds = 2\nu^H(t)$. In summary, we have

$$\begin{cases} v_X^2(t) = E[X_t^2] = \nu(t) \\ v_Z^2(t) = E[Z_t^2] = 2\nu^H(t) \\ c_{XZ}(t) = E[X_t Z_t] = \nu^h(t) \\ \rho_{XZ}^2(t) = \frac{(E[X_t Z_t])^2}{v_X^2(t) v_Z^2(t)} = \frac{(\nu^h(t))^2}{2\nu(t)\nu^H(t)} \end{cases}$$

Therefore, the covariance matrix Σ_t of the pair (Z_t, X_t) is

$$\Sigma_t = \begin{pmatrix} E[Z_t^2] & E[X_t Z_t] \\ E[X_t Z_t] & E[X_t^2] \end{pmatrix} = \begin{pmatrix} 2\nu^H(t) & \nu^h(t) \\ \nu^h(t) & \nu(t) \end{pmatrix}$$

and its inverse is

$$\Sigma_t^{-1} = \frac{1}{1-\rho_{XZ}^2(t)} \begin{pmatrix} \frac{1}{v_Z^2(t)} & -\frac{\rho_{XZ}(t)}{v_X(t) v_Z(t)} \\ -\frac{\rho_{XZ}(t)}{v_X(t) v_Z(t)} & \frac{1}{v_X^2(t)} \end{pmatrix}$$

The joint density function of the pair (X_t, Z_t) is therefore

$$g(x,z;t) = \frac{|\Sigma_t|^{-\frac{1}{2}}}{2\pi} e^{-\frac{1}{2}(x,z)\Sigma_t^{-1}(x,z)'}$$
$$= \boxed{\frac{1}{2\pi v_X(t) v_Z(t)\sqrt{1-\rho_{XZ}^2(t)}} \exp\left\{-\frac{1}{2\left(1-\rho_{XZ}^2(t)\right)}\left[\frac{x^2}{v_X^2(t)} - 2\rho_{XZ}(t)\frac{xz}{v_X(t) v_Z(t)} + \frac{z^2}{v_Z^2(t)}\right]\right\}} \quad (3)$$

To compute the other quantity, note when conditioning on $X_t = x$,

$$Z_t \sim N\left(\frac{c_{XZ}(t)}{v_X^2(t)}x, v_Z^2(t) - \frac{c_{XZ}^2(t)}{v_X^2(t)}\right) = N\left(x\frac{\nu^h(t)}{\nu(t)}, 2\nu^H(t) - \frac{(\nu^h(t))^2}{\nu(t)}\right).$$

So

$$\boxed{E^Q\left[e^{-Z_t}\middle| X_t\right] = \exp\left\{-X_t\frac{\nu^h(t)}{\nu(t)} + \nu^H(t) - \frac{(\nu^h(t))^2}{2\nu(t)}\right\}} \tag{4}$$

1.4 Pricing formula of European contingent claim

To price a European contingent claim with payoff $f(X_T)$ at terminal time T, we note by formula (4)

$$\begin{aligned}
V_0 &= E^Q\left[e^{-\int_0^T r_s ds}f(X_T)\right] = e^{-\int_0^T \theta_s ds}E^Q\left[e^{-\int_0^T X_s ds}f(X_T)\right] = e^{-\int_0^T \theta_s ds}E^Q\left[E^Q\left[e^{-\int_0^T X_s ds}\middle| X_T\right]f(X_T)\right] \\
&= \exp\left\{-\int_0^T \theta_s ds + \nu^H(T) - \frac{(\nu^h(T))^2}{2\nu(T)}\right\}E^Q\left[f(X_T)\exp\left\{-X_T\frac{\nu^h(T)}{\nu(T)}\right\}\right] \\
&= \boxed{P(0,T)\exp\left\{-\frac{(\nu^h(T))^2}{2\nu(T)}\right\}E^Q\left[f(X_T)\exp\left\{-X_T\frac{\nu^h(T)}{\nu(T)}\right\}\right]} \qquad (5)
\end{aligned}$$

1.5 Dynamics of short rate r_t and state variable X_t under forward measure

Denote by Q^T the T-forward measure. The Radon-Nikodym derivative D of T-forward measure Q^T w.r.t. the risk-neutral measure Q is defined as

$$D_t := \frac{dQ^T|_{\mathcal{F}_t}}{dQ|_{\mathcal{F}_t}} = \frac{P(t,T)/P(0,T)}{e^{\int_0^t r_u du}}.$$

Therefore, by formula (2)

$$\begin{aligned}
d\ln D_t &= d\ln P(t,T) - d\int_0^t r_u du \\
&= d\left(\ln P(0,T) - \ln P(0,t) - H(T-t)\left[X_t + \nu^h(t) + \frac{1}{2}\nu(t)H(T-t)\right]\right) - r_t dt \\
&= (\cdots)dt - H(T-t)\sigma_t dW_t.
\end{aligned}$$

Since D_t is a martingale under risk-neutral measure Q, we conclude dD_t/D_t does not have drift term under Q. Therefore,

$$dD_t = -D_t H(T-t)\sigma_t dW_t$$

Define $L_t = -\int_0^t H(T-u)\sigma_u dW_u$. Then $D_t = \mathcal{E}(L_t) := \exp\left\{L_t - \frac{1}{2}\langle L\rangle_t\right\}$. By Girsanov's Theorem,

$$W_t^T := W_t - \langle W, L\rangle_t = W_t + \int_0^t H(T-u)\sigma_u du$$

is a Brownian motion under Q^T. Therefore, under the T-forward measure Q^T, the dynamics of state variable X becomes (see Brigo and Mercurio [2], Lemma 4.2)

$$X_t = e^{-\kappa(t-s)}X_s + \int_s^t e^{-\kappa(t-u)}\sigma_u dW_u^T - \int_s^t e^{-\kappa(t-u)}H(T-u)\sigma_u^2 du,\ X_0 = 0.$$

Note $\nu(t)' = -2\kappa\nu(t) + \sigma_t^2$ and $h(t) + \kappa H(t) = 1$, we have

$$\begin{aligned}
& \int_0^t e^{-\kappa(t-u)} H(T-u)\sigma_u^2 du = \int_0^t h(t-u)H(T-u)\Big[\nu(u)' + 2\kappa\nu(u)\Big]du \\
= \;& H(T-t)\nu(t) - \int_0^t \nu(u) d_u\Big[h(t-u)H(T-u)\Big] + 2\kappa\int_0^t h(t-u)H(T-u)\nu(u)du \\
= \;& H(T-t)\nu(t) - \int_0^t \nu(u)\Big[\kappa h(t-u)H(T-u) - h(t-u)h(T-u)\Big]du + 2\kappa\int_0^t h(t-u)H(T-u)\nu(u)du \\
= \;& H(T-t)\nu(t) + \int_0^t \nu(u)\Big[\kappa h(t-u)H(T-u) + h(t-u)h(T-u)\Big]du \\
= \;& H(T-t)\nu(t) + \int_0^t \nu(u)h(t-u)du \\
= \;& H(T-t)\nu(t) + \nu^h(t).
\end{aligned}$$

Therefore

$$X_t = h(t-s)X_s + \int_s^t h(t-u)\sigma_u dW_u^T - \left\{\Big[H(T-t)\nu(t) + \nu^h(t)\Big] - h(t-s)\Big[H(T-s)\nu(s) + \nu^h(s)\Big]\right\}.$$

Conditioning on $\mathcal{F}_s$, X_t $(t > s)$ is Gaussian with mean $h(t-s)X_s - \Big[H(T-t)\nu(t) + \nu^h(t)\Big] + h(t-s)\Big[H(T-s)\nu(s) + \nu^h(s)\Big]$ and variance $\int_s^t e^{-2\kappa(t-u)}\sigma_u^2 du = \nu(t) - h(t-s;2\kappa)\nu(s)$.

In summary, under the T-forward measure Q^T, the model dynamics is

$$\begin{cases}
r_t = \theta_t + X_t \\
X_t = h(t-s)X_s + \int_s^t h(t-u)\sigma_u dW_u^T - \left\{\Big[H(T-t)\nu(t) + \nu^h(t)\Big] - h(t-s)\Big[H(T-s)\nu(s) + \nu^h(s)\Big]\right\} \\
E^T[X_t|X_s] = h(t-s)X_s - \Big[H(T-t)\nu(t) + \nu^h(t)\Big] + h(t-s)\Big[H(T-s)\nu(s) + \nu^h(s)\Big] \\
Var^T[X_t|X_s] = \nu(t) - h(t-s;2\kappa)\nu(s).
\end{cases} \tag{6}$$

where W^T is a standard Brownian motion under forward measure Q^T, and X_t is Guassian conditioning on X_s $(0 \le s < t)$.

1.6 Pricing formula of caplet

1.6.1 Formula

Denote by $F(t;T_s,T_e)$ the time-t forward Libor rate with expiry T_s and maturity T_e $(t \le T_s < T_e)$:

$$F(t;,T_s,T_e) = \frac{1}{\tau}\frac{P(t,T_s) - P(t,T_e)}{P(t,T_e)}$$

where τ is the year fraction of $[T_s,T_e]$. Let $t_f \le T_s$ be the rate fixing time, then the time-t price of the caplet is

$$V_t = \begin{cases}
P(t,T_e)\tau \mathrm{Bl}\left(\frac{1+\tau K}{\tau}, \frac{1}{\tau}\frac{P(t,T_s)}{P(t,T_e)}, \sigma_B, 1\right) & \text{if } t < t_f \\
P(t,T_e)\tau(F(t_f;T_s,T_e) - K)^+ & \text{if } t \ge t_f
\end{cases} \tag{7}$$

where

$$\sigma_B = H(T_e - T_s)\sqrt{h(T_s - t_f;2\kappa)\nu(t_f) - h(T_s - t;2\kappa)\nu(t)}$$

and

$$\begin{aligned}\text{Bl}(K,F,v,w) &= Fw\Phi(wd_1(K,F,v)) - Kw\Phi(wd_2(K,F,v))\\ d_1(K,F,v) &= \frac{\ln(F/K)+v^2/2}{v}\\ d_2(K,F,v) &= \frac{\ln(F/K)-v^2/2}{v}.\end{aligned}$$

In particular,

$$\boxed{V_0 = P(0,T_e)\tau\text{Bl}\left(K+\frac{1}{\tau}, F(0;T_s,T_e)+\frac{1}{\tau}, \big[H(T_e-t_f)-H(T_s-t_f)\big]\sqrt{\nu(t_f)}, 1\right).}$$

1.6.2 Derivation

To see a derivation of formula (7), we define

$$S_t = S_t(T_s,T_e) := \frac{P(t,T_s)}{P(t,T_e)},\ t<T_s<T_e$$

so that we can take advantage of Black-Scholes formula. Indeed, we note under the T_e-forward measure Q^{T_e}, S is necessarily a martingale. Using pricing formula of zero coupon bond, we have

$$\begin{aligned}d\ln S_t &= d\left(-\big[H(T_s-t)-H(T_e-t)\big](X_t+\nu^h(t)) - \frac{1}{2}\nu(t)\big[H^2(T_s-t)-H^2(T_e-t)\big]\right)\\ &= (\cdots)dt - \big[H(T_s-t)-H(T_e-t)\big]dX_t\\ &= (\cdots)dt + H(T_e-T_s)h(T_s-t)\sigma_t dW_t^{T_e}\end{aligned}$$

Therefore, we must have

$$dS_t = S_t H(T_e-T_s)h(T_s-t)\sigma_t dW_t^T,\ S_0 = \frac{P(0,T_s)}{P(0,T_e)}$$

and the forward Libor rate can be written in the form of

$$F(t;T_s,T_e) = \frac{1}{\tau}(S_t(T_s,T_e)-1),$$

where τ is the year fraction of $[T_s,T_e]$.

Denote by t_f ($t_f<T_s<T_e$) the rate fixing time of the Libor for $[T_s,T_e]$. Then the price of the caplet at time t is (assuming unit notional)

$$V_t = \tau\cdot P(t,T_e)E_t^{T_e}\left[(F(t_f;T_s,T_e)-K)^+\right] = P(t,T_e)E_t^{T_e}\left[(S_{t_f}-(1+\tau K))^+\right].$$

Using Black-Scholes formula $\text{Bl}(K,F,v,w)$

$$\begin{aligned}\text{Bl}(K,F,v,w) &= Fw\Phi(wd_1(K,F,v)) - Kw\Phi(wd_2(K,F,v))\\ d_1(K,F,v) &= \frac{\ln(F/K)+v^2/2}{v}\\ d_2(K,F,v) &= \frac{\ln(F/K)-v^2/2}{v},\end{aligned}$$

we can write

$$V_t = \begin{cases} P(t,T_e)\text{Bl}\left(1+\tau K, \frac{P(t,T_s)}{P(t,T_e)}, \sigma_B, 1\right) & \text{if } t<t_f\\ P(t,T_e)\tau(F(t_f;T_s,T_e)-K)^+ & \text{if } t\ge t_f\end{cases}$$

where

$$\sigma_B = H(T_e - T_s)\sqrt{h(T_s - t_f; 2\kappa)\nu(t_f) - h(T_s - t; 2\kappa)\nu(t)}.$$

For sake of model calibration to caplets, we can write the above formula equivalently as

$$V_t = \begin{cases} P(t,T_e)\tau \mathrm{Bl}\left(\frac{1+\tau K}{\tau}, \frac{1}{\tau}\frac{P(t,T_s)}{P(t,T_e)}, \sigma_B, 1\right) & \text{if } t < t_f \\ P(t,T_e)\tau(F(t_f; T_s, T_e) - K)^+ & \text{if } t \geq t_f \end{cases}$$

1.7 Calibration to caps market

We assume there is a tenure structure $0 = T_0 < \cdots, T_n$ and the Libor for $[T_{i-1}, T_i]$ is set at a time $t_i \leq T_{i-1}$. We take as given the Black volatilities of caplets for various strikes. Suppose the i-th caplet has strike K_i and denote by τ_i the year fraction of $[T_{i-1}, T_i]$. The market price for the i-th caplet is

$$V_i^{mkt} = P(0,T_i)\tau_i \mathrm{Bl}(K_i, F(0; T_{i-1}, T_i), \sigma_i^{mkt}\sqrt{t_i}, 1)$$

where σ_i^{mkt} is market quoted Black volatility. Recall the model price of the i-th caplet is given by

$$V^{model} = P(0,T_i)\tau_i \mathrm{Bl}\left(K_i + \frac{1}{\tau_i}, F(0; T_{i-1}, T_i) + \frac{1}{\tau_i}, \left[H(T_i - t_i) - H(T_{i-1} - t_i)\right]\sqrt{\nu(t_i)}, 1\right).$$

If we assume the model parameter κ is exogenously given, we can solve the following equation for $\nu(t_i)$ $(i = 1, \cdots, n)$:

$$\boxed{\mathrm{Bl}(K_i, F(0; T_{i-1}, T_i), \sigma_i^{mkt}\sqrt{t_i}, 1) = \mathrm{Bl}\left(K_i + \frac{1}{\tau_i}, F(0; T_{i-1}, T_i) + \frac{1}{\tau_i}, \left[H(T_i - t_i) - H(T_{i-1} - t_i)\right]\sqrt{\nu(t_i)}, 1\right)}$$

where $H(T_i - t_i) - H(T_{i-1} - t_i)$ can be further written as $h(T_{i-1} - t_i)H(T_i - T_{i-1})$.

If we assume the model's volatility process $(\sigma_t)_{t\geq 0}$ is piecewise constant ($t_0 = 0$)

$$\sigma_t = \sum_{i=1}^n \sigma_{i-1} 1_{\{t_{i-1} \leq t < t_i\}} + \sigma_{n-1} 1_{\{t_n \leq t\}},$$

we have the following system of equations to solve for each σ_{i-1} $(i = 1, \cdots, n)$:

$$\boxed{\sigma_{i-1}^2 = \frac{e^{2\kappa t_i}\nu(t_i) - e^{2\kappa t_{i-1}}\nu(t_{i-1})}{\int_{t_{i-1}}^{t_i} e^{2\kappa u} du} = \frac{\nu(t_i) - e^{-2\kappa(t_i - t_{i-1})}\nu(t_{i-1})}{\int_0^{t_i - t_{i-1}} e^{-2\kappa u} du} = \frac{\nu(t_i) - h(2\kappa; t_i - t_{i-1})\nu(t_{i-1})}{H(2\kappa; t_i - t_{i-1})}}$$

Once σ_0, $\cdots$, σ_{n-1} have been determined, we can easily implement $\nu(t)$, $\nu^h(t)$ and $\nu^H(t)$ for general $t \geq 0$.

1.8 Pricing formulas of Asian swaps/caps/floors

To simplify notations, we focus on the pricing of a single payment for the coupon period $[T_s, T_e]$. We assume $t_1, \cdots, t_n$ are the rate fixing times for the Libor rates being averaged, and we assume $[a_i, b_i]$ $(i = 1, \cdots, n)$ is the corresponding expiry-maturity pairs. Denote by w_i the weight for the i-th Libor rate. Assuming unit notional, we need to value the time-t price of a payment made at time T $(t < T_e \leq T)$:

$$\begin{cases} \tau(T_s, T_e)\sum_{i=1}^n w_i F(t_i; a_i, b_i) & \text{for floater} \\ \tau(T_s, T_e)\left(\sum_{i=1}^n w_i F(t_i; a_i, b_i) - K\right)^+ & \text{for caplet} \\ \tau(T_s, T_e)\left(K - \sum_{i=1}^n w_i F(t_i; a_i, b_i)\right)^+ & \text{for flooret} \end{cases}$$

where $\tau(T_s, T_e)$ is the year fraction of $[T_s, T_e]$ and $F(t_i; a_i, b_i)$ is the forward Libor rate for coupon period $[a_i, b_i]$:

$$F(t_i; a_i, b_i) = \frac{1}{\tau_i}\frac{P(t_i, a_i) - P(t_i, b_i)}{P(t_i, b_i)}.$$

Here τ_i is the year fraction of $[a_i, b_i]$. Note in general, we have $t_i \leq a_i < T_e$, but we could have both $b_i \leq T_e$ and $b_i > T_e$.

1.8.1 Price of Asian floater

The time-t price of the floating leg for the coupon period $[T_s, T_e]$ is

$$\boxed{V_t = P(t,T)\tau(T_s,T_e)\left(\sum_{i=1}^{n} w_i F(t_i;a_i,b_i)1_{\{t_i\le t\}} + \sum_{i=1}^{n}\frac{w_i}{\tau_i}\left[A_i e^{-B_i X_t}-1\right]1_{\{t<t_i\}}\right)}$$

where

$$\begin{cases} A_i = \frac{P(0,a_i)}{P(0,b_i)}\exp\left\{-\frac{1}{2}\nu(t_i)\left[H^2(a_i-t_i)-H^2(b_i-t_i)\right]-\lambda_i\alpha_i+\frac{1}{2}\lambda_i^2\beta_i\right\} \\ B_i = \lambda_i h(t_i-t) \\ \lambda_i = H(a_i-t_i)-H(b_i-t_i) \\ \alpha_i = -\left[H(T-t_i)\nu(t_i)+\nu^h(t_i)\right]+h(t_i-t)\left[H(T-t)\nu(t)+\nu^h(t)\right] \\ \beta_i = \nu(t_i)-h(t_i-t;2\kappa)\nu(t) \end{cases}$$

If l_i is the leverage and s_i is the spread for the i-th Libor, the price formula is revised to

$$\boxed{V_t = P(t,T)\tau(T_s,T_e)\left(\sum_{i=1}^{n} w_i\big(l_iF(0;a_i,b_i)+s_i\big)1_{\{t_i\le t\}} + \sum_{i=1}^{n}\frac{w_il_i}{\tau_i}\left[A_ie^{-B_iX_t}-\left(1-\frac{s_i\tau_i}{l_i}\right)\right]1_{\{t<t_i\}}\right)} \quad (8)$$

In particular,

$$\boxed{V_0 = P(0,T)\tau(T_s,T_e)\sum_{i=1}^{n}\frac{w_il_i}{\tau_i}\left[A_i-\left(1-\frac{s_i\tau_i}{l_i}\right)\right]}$$

with

$$\begin{cases} A_i = \frac{P(0,a_i)}{P(0,b_i)}\exp\left\{-\frac{1}{2}\nu(t_i)[H^2(a_i-t_i)-H^2(b_i-t_i)]-\lambda_i\alpha_i+\frac{1}{2}\lambda_i^2\nu(t_i)\right\} \\ \lambda_i = H(a_i-t_i)-H(b_i-t_i) \\ \alpha_i = -[H(T-t_i)\nu(t_i)+\nu^h(t_i)] \end{cases}$$

To see a derivation of the above formulas, we note the time-t price of the floater is given by

$$V_t = P(t,T)\tau(T_s,T_e)\sum_{i=1}^{n} w_iE_t^T[F(t_i;a_i,b_i)] = P(t,T)\tau(T_s,T_e)\sum_{i=1}^{n}\frac{w_i}{\tau_i}\left(E_t^T\left[\frac{P(t_i,a_i)}{P(t_i,b_i)}\right]-1\right).$$

Recall

$$\frac{P(t_i,a_i)}{P(t_i,b_i)} = \frac{P(0,a_i)}{P(0,b_i)}\exp\left\{-\left[H(a_i-t_i)-H(b_i-t_i)\right]X_{t_i}-\frac{1}{2}\nu(t_i)\left[H^2(a_i-t_i)-H^2(b_i-t_i)\right]\right\}$$

If $t\ge t_i$, $E_t^T\left[F(t_i;a_i,b_i)\right]=F(t_i;a_i,b_i)$. So without loss of generality, we assume in the below $t<t_i$. Then for each $t_i\in(t,T]$, the dynamics of X_{t_i} under the T-forward measure Q^T is

$$X_{t_i} = h(t_i-t)X_t+\int_t^{t_i}h(t_i-u)\sigma_u dW_u^T-\left\{\left[H(T-t_i)\nu(t_i)+\nu^h(t_i)\right]-h(t_i-t)\left[H(T-t)\nu(t)+\nu^h(t)\right]\right\}.$$

Conditioning on $\mathcal{F}_t$, X_{t_i} is Gaussian with mean $h(t_i-t)X_t-\left[H(T-t_i)\nu(t_i)+\nu^h(t_i)\right]+h(t_i-t)\left[H(T-t)\nu(t)+\nu^h(t)\right]$ and variance $\nu(t_i)-h(t_i-t;2\kappa)\nu(t)$. Therefore

$$\begin{aligned} & E_t^T\left[e^{-\lambda X_{t_i}}\right] \\ = & \exp\left\{-\lambda h(t_i-t)X_t+\lambda[H(T-t_i)\nu(t_i)+\nu^h(t_i)]-\lambda h(t_i-t)[H(T-t)\nu(t)+\nu^h(t)]\right. \\ & \left.+\frac{1}{2}\lambda^2[\nu(t_i)-h(t_i-t;2\kappa)\nu(t)]\right\}. \end{aligned}$$

This gives us the following formula

$$E_t^T\left[\frac{P(t_i,a_i)}{P(t_i,b_i)}\right] = \frac{P(0,a_i)}{P(0,b_i)}\exp\left\{-\frac{1}{2}\nu(t_i)\big[H^2(a_i-t_i)-H^2(b_i-t_i)\big]\right\}\cdot\exp\left\{-\lambda_i h(t_i-t)X_t-\lambda_i\alpha_i+\frac{1}{2}\lambda_i^2\beta_i\right\}$$

where

$$\begin{cases}\lambda_i = H(a_i-t_i)-H(b_i-t_i)\\ \alpha_i = -\big[H(T-t_i)\nu(t_i)+\nu^h(t_i)\big]+h(t_i-t)\left[H(T-t)\nu(t)+\nu^h(t)\right]\\ \beta_i = \nu(t_i)-h(t_i-t;2\kappa)\nu(t)\end{cases}$$

Combined, we have

$$V_t = P(t,T)\tau(T_s,T_e)\left(\sum_{i=1}^n w_i F(t_i;a_i,b_i)1_{\{t_i\le t\}}+\sum_{i=1}^n\frac{w_i}{\tau_i}\left[A_i e^{-B_iX_t}-1\right]1_{\{t<t_i\}}\right)$$

where

$$\begin{cases}A_i = \frac{P(0,a_i)}{P(0,b_i)}\exp\left\{-\frac{1}{2}\nu(t_i)\big[H^2(a_i-t_i)-H^2(b_i-t_i)\big]-\lambda_i\alpha_i+\frac{1}{2}\lambda_i^2\beta_i\right\}\\ B_i = \lambda_i h(t_i-t)\\ \lambda_i = H(a_i-t_i)-H(b_i-t_i)\\ \alpha_i = -\big[H(T-t_i)\nu(t_i)+\nu^h(t_i)\big]+h(t_i-t)\left[H(T-t)\nu(t)+\nu^h(t)\right]\\ \beta_i = \nu(t_i)-h(t_i-t;2\kappa)\nu(t)\end{cases}$$

In particular, we have

$$\begin{aligned}V_0 &= P(0,T)\tau(T_s,T_e)\sum_{i=1}^n\frac{w_i}{\tau_i}\left[\frac{P(0,a_i)}{P(0,b_i)}\exp\left\{-\frac{1}{2}\nu(t_i)\big[H^2(a_i-t_i)-H^2(b_i-t_i)\big]\right.\right.\\ &\quad\left.\left.+\lambda_i\big[H(T-t_i)\nu(t_i)+\nu^h(t_i)\big]+\frac{1}{2}\lambda_i^2\nu(t_i)\right\}-1\right]\end{aligned}$$

Sometimes, we will have leverage and spread for Libor rates. Denote by l_i the leverage and s_i the spread for the i-th Libor, the time-t price of the floater is given by

$$V_t = P(t,T)\tau(T_s,T_e)\sum_{i=1}^n w_i E_t^T\left[l_iF(t_i;a_i,b_i)+s_i\right]$$

So the price is revised to

$$V_t = P(t,T)\tau(T_s,T_e)\left(\sum_{i=1}^n w_i\big(l_iF(0;a_i,b_i)+s_i\big)1_{\{t_i\le t\}}+\sum_{i=1}^n\frac{w_il_i}{\tau_i}\left[A_ie^{-B_iX_t}-\left(1-\frac{s_i\tau_i}{l_i}\right)\right]1_{\{t<t_i\}}\right)$$

1.8.2 Price of Asian caplet/flooret

We let w take either the value 1 or the value -1, with 1 for caplet and -1 for flooret. Then the time-t value of the contract is

$$\boxed{V_t = P(t,T)\tau(T_s,T_e)E_t^T\left[\big(w(e^{\xi}-K')\big)^+\right] = P(t,T)\tau(T_s,T_e)\mathrm{Bl}(K',e^{\hat\mu+\frac{1}{2}\hat\sigma^2},\hat\sigma,w)} \tag{9}$$

where $\hat{\mu}$ and $\hat{\sigma}^2$ are obtained as below (i_0 is the first i such that $t_i > t$)

$$\begin{cases} \hat{\sigma}^2 = \ln \frac{V}{M^2} \\ \hat{\mu} = \ln M - \frac{1}{2}\hat{\sigma}^2 \\ M = \sum_{i=i_0}^n w_i' e^{\mu_i + \frac{1}{2}\Sigma_{ii}} \\ V = \sum_{i,j=i_0}^n w_i' w_j' e^{\mu_i + \mu_j + \frac{1}{2}(\Sigma_{ii} + \Sigma_{jj}) + \Sigma_{ij}} \\ K' = \sum_{i=i_0}^n \frac{w_i}{\tau_i} + K + \sum_{i=1}^{i_0 - 1} w_i F(t_i; a_i, b_i) \\ w_i' = \frac{w_i}{\tau_i} \frac{P(0,a_i)}{P(0,b_i)} \exp\left\{ -\frac{1}{2}\nu(t_i)\left[H^2(a_i - t_i) - H^2(b_i - t_i)\right] - \left[H(a_i - t_i) - H(b_i - t_i)\right]\nu^h(t_i) \right\} \\ \mu_i = -\left[H(a_i - t_i) - H(b_i - t_i)\right]\left(h(t_i - t)X_t - [H(T - t_i)\nu(t_i) + \nu^h(t_i)] + h(t_i - t)[H(T-t)\nu(t) + \nu^h(t)]\right) \\ \Sigma_{ij}(t) = \left[H(a_i - t_i) - H(b_i - t_i)\right]\left[H(a_j - t_j) - H(b_j - t_j)\right] h(t_i + t_j - 2t_i \wedge t_j)\left[\nu(t_i \wedge t_j) - h(t_i \wedge t_j - t; 2\kappa)\nu(t)\right] \end{cases}$$

In particular, we have

$$\boxed{V_0 = P(0,T)\tau(T_s, T_e) E_0^T[(w(e^\xi - K'))^+] = P(0,T)\tau(T_s,T_e)\mathrm{Bl}(K', e^{\hat{\mu} + \frac{1}{2}\hat{\sigma}^2}, \hat{\sigma}, w)}$$

where

$$\begin{cases} \hat{\sigma}^2 = \ln \frac{V}{M^2} \\ \hat{\mu} = \ln M - \frac{1}{2}\hat{\sigma}^2 \\ M = \sum_{i=1}^n w_i' e^{\mu_i + \frac{1}{2}\Sigma_{ii}} \\ V = \sum_{i,j=1}^n w_i' w_j' e^{\mu_i + \mu_j + \frac{1}{2}(\sum_{ii} + \sum_{jj}) + \sum_{ij}} \\ K' = \sum_{i=1}^n \frac{w_i}{\tau_i} + K \\ w_i' = \frac{w_i}{\tau_i} \frac{P(0,a_i)}{P(0,b_i)} \exp\left\{ -\frac{1}{2}\nu(t_i)\left[H^2(a_i - t_i) - H^2(b_i - t_i)\right] - \left[H(a_i - t_i) - H(b_i - t_i)\right]\nu^h(t_i) \right\} \\ \mu_i = [H(a_i - t_i) - H(b_i - t_i)][H(T - t_i)\nu(t_i) + \nu^h(t_i)] \\ \Sigma_{ij} = \left[H(a_i - t_i) - H(b_i - t_i)\right]\left[H(a_j - t_j) - H(b_j - t_j)\right] h(t_i + t_j - 2t_i \wedge t_j)\nu(t_i \wedge t_j) \end{cases}$$

To derive formula (9), we let w take either the value 1 or the value -1. We need to value

$$\text{the time-}t\text{ price of } \boxed{\tau(T_s, T_e)\left(w\sum_{i=1}^n w_i F(t_i; a_i, b_i) - wK\right)^+} \text{ which is paid at } T\ (t < T_e \le T).$$

We first compute covariance matrix of X_{t_i}'s under Q^T, conditioning on $\mathcal{F}_t$. Given the dynamics of X_{t_i} under Q^T

$$X_{t_i} = h(t_i - t)X_t + \int_t^{t_i} h(t_i - u)\sigma_u dW_u^T - \left\{\left[H(T - t_i)\nu(t_i) + \nu^h(t_i)\right] - h(t_i - t)\left[H(T-t)\nu(t) + \nu^h(t)\right]\right\},$$

this is easily obtained as

$$\begin{aligned} Cov_t^T[X_{t_i} X_{t_j}] &= E_t^T\left[\int_t^{t_i} h(t_i - u)\sigma_u dW_u^T \int_t^{t_j} h(t_j - v)\sigma_v dW_v^T\right] = e^{-\kappa(t_i + t_j)} \int_t^{t_i \wedge t_j} e^{2\kappa u}\sigma_u^2 du \\ &= h(t_i + t_j - 2t_i \wedge t_j)\left[\nu(t_i \wedge t_j) - h(t_i \wedge t_j - t; 2\kappa)\nu(t)\right]. \end{aligned}$$

Denote by i_0 the first i such that $t_i > t$. Then

$$\begin{aligned} \sum_{i=1}^n w_i F(t_i; a_i, b_i) - K &= \sum_{i=i_0}^n w_i F(t_i; a_i, b_i) - \left(K + \sum_{i=1}^{i_0 - 1} w_i F(t_i; a_i, b_i)\right) \\ &= \sum_{i=i_0}^n \frac{w_i}{\tau_i}\frac{P(t_i, a_i)}{P(t_i, b_i)} - \left(\sum_{i=i_0}^n \frac{w_i}{\tau_i} + K + \sum_{i=1}^{i_0 - 1} w_i F(t_i; a_i, b_i)\right) \end{aligned}$$

Define $K' = \sum_{i=i_0}^{n} \frac{w_i}{\tau_i} + K + \sum_{i=1}^{i_0-1} w_i F(t_i; a_i, b_i)$ and for $i \geq i_0$, define

$$\begin{aligned} w_i' &= \frac{w_i}{\tau_i} \frac{P(0,a_i)}{P(0,b_i)} \exp\left\{ -\frac{1}{2}\nu(t_i)\big[H^2(a_i - t_i) - H^2(b_i - t_i)\big] - \big[H(a_i - t_i) - H(b_i - t_i)\big]\nu^h(t_i) \right\}, \\ Z_i &= -\big[H(a_i - t_i) - H(b_i - t_i)\big] X_{t_i}. \end{aligned}$$

We then have

$$\sum_{i=1}^{n} w_i F(t_i; a_i, b_i) - K = \sum_{i=i_0}^{n} w_i' e^{Z_i} - K'.$$

Conditioning on $\mathcal{F}_t$, Z_i is Gaussian with mean

$$\mu_i = -\big[H(a_i - t_i) - H(b_i - t_i)\big] \big(h(t_i - t) X_t - [H(T - t_i)\nu(t_i) + \nu^h(t_i)] + h(t_i - t)[H(T - t)\nu(t) + \nu^h(t)] \big)$$

and covariance

$$\Sigma_{ij}(t) = \big[H(a_i - t_i) - H(b_i - t_i)\big]\big[H(a_j - t_j) - H(b_j - t_j)\big] h(t_i + t_j - 2t_i \wedge t_j)\big[\nu(t_i \wedge t_j) - h(t_i \wedge t_j - t; 2\kappa)\nu(t)\big].$$

By the computation of Appendix B,

$$\sum_{i=1}^{n} w_i F(t_i; a_i, b_i) - K = \sum_{i=i_0}^{n} w_i' e^{Z_i} - K' \approx e^{\xi} - K'$$

where $\xi \sim N(\hat{\mu}, \hat{\sigma}^2)$ with

$$\begin{cases} \hat{\sigma}^2 = \ln \frac{V}{M^2} \\ \hat{\mu} = \ln M - \frac{1}{2}\hat{\sigma}^2 \\ M = \sum_{i=i_0}^{n} w_i' e^{\mu_i + \frac{1}{2}\Sigma_{ii}} \\ V = \sum_{i,j=i_0}^{n} w_i' w_j' e^{\mu_i + \mu_j + \frac{1}{2}(\Sigma_{ii} + \Sigma_{jj}) + \Sigma_{ij}} \end{cases}$$

Then the time-t value of the contract is

$$V_t = P(t,T)\tau(T_s, T_e) E_t^T \left[\big(w(e^{\xi} - K') \big)^+ \right] = P(t,T)\tau(T_s,T_e)\mathrm{Bl}(K', e^{\hat{\mu} + \frac{1}{2}\hat{\sigma}^2}, \hat{\sigma}, w).$$

1.9 Pricing cross-currency European contingent claims under HW1F

Suppose we have two currencies. Traded on the market are foreign money market M^f and a family of foreign zero-coupon bonds $\mathcal{P}^f$ with all maturities, as well as domestic money market M^d and a family of domestic zero-coupon bonds $\mathcal{P}^d$ with all maturities. Foreign assets are denominated in foreign currency and domestic assets are denominated in domestic currency. We denote by Y the spot exchange rate that gives units of domestic currency per unit of foreign currency.

Denominating everything in domestic currency, we have the tradable assets

$$\left(M^d, \mathcal{P}^d, Y M^f, Y \mathcal{P}^f \right).$$

By the arbitrage theory, we can find a probability measure Q^d such that $\left(\frac{\mathcal{P}^d}{M^d}, \frac{YM^f}{M^d}, \frac{Y\mathcal{P}^f}{M^d} \right)$ are Q^d-martingales. We assume under Q^d, the domestic short rate follows one-factor Hull-White model

$$\boxed{r_t^d = \theta_t^d + X_t^d, \; dX_t^d = -\kappa^d X_t^d dt + \sigma_t^d dW_t^d, \; X_0^d = 0}$$

and the spot exchange rate Y satisfies the SDE

$$dY_t = Y_t(\mu_t^e dt + \sigma_t^e dW_t^e),$$

where σ_t^e is a deterministic function of time t and W^e is a standard Brownian motion with $dW_t^e dW_t^d = \rho_{de} dt$ (ρ_{de} is a constant). To avoid arbitrage, it's necessary that the prices of domestic zero-coupon bonds are given by $P^d(t,T) = E^d\left[\exp\left\{-\int_t^T r^d_s ds\right\}\Big|_{\mathcal{F}_t}\right]$ (E^d denotes the expectation under Q^d) and the exchange rate satisfies

$$\boxed{dY_t = Y_t\left[(r^d_t - r^f_t)dt + \sigma^e_t dW^e_t\right]}$$

To take advantage of the single currency pricing infrastructure in each currency, we now denominate everything in foreign currency:

$$\left(\frac{M^d}{Y}, \frac{\mathcal{P}^d}{Y}, M^f, P^f\right).$$

We assume there is a probability measure Q^f under which $\left(\frac{M^d}{YM^f}, \frac{\mathcal{P}^d}{YM^f}, \frac{\mathcal{P}^f}{M^f}\right)$ are martingales. We assume under Q^f the foreign short rate also follows one-factor Hull-White model

$$\boxed{r^f_t = \theta^f_t + X^f_t,\ dX^f_t = -\kappa^f X^f_t dt + \sigma^f_t dW^f_t,\ X^f_0 = 0}$$

where W^f is a standard Brownian motion under Q^f, with $dW^f_t dW^d_t = \rho_{df} dt$ and $dW^f_t dW^e_t = \rho_{ef} dt$ (ρ_{df} and ρ_{ef} are constants).[1]

Motivated by risk neutral pricing under domestic money market account measure, we need to know *what the behavior of* r^f_t *is under* Q^d*?*

1.9.1 Dynamics of foreign rate under domestic money market account measure

We make the following key observation: Q^f is such that the domestic currency denominated assets

$$(M^d, \mathcal{P}^d, YM^f, Y\mathcal{P}^f)$$

discounted by the numéraire YM^f are Q^f-martingales. Therefore, Q^f is the martingale measure associated with the numéraire YM^f.

Then we can "read out" the dynamics of r^f_t directly. Indeed, the Radon-Nikodym derivative of Q^d with respect to Q^f is

$$\begin{aligned}
\frac{dQ^d|_{\mathcal{F}_t}}{dQ^f|_{\mathcal{F}_t}} &= \frac{1}{Y_t}\exp\left\{\int_0^t (r^d_s - r^f_s)ds\right\} \\
&= \exp\left\{-\int_0^t \sigma^e_s dW^e_s + \frac{1}{2}\int_0^t (\sigma^e_s)^2 ds\right\} \\
&= \exp\left\{-\int_0^t \sigma^e_s (dW^e_s - \sigma^e_s ds) - \frac{1}{2}\int_0^t (\sigma^e_s)^2 ds\right\} \\
&= \mathcal{E}\left(-\int_0^t \sigma^e_s d(W^e_s - \int_0^s \sigma^e_u du)\right),
\end{aligned}$$

where $\mathcal{E}$ stands for the Dolèans-Dade exponential and $W^e_t - \int_0^t \sigma^e_s ds$ can be verified to be a Brownian motion under Q^f. So by Girsanov's Theorem,

$$W^f_t - \langle W^f, -\int_0^\cdot \sigma^e_s d(W^e_s - \int_0^s \sigma^e_u du)\rangle_t = W^f_t + \rho_{ef}\int_0^t \sigma^e_s ds := \hat{W}^f_t$$

is a Q^d-BM. Then the dynamics of foreign rate r^f_t under domestic money market account measure Q^d becomes

$$\boxed{r^f_t = \hat{\theta}^f_t + \hat{X}^f_t,\ d\hat{X}^f_t = -\kappa^f \hat{X}^f_t dt + \sigma^f_t d\hat{W}^f_t,\ \hat{X}^f_0 = 0}$$

where $\hat{\theta}^f_t = \theta^f_t - \rho_{ef} e^{-\kappa^f t}\int_0^t e^{\kappa^f s}\sigma^f_s \sigma^e_s ds$.

[1] Since the quadratic variation of W^d and W^f can be calculated as the a.s. limit of $\sum_i (W^d_{t_i} - W^d_{t_{i-1}})(W^f_{t_i} - W^f_{t_{i-1}})$ and since Q^d and Q^f are equivalent, $\langle W^d, W^f\rangle$ is uniquely defined regardless the measure under consideration. Therefore, the notion of instantaneous correlation ρ_{df} is well-defined. Similar argument holds for ρ_{ef}.

1.9.2 Pricing of European contingent claims

We summarize the dynamics of domestic rate, foreign rate, and exchange rate under the domestic money market account measure Q^d as follows (W^d, $\hat{W}^f$, and W^e are all standard BM under Q^d):

$$\boxed{\begin{cases} r_t^d = \theta_t^d + X_t^d,\ dX_t^d = -\kappa^d X_t^d dt + \sigma_t^d dW_t^d,\ X_0^d = 0 \\ r_t^f = \hat{\theta}_t^f + \hat{X}_t^f,\ d\hat{X}_t^f = -\kappa^f \hat{X}_t^f dt + \sigma_t^f d\hat{W}_t^f,\ \hat{X}_0^f = 0,\ \hat{\theta}_t^f = \theta_t^f - \rho_{ef} e^{-\kappa^f t} \int_0^t e^{\kappa^f s} \sigma_s^f \sigma_s^e ds \\ Y_t = Y_0 e^{\theta_t^e + X_t^e},\ \theta_t^e = \int_0^t (\theta_s^d - \hat{\theta}_s^f) ds - \frac{1}{2} \int_0^t (\sigma_s^e)^2 ds,\ X_t^e = \int_0^t (X_s^d - \hat{X}_s^f) ds + \int_0^t \sigma_s^e dW_s^e \\ dW_t^d dW_t^e = \rho_{de} dt,\ dW_t^d d\hat{W}_t^f = \rho_{df} dt,\ dW_t^e d\hat{W}_t^f = \rho_{ef} dt \end{cases}} \tag{10}$$

Suppose we have a European contingent claim ξ having payoff $\xi = f(g_1(X_T^d), g_2(X_T^f)Y_T)$ at time T, where f, g_1 and g_2 are all deterministic functions. Then risk neutral pricing gives the claims's price at time 0 as (E^d stands for the expectation under domestic money market account measure Q^d)

$$\begin{aligned} V_0 &= E^d\left[e^{-\int_0^T r_t^d dt} f(g_1(X_T^d), g_2(X_T^f)Y_T)\right] \\ &= e^{-\int_0^T \theta_t^d dt} E^d\left[e^{-\int_0^T X_t^d dt} f\Big(g_1(X_T^d), g_2(\hat{X}_T^f + \hat{\theta}_T^f - \theta_T^f) Y_0 e^{\theta_T^e + X_T^e}\Big)\right] \\ &= P^d(0,T) e^{-\nu_d^H(T)} E^d\Bigg[e^{-\int_0^T X_t^d dt} f\Big(g_1(X_T^d), g_2(\hat{X}_T^f - \rho_{ef} e^{-\kappa^f t} \int_0^t e^{\kappa^f s} \sigma_s^f \sigma_s^e ds) Y_0 e^{X_T^e} \\ &\quad \cdot \frac{P^f(0,T)}{P^d(0,T)} e^{-\nu_f^H(T) + \nu_d^H(T) + \rho_{ef} \int_0^T e^{-\kappa^f t} \int_0^t e^{\kappa^f s} \sigma_s^f \sigma_s^e ds - \frac{1}{2} \int_0^T (\sigma_t^e)^2 dt}\Big)\Bigg] \\ &= A(T) E^d\left[e^{-\int_0^T X_t^d dt} f\big(g_1(X_T^d), g_2(\hat{X}_T^f - B(T)) \cdot C(T) Y_0 e^{X_T^e}\big)\right], \end{aligned}$$

where

$$\begin{cases} A(T) = P^d(0,T) e^{-\nu_d^H(T)} \\ B(T) = \rho_{ef} e^{-\kappa^f T} \int_0^T e^{\kappa^f s} \sigma_s^f \sigma_s^e ds \\ C(T) = \frac{P^f(0,T)}{P^d(0,T)} e^{-\nu_f^H(T) + \nu_d^H(T) + \rho_{ef} \int_0^T e^{-\kappa^f t} \int_0^t e^{\kappa^f s} \sigma_s^f \sigma_s^e ds - \frac{1}{2} \int_0^T (\sigma_t^e)^2 dt} \end{cases}$$

So the price formula hinges on the joint density function of the centered Gaussian random vector $(Z_T, X_T^d, \hat{X}_T^f, X_T^e)$ ($Z_T = \int_0^T X_t^d dt$). With a "conditioning" trick, we can reduce the problem's dimension by 1 (see Section 1.9.3) and obtain the time-0 price of contingent claim as

$$\boxed{V_0 = A(T) e^{\frac{1}{2}\sigma^2(T)} E^d\left[e^{-\mu(X_T^d, X_T^e, \hat{X}_T^f, T)} f\Big(g_1(X_T^d), g_2(\hat{X}_T^f - B(T)) \cdot C(T) Y_0 e^{X_T^e}\Big)\right]} \tag{11}$$

where

$$\mu(x_d, x_e, x_f, t) = (c_{zd}(t), c_{ze}(t), c_{zf}(t)) \begin{pmatrix} v_d^2(t) & c_{de}(t) & c_{df}(t) \\ c_{ed}^2(t) & v_e^2(t) & c_{ef}(t) \\ c_{fd}(t) & c_{fe}(t) & v_f^2(t) \end{pmatrix}^{-1} \begin{pmatrix} x_d \\ x_e \\ x_f \end{pmatrix}$$

and

$$\sigma^2(t) = v_z^2(t) - (c_{zd}(t), c_{ze}(t), c_{zf}(t)) \begin{pmatrix} v_d^2(t) & c_{de}(t) & c_{df}(t) \\ c_{ed}^2(t) & v_e^2(t) & c_{ef}(t) \\ c_{fd}(t) & c_{fe}(t) & v_f^2(t) \end{pmatrix}^{-1} \begin{pmatrix} c_{dz}(t) \\ c_{ez}(t) \\ c_{fz}(t) \end{pmatrix}$$

with the covariance matrix given in Appendix 1.9.3:

$$\Sigma_t = E\left[\begin{pmatrix} Z_t \\ X_t^d \\ X_t^e \\ X_t^f \end{pmatrix} (Z_t, X_t^d, X_t^e, X_t^f)\right] = \begin{pmatrix} v_z^2(t) & c_{zd}(t) & c_{ze}(t) & c_{zf}(t) \\ c_{dz}(t) & v_d^2(t) & c_{de}(t) & c_{df}(t) \\ c_{ez}(t) & c_{ed}^2(t) & v_e^2(t) & c_{ef}(t) \\ c_{fz}(t) & c_{fd}(t) & c_{fe}(t) & v_f^2(t) \end{pmatrix}$$

1.9.3 Joint density of $(\int_0^t X_s^d ds, X_t^d, X_t^e, \hat{X}_t^f)$ and value of $E^d\left[e^{-\int_0^t X_s^d ds}\Big| X_t^d, X_t^e, \hat{X}_t^f\right]$

We define $Z_t = \int_0^t X_s^d ds$. Recall

$$X_t^d = e^{-\kappa^d t}\int_0^t e^{\kappa^d s}\sigma_s^d dW_s^d,\ X_t^e = \int_0^t (X_s^d - \hat{X}_s^f)ds + \int_0^t \sigma_s^e dW_s^e,\ \hat{X}_t^f = e^{-\kappa^f t}\int_0^t e^{\kappa^f s}\sigma_s^f d\hat{W}_s^f,$$

where W^d, $\hat{W}^f$ and W^e are all standard Brownian motion under Q^d with the following instantaneous correlation:

$$dW_t^d d\hat{W}_t^f = \rho_{df}dt,\ dW_t^d dW_t^e = \rho_{de}dt,\ d\hat{W}_t^f dW_t^e = \rho_{fe}dt.$$

Note $(Z_t, X_t^d, \hat{X}_t^f, X_t^e)$ are jointly Gaussian, so it's sufficient to calculate the covariance matrix. We can easily deduce the following formulas:

$$\begin{cases} v_z^2(t) = E^d[(Z_t)^2] = 2\nu_d^H(t) \\ v_d^2(t) = E^d[(X_t^d)^2] = \nu_d(t) \\ v_e^2(t) = 2c_{ze}(t) - v_z^2(t) + 2\nu_f^H(t) + \int_0^t (\sigma_s^e)^2 ds - 2\rho_{ef}\int_0^t e^{-\kappa^f\xi}\int_0^\xi e^{\kappa^f s}\sigma_s^e\sigma_s^f ds d\xi \\ v_f^2(t) = E^d[(\hat{X}_t^f)^2] = \nu_f(t) \\ c_{zd}(t) = E^d[X_t^d Z_t] = \nu_d^h(t) \\ c_{ze}(t) = \rho_{de}\int_0^t e^{-\kappa^d s}\int_0^s e^{\kappa^d u}\sigma_u^d\sigma_u^e du ds - \rho_{df}\int_0^t\int_0^s e^{-\kappa^d(s-\xi)}\left(e^{-(\kappa^d+\kappa^f)\xi}\int_0^\xi e^{(\kappa^d+\kappa^f)u}\sigma_u^d\sigma_u^f du\right)d\xi ds \\ -\rho_{df}\int_0^t\int_0^s e^{-\kappa^f(s-\xi)}\left(e^{-(\kappa^d+\kappa^f)\xi}\int_0^\xi e^{(\kappa^d+\kappa^f)u}\sigma_u^d\sigma_u^f du\right)d\xi ds + v_z^2(t) \\ c_{zf}(t) = \rho_{df}\int_0^t e^{-\kappa^f(t-s)}\left(e^{-(\kappa^d+\kappa^f)s}\int_0^s e^{(\kappa^d+\kappa^f)u}\sigma_u^d\sigma_u^f du\right)ds \\ c_{de}(t) = \nu_d^h(t) - \rho_{df}\int_0^t e^{-\kappa^d(t-s)}\left(e^{-(\kappa^d+\kappa^f)s}\int_0^s e^{(\kappa^d+\kappa^f)u}\sigma_u^d\sigma_u^f du\right)ds + \rho_{de}e^{-\kappa^d t}\int_0^t e^{\kappa^d s}\sigma_s^d\sigma_s^e ds \\ c_{df}(t) = E^d[X_t^d X_t^f] = \rho_{df}e^{-(\kappa^d+\kappa^f)t}\int_0^t e^{(\kappa^d+\kappa^f)s}\sigma_s^d\sigma_s^f ds \\ c_{ef}(t) = \rho_{df}\int_0^t e^{-\kappa^f(t-s)}\left(e^{-(\kappa^d+\kappa^f)s}\int_0^s e^{(\kappa^d+\kappa^f)u}\sigma_u^d\sigma_u^f du\right)ds - \nu_f^h(t) + \rho_{ef}e^{-\kappa^f t}\int_0^t e^{\kappa^f s}\sigma_s^e\sigma_s^f ds \end{cases} \quad (12)$$

Denote the covariance matrix by

$$\Sigma_t = \begin{pmatrix} v_z^2(t) & c_{zd}(t) & c_{ze}(t) & c_{zf}(t) \\ c_{dz}(t) & v_d^2(t) & c_{de}(t) & c_{df}(t) \\ c_{ez}(t) & c_{ed}^2(t) & v_e^2(t) & c_{ef}(t) \\ c_{fz}(t) & c_{fd}(t) & c_{fe}(t) & v_f^2(t) \end{pmatrix}.$$

The joint density function of the pair $(X_t^d, X_t^e, X_t^f, Z_t)$ is therefore

$$g(x_z, x_d, x_e, x_f; t) = \frac{|\Sigma_t|^{-\frac{1}{2}}}{4\pi^2}e^{-\frac{1}{2}(x_z,x_d,x_e,x_f)\Sigma_t^{-1}(x_z,x_d,x_e,x_f)'}.$$

Now we want to calculate the conditional expectation

$$E^d\left[e^{-Z_t}\big| X_t^d = x_d, X_t^e = x_e, \hat{X}_t^f = x_f\right].$$

We note conditioning on $(X_t^d, X_t^e, \hat{X}_t^f)$, $Z_t \sim N(\mu, \sigma^2)$, with (see Anderson [1])

$$\mu(x_d, x_e, x_f, t) = (v_{zd}(t), v_{ze}(t), v_{zf}(t))\begin{pmatrix} v_d^2(t) & v_{de}(t) & v_{df}(t) \\ v_{ed}^2(t) & v_e^2(t) & v_{ef}(t) \\ v_{fd}(t) & v_{fe}(t) & v_f^2(t) \end{pmatrix}^{-1}\begin{pmatrix} x_d \\ x_e \\ x_f \end{pmatrix}$$

and

$$\sigma^2(t) = v_z^2(t) - (v_{zd}(t), v_{ze}(t), v_{zf}(t))\begin{pmatrix} v_d^2(t) & v_{de}(t) & v_{df}(t) \\ v_{ed}^2(t) & v_e^2(t) & v_{ef}(t) \\ v_{fd}(t) & v_{fe}(t) & v_f^2(t) \end{pmatrix}^{-1}\begin{pmatrix} v_{dz}(t) \\ v_{ez}(t) \\ v_{fz}(t) \end{pmatrix}$$

Therefore

$$E^d\left[e^{-Z_t}\big|\, X_t^d = x_d, X_t^e = x_e, \hat{X}_t^f = x_f\right] = \exp\left\{-\mu(x_d, x_e, x_f, t) + \frac{1}{2}\sigma^2(t)\right\}.$$

To verify the above formulas, we note

Variance of X_t^d: We note $v_d^2(t) = E^d[(X_t^d)^2] = e^{-2\kappa^d t}\int_0^t e^{2\kappa^d s}(\sigma_s^d)^2 ds = \nu_d(t)$.

Variance of $\hat{X}_t^f$: We note $v_f^2(t) = E^d[(\hat{X}_t^f)^2] = e^{-2\kappa^f t}\int_0^t e^{2\kappa^f s}(\sigma_s^f)^2 ds = \nu_f(t)$.

Covariance of X_t^d and Z_t: We note

$$\begin{aligned} c_{zd}(t) &= E^d[X_t^d Z_t] = E^d[\int_0^t (X_s^d)^2 ds] + E^d[\int_0^t Z_s(-\kappa^d X_s^d ds + \sigma_s^d dW_s^d)] \\ &= \int_0^t v_d^2(s)ds - \kappa^d \int_0^t c_{zd}(s)ds. \end{aligned}$$

Solving the consequent differential equation, $\frac{d}{dt}c_{zd}(t) = -\kappa^d c_{zd}(t) + \nu_d(t)$, we get $c_{zd}(t) = \nu_d^h(t)$.

Variance of Z_t: We note $v_z^2(t) = E^d[(Z_t)^2] = 2\int_0^t E^d[Z_s X_s^d]ds = 2\nu_d^H(t)$.

Covariance of X_t^d and $\hat{X}_t^f$: We note

$$\begin{aligned} c_{df}(t) &= E^d[X_t^d \hat{X}_t^f] = e^{-(\kappa^d+\kappa^f)t}E^d[\int_0^t e^{\kappa^d s}\sigma_s^d dW_s^d \int_0^t e^{\kappa^f s}\sigma_s^f d\hat{W}_s^f] \\ &= \rho_{df}e^{-(\kappa^d+\kappa^f)t}\int_0^t e^{(\kappa^d+\kappa^f)s}\sigma_s^d\sigma_s^f ds. \end{aligned}$$

Covariance of X_t^d and X_t^e: We note

$$c_{de}(t) = E^d[X_t^d X_t^e] = E^d\left[\int_0^t X_s^d(X_s^d - \hat{X}_s^f)ds + \int_0^t X_s^e(-\kappa^d X_s^d ds)\right] + \rho_{de}\int_0^t \sigma_s^d\sigma_s^e ds$$

So $c_{de}(t)$ satisfies the differential equation

$$\frac{d}{dt}c_{de}(t) = v_d^2(t) - c_{df}(t) - \kappa^d c_{de}(t) + \rho_{de}\sigma_t^d\sigma_t^e.$$

Therefore

$$\begin{aligned} & c_{de}(t) \\ =\ & e^{-\kappa^d t}\int_0^t e^{\kappa^d s}[v_d^2(s) - c_{df}(s) + \rho_{de}\sigma_s^d\sigma_s^e]ds \\ =\ & \nu_d^h(t) - e^{-\kappa^d t}\int_0^t \rho_{df}e^{-\kappa^f s}\int_0^s e^{(\kappa^d+\kappa^f)s}\sigma_u^d\sigma_u^f du ds + \rho_{de}e^{-\kappa^d t}\int_0^t e^{\kappa^d s}\sigma_s^d\sigma_s^e ds \\ =\ & \nu_d^h(t) - \rho_{df}\int_0^t e^{-\kappa^d(t-s)}\left(e^{-(\kappa^d+\kappa^f)s}\int_0^s e^{(\kappa^d+\kappa^f)u}\sigma_u^d\sigma_u^f du\right)ds + \rho_{de}e^{-\kappa^d t}\int_0^t e^{\kappa^d s}\sigma_s^d\sigma_s^e ds. \end{aligned}$$

Covariance of Z_t and $\hat{X}_t^f$: We note

$$dc_{zf}(t) = E^d[d(Z_t\hat{X}_t^f)] = E^d[\hat{X}_t^f X_t^d dt + Z_t(-\kappa^f\hat{X}_t^f)dt] = c_{df}(t)dt - \kappa^f c_{zf}(t)dt.$$

So

$$\begin{aligned} c_{zf}(t) &= e^{-\kappa^f t}\int_0^t e^{\kappa^f s}c_{df}(s)ds = e^{-\kappa^f t}\int_0^t \rho_{df}e^{-\kappa^d s}\int_0^s e^{(\kappa^d+\kappa^f)u}\sigma_u^d\sigma_u^f du ds \\ &= \rho_{df}\int_0^t e^{-\kappa^f(t-s)}\left(e^{-(\kappa^d+\kappa^f)s}\int_0^s e^{(\kappa^d+\kappa^f)u}\sigma_u^d\sigma_u^f du\right)ds. \end{aligned}$$

Covariance of X_t^e and $\hat{X}_t^f$: We note

$$\begin{aligned} dc_{ef}(t) &= E^d[d(X_t^e \hat{X}_t^f)] = E^d[\hat{X}_t^f(X_t^d - \hat{X}_t^f)dt + X_t^e(-\kappa^f \hat{X}_t^f dt) + \sigma_t^e dW_t^e \sigma_t^f d\hat{W}_t^f] \\ &= c_{df}(t)dt - v_f^2(t)dt - \kappa^f c_{ef}(t)dt + \rho_{ef}\sigma_t^e \sigma_t^f dt. \end{aligned}$$

Therefore

$$\begin{aligned} & c_{ef}(t) \\ = \;& e^{-\kappa^f t}\int_0^t e^{\kappa^f s}[c_{df}(s) - v_f^2(s) + \rho_{ef}\sigma_s^e\sigma_s^f]ds \\ = \;& e^{-\kappa^f t}\int_0^t \left(\rho_{df} e^{-\kappa^d s}\int_0^s e^{(\kappa^d+\kappa^f)u}\sigma_u^d\sigma_u^f du - e^{-\kappa^f s}\int_0^s e^{2\kappa^f u}(\sigma_u^f)^2 du\right) ds + \rho_{ef} e^{-\kappa^f t}\int_0^t e^{\kappa^f s}\sigma_s^e\sigma_s^f ds \\ = \;& \rho_{df}\int_0^t e^{-\kappa^f(t-s)}\left(e^{-(\kappa^d+\kappa^f)s}\int_0^s e^{(\kappa^d+\kappa^f)u}\sigma_u^d\sigma_u^f du\right) - \nu_f^h(t) + \rho_{ef} e^{-\kappa^f t}\int_0^t e^{\kappa^f s}\sigma_s^e\sigma_s^f ds. \end{aligned}$$

Covariance of Z_t and X_t^e: We note

$$dc_{ze}(t) = E^d[d(Z_t X_t^e)] = E^d[X_t^d X_t^e + Z_t(X_t^d - \hat{X}_t^f)]dt = (c_{de}(t) + c_{zd}(t) - c_{zf}(t))dt.$$

Therefore

$$\begin{aligned} & c_{ze}(t) \\ = \;& \int_0^t [c_{de}(s) + c_{zd}(s) - c_{zf}(s)]\, ds \\ = \;& \int_0^t \Bigg[2\nu_d^h(s) + \rho_{de}e^{-\kappa^d s}\int_0^s e^{\kappa^d u}\sigma_u^d\sigma_u^e du - \rho_{df}\int_0^s e^{-\kappa^d(s-\xi)}\left(e^{-(\kappa^d+\kappa^f)\xi}\int_0^\xi e^{(\kappa^d+\kappa^f)u}\sigma_u^d\sigma_u^f du\right)d\xi \\ & -\rho_{df}\int_0^s e^{-\kappa^f(s-\xi)}\left(e^{-(\kappa^d+\kappa^f)\xi}\int_0^\xi e^{(\kappa^d+\kappa^f)u}\sigma_u^d\sigma_u^f du\right)d\xi\Bigg]\, ds \\ = \;& 2\nu_d^H(t) + \rho_{de}\int_0^t e^{-\kappa^d s}\int_0^s e^{\kappa^d u}\sigma_u^d\sigma_u^e du ds \\ & -\rho_{df}\int_0^t\int_0^s\left(e^{-\kappa^d(s-\xi)} + e^{-\kappa^f(s-\xi)}\right)\left(e^{-(\kappa^d+\kappa^f)\xi}\int_0^\xi e^{(\kappa^d+\kappa^f)u}\sigma_u^d\sigma_u^f du\right)d\xi ds \end{aligned}$$

Variance of X_t^e: We note

$$\begin{aligned} \frac{d}{dt}v_e^2(t) &= 2E^d[X_t^e dX_t^e] + E^d[(dX_t^e)^2] = 2E^d[X_t^e(X_t^d - \hat{X}_t^f)dt] + E^d[(\sigma_t^e)^2 dt] \\ &= (2c_{de}(t) - 2c_{ef}(t) + (\sigma_t^e)^2)dt. \end{aligned}$$

Therefore

$$\begin{aligned} & v_e^2(t) \\ = \;& \int_0^t [2c_{de}(s) - 2c_{ef}(s) + (\sigma_s^e)^2]ds \\ = \;& 2\nu_d^H(t) + 2\nu_f^H(t) + \int_0^t(\sigma_s^e)^2 ds - 2\rho_{df}\int_0^t\int_0^\xi e^{-\kappa^d(\xi-s)}\left(e^{-(\kappa^d+\kappa^f)s}\int_0^s e^{(\kappa^d+\kappa^f)u}\sigma_u^d\sigma_u^f du\right)dsd\xi \\ & -2\rho_{df}\int_0^t\int_0^\xi e^{-\kappa^f(\xi-s)}\left(e^{-(\kappa^d+\kappa^f)s}\int_0^s e^{(\kappa^d+\kappa^f)u}\sigma_u^d\sigma_u^f du\right)dsd\xi + 2\rho_{de}\int_0^t e^{-\kappa^d\xi}\int_0^\xi e^{\kappa^d s}\sigma_s^d\sigma_s^e dsd\xi \\ & -2\rho_{ef}\int_0^t e^{-\kappa^f\xi}\int_0^\xi e^{\kappa^f s}\sigma_s^e\sigma_s^f dsd\xi \\ = \;& 2c_{ze}(t) - v_z^2(t) + 2\nu_f^H(t) + \int_0^t(\sigma_s^e)^2 ds - 2\rho_{ef}\int_0^t e^{-\kappa^f\xi}\int_0^\xi e^{\kappa^f s}\sigma_s^e\sigma_s^f dsd\xi \end{aligned}$$

1.9.4 Pricing of cross-currency swaption

Suppose two parties are going to exchange cash flows at times $0 < T_1 < T_2 < \cdots < T_N$: the cash flow for foreign investor is $(C_i^f)_{i=1}^N$ which is denominated in foreign currency, and the cash flow for domestic investor is $(C_i^d)_{i=1}^N$ which is denominated in domestic currency. For generality, we assume $(C_i^f)_{i=1}^N$ and $(C_i^d)_{i=1}^N$ may depend on state variable (e.g. C_i^f and C_i^d are LIBOR rates). Suppose the option is European with maturity $t < T_1$. Then the value of the contingent claim at option maturity t, denominated in domestic currency and from the standpoint of domestic investor, is

$$\max\left\{ Y_t \sum_{i=1}^N C_i^f P^f(t, T_i; X_t^f) - \sum_{i=1}^N C_i^d P^d(t, T_i; X_t^d), 0 \right\}.$$

Therefore, we have the time-0 value of the option as

$$\begin{aligned} V_0 &= E\left[e^{-\int_0^t r_s^d ds} \left(Y_t \sum_{i=1}^N C_i^f(X_t^f) P^f(t, T_i; X_t^f) - \sum_{i=1}^N C_i^d(X_t^d) P^d(t, T_i; X_t^d) \right)^+ \right] \\ &= \boxed{A(t) e^{\frac{1}{2}\sigma^2(t)} E^d \left[e^{-\mu(X_t^d, X_t^e, \hat{X}_t^f, t)} f\left(g_1(X_t^d), g_2(\hat{X}_t^f - B(t)) \cdot C(t) Y_0 e^{X_t^e} \right) \right]} \qquad (13) \end{aligned}$$

where $A(t)$, $B(t)$, and $C(t)$ are as given in formula (11) and

$$\begin{cases} g_1(x) = \sum_{i=1}^N C_i^d(x) P^d(t, T_i; x) \\ g_2(x) = \sum_{i=1}^N C_i^f(x) P^f(t, T_i; x) \\ f(x, y) = \max\{y - x, 0\} \end{cases}$$

Formula (13) can be evaluated by the results in Section 1.9.3.

Since the calculation of $\mu(x_d, x_e, x_f, t)$ involves matrix inversion, sometimes it might be numerically more efficient to work without the "conditioning". More precisely, we evaluate the price via 4-dimensional integration instead of 3-dimensional integration:

$$V_0 = A(t) E^d \left[e^{-\int_0^t X_s^d ds} \left(C(t) Y_0 e^{X_t^e} \sum_{i=1}^N C_i^f(\hat{X}_t^f - B(t)) P^f(t, T_i; \hat{X}_t^f - B(t)) - \sum_{i=1}^N C_i^d(X_t^d) P^d(t, T_i; X_t^d) \right)^+ \right]$$

1.10 Implementation

1.10.1 Implementation of ν

We first recall the definition of $h(t; \kappa)$ and $H(t; \kappa)$:

$$h(t; \kappa) = e^{-\kappa t}, \; H(t; \kappa) = \int_0^t h(s; \kappa) ds = \begin{cases} t & \kappa = 0 \\ \frac{1 - e^{-\kappa t}}{\kappa} & \kappa \neq 0 \end{cases}$$

Suppose we have a sequence of time points $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n := \infty$ and a sequence of constants σ_0, σ_1, $\cdots$, σ_{n-1}, such that the one-factor Hull-White model under consideration has constant volatility σ_{i-1} on the interval $[t_{i-1}, t_i)$ $(i = 1, 2, \cdots, n)$. Then $\nu(t; \kappa) = e^{-2\kappa t} \int_0^t e^{2\kappa s} \sigma_s^2 ds$ can be computed by the following algorithm.

First, we compute $\nu(t_0; \kappa)$, $\nu(t_1; \kappa)$, $\cdots$, $\nu(t_{n-1}; \kappa)$ recursively as follows: $\nu(t_0; \kappa) = \nu(0; \kappa) = 0$. For $i = 1, \cdots, n$,

$$\begin{aligned} \nu(t_i; \kappa) &= e^{-2\kappa t_i} \left(\int_0^{t_{i-1}} e^{2\kappa s} \sigma_s^2 ds + \int_{t_{i-1}}^{t_i} e^{2\kappa s} \sigma_s^2 ds \right) \\ &= e^{-2\kappa(t_i - t_{i-1})} \nu(t_{i-1}; \kappa) + e^{-2\kappa t_i} \cdot \sigma_{i-1}^2 \int_{t_{i-1}}^{t_i} e^{2\kappa s} ds \\ &= h(t_i - t_{i-1}; 2\kappa) \nu(t_{i-1}; \kappa) + \sigma_{i-1}^2 H(t_i - t_{i-1}; 2\kappa). \end{aligned}$$

If the values of $\nu(t_0;\kappa)$, $\nu(t_1;\kappa)$, $\cdots$, $\nu(t_{n-1};\kappa)$ are already obtained from calibration, we can save the results and omit this step of computation. Second, for general $t \geq 0$. Suppose $t \in [t_{i-1}, t_i)$, we have

$$\boxed{\nu(t;\kappa) = h(t-t_{i-1};2\kappa)\nu(t_{i-1};\kappa) + H(t-t_{i-1};2\kappa)\sigma_{i-1}^2}$$

1.10.2 Implementation of ν^h

We recall that

$$\nu^h(t;\kappa) = \int_0^t e^{-\kappa(t-s)}\nu(s;\kappa)ds.$$

We compute $\nu^h(t;\kappa)$ recursively as follows: $\nu^h(t_0;\kappa) = \nu^h(0;\kappa) = 0$. For $t \in [t_{i-1}, t_i)$ $(i = 1, \cdots, n)$, we have

$$\begin{aligned}
\nu^h(t;\kappa) &= \int_0^t e^{-\kappa(t-s)}\nu(s;\kappa)ds = e^{-\kappa t}\left[\int_0^{t_{i-1}} e^{\kappa s}\nu(s;\kappa)ds + \int_{t_{i-1}}^t e^{\kappa s}\nu(s;\kappa)ds\right] \\
&= e^{-\kappa(t-t_{i-1})}\nu^h(t_{i-1};\kappa) + e^{-\kappa t}\int_{t_{i-1}}^t e^{\kappa s}\nu(s;\kappa)ds
\end{aligned}$$

Therefore, for $t \in [t_{i-1}, t_i)$,

$$\boxed{\nu^h(t;\kappa) = h(t-t_{i-1};\kappa)\nu^h(t_{i-1};\kappa) + \nu(t_{i-1};\kappa)h(t-t_{i-1};\kappa)H(t-t_{i-1};\kappa) + \frac{1}{2}\sigma_{i-1}^2 H^2(t-t_{i-1};\kappa)}$$

1.10.3 Implementation of ν^H

We consider a more general implementation of ν^H that evaluates

$$\nu^H(t;\kappa,\kappa') = \int_0^t H(t-s,\kappa')\nu(s;\kappa)ds,$$

where $\nu(t;\kappa) = e^{-2\kappa t}\int_0^t e^{2\kappa s}\sigma_s^2 ds$ and $H(t,\kappa') = \int_0^t e^{-\kappa' s}ds$. When $\kappa' \neq 0$, we have

$$\nu^H(t;\kappa,\kappa') = \int_0^t \frac{1-e^{-\kappa'(t-s)}}{\kappa'}\nu(s;\kappa)ds = \frac{1}{\kappa'}\left[\nu^h(t;\kappa,0) - \nu^h(t;\kappa,\kappa')\right].$$

When $\kappa' = 0$, we have

$$\nu^H(t;\kappa,0) = \int_0^t (t-s)\nu(s;\kappa)ds = t\int_0^t \nu(s)ds - \int_0^t s\nu(s)ds.$$

We compute $\nu^0(t) := \int_0^t \nu(s)ds$ and $\nu^1(t) := \int_0^t s\nu(s)ds$ recursively as follows: $\nu^0(t_0) = \nu^1(t_0) = 0$. For $t \in [t_{i-1}, t_i)$, we have

$$\begin{aligned}
\nu^0(t) &= \nu^0(t_{i-1}) + \int_{t_{i-1}}^t \nu(s)ds \\
&= \begin{cases} \nu^0(t_{i-1}) + \int_{t_{i-1}}^t [\nu(t_{i-1}) + \sigma_{i-1}^2(s-t_{i-1})]ds & \kappa = 0 \\ \nu^0(t_{i-1}) + \int_{t_{i-1}}^t \left[e^{-2\kappa(s-t_{i-1})}\nu(t_{i-1}) + \sigma_{i-1}^2\frac{1-e^{-2\kappa(s-t_{i-1})}}{2\kappa}\right]ds & \kappa \neq 0 \end{cases} \\
&= \begin{cases} \nu^0(t_{i-1}) + \nu(t_{i-1})(t-t_{i-1}) + \frac{\sigma_{i-1}^2}{2}(t-t_{i-1})^2 & \kappa = 0 \\ \nu^0(t_{i-1}) + \left[\nu(t_{i-1}) - \frac{\sigma_{i-1}^2}{2\kappa}\right]\frac{1-e^{-2\kappa(t-t_{i-1})}}{2\kappa} + \frac{\sigma_{i-1}^2}{2\kappa}(t-t_{i-1}) & \kappa \neq 0 \end{cases}
\end{aligned}$$

and

$$
\begin{aligned}
\nu^1(t) &= \nu^1(t_{i-1}) + \int_{t_{i-1}}^t s\nu(s)ds \\
&= \begin{cases} \nu^1(t_{i-1}) + \int_{t_{i-1}}^t [s\nu(t_{i-1}) + \sigma_{i-1}^2 s(s-t_{i-1})]ds & \kappa = 0 \\ \nu^1(t_{i-1}) + \int_{t_{i-1}}^t s\left[e^{-2\kappa(s-t_{i-1})}\nu(t_{i-1}) + \sigma_{i-1}^2 \frac{1-e^{-2\kappa(s-t_{i-1})}}{2\kappa}\right]ds & \kappa \neq 0 \end{cases} \\
&= \begin{cases} \nu^1(t_{i-1}) + \frac{\nu(t_{i-1})}{2}(t^2 - t_{i-1}^2) + \sigma_{i-1}^2\left[\frac{t^3-t_{i-1}^3}{3} - \frac{t_{i-1}}{2}(t^2-t_{i-1}^2)\right] & \kappa = 0 \\ \nu^1(t_{i-1}) + \frac{\sigma_{i-1}^2}{4\kappa}(t^2-t_{i-1}^2) + \frac{\nu(t_{i-1}) - \frac{\sigma_{i-1}^2}{2\kappa}}{2\kappa}\left[t_{i-1} - te^{-2\kappa(t-t_{i-1})} + \frac{1-e^{-2\kappa(t-t_{i-1})}}{2\kappa}\right] & \kappa \neq 0 \end{cases}
\end{aligned}
$$

Once we have computed $\nu^0(t)$ and $\nu^1(t)$, we can accordingly compute ν^H as $\nu^H = t\nu^0(t) - \nu^1(t)$.

2 Two-factor Hull-White model

In the two-factor Hull-White short rate model, it is assumed that the short rate r follows the following dynamics under risk-neutral measure with money market account as the numeraire:

$$r(t) = X_1(t) + X_2(t) + \theta(t),\ r(0) = r_0,$$

where the processes $X_1(t)$ and $X_2(t)$ satisfy

$$\begin{cases} dX_1(t) = -\kappa_1 X_1(t)dt + \sigma_1(t)dW_1(t),\ X_1(0) = 0 \\ dX_2(t) = -\kappa_2 X_2(t)dt + \sigma_2(t)dW_2(t),\ X_2(0) = 0 \end{cases}$$

where (W_1, W_2) is a two-dimensional Brownian motion with instantaneous correlation ρ:

$$dW_1(t)dW_2(t) = \rho dt,$$

r_0, κ_1, κ_2 are positive constants, $\sigma_1(t)$ and $\sigma_2(t)$ are positive deterministic functions, and $\rho \in [-1, 1]$. The function $\theta(t)$ is a deterministic function that will be used to fit the initial yield curve.

2.1 Dynamics of $x_1(t)$ and $x_2(t)$ under risk-neutral measure

We have

$$\boxed{\begin{cases} x_1(t) = e^{-\kappa_1(t-s)}x_1(s) + \int_s^t e^{-\kappa_1(t-u)}\sigma_1(u)dW_1(u),\ x_1(0) = 0 \\ x_2(t) = e^{-\kappa_2(t-s)}x_2(s) + \int_s^t e^{-\kappa_2(t-u)}\sigma_2(u)dW_2(u),\ x_2(0) = 0 \end{cases}} \tag{14}$$

Conditioning on $\mathcal{F}_s$, the random vector $(x_1(t), x_2(t))'$ is Gaussian with mean

$$\mu(s,t) = E[(x_1(t), x_2(t))' \mid \mathcal{F}_s] = (\mu_1(s,t), \mu_2(s,t))' = (h_1(t-s)x_1(s), h_2(t-s)x_2(s))'$$

and covariance matrix

$$
\begin{aligned}
\Sigma(s,t) &= \begin{pmatrix} \int_s^t e^{-2\kappa_1(t-u)}\sigma_1^2(u)du & \int_s^t e^{-(\kappa_1+\kappa_2)(t-u)}\sigma_1(u)\sigma_2(u)\rho du \\ \int_s^t e^{-(\kappa_1+\kappa_2)(t-u)}\sigma_1(u)\sigma_2(u)\rho du & \int_s^t e^{-2\kappa_2(t-u)}\sigma_2^2(u)du \end{pmatrix} \\
&= \begin{pmatrix} \nu_{11}(s,t) & \nu_{12}(s,t) \\ \nu_{21}(s,t) & \nu_{22}(s,t) \end{pmatrix}
\end{aligned}
$$

2.2 Dynamics of $x_1(t)$ and $x_2(t)$ under T-forward measure

Denote by Q^T the T-forward measure. Note the Radon-Nikodym derivative of T-forward measure Q^T w.r.t. the risk-neutral measure Q is

$$\zeta_t = E_t^Q\left[\frac{dQ^T}{dQ}\right] = \frac{P(t,T)/P(0,T)}{e^{\int_0^t r(u)du}}.$$

Therefore

$$\begin{aligned}
d\ln\zeta_t &= d\ln P(t,T) - d\int_0^t r(u)du = d\left[-\int_t^T \varphi(u)du - \sum_{i=1}^2 H_i(T-t)x_i(t) + \frac{1}{2}V(t,T)\right] - r(t)dt \\
&= \varphi(t)dt + \sum_{i=1}^2 h_i(T-t)x_i(t)dt - \sum_{i=1}^2 H_i(T-t)dx_i(t) + \frac{1}{2}\frac{\partial}{\partial t}V(t,T) - r(t)dt \\
&= \left[-\sum_{i=1}^2 x_i(t) + \sum_{i=1}^2 h_i(T-t)x_i(t) + \frac{1}{2}\frac{\partial}{\partial t}V_t(t,T) + \sum_{i=1}^2 H_i(T-t)\kappa_i x_i(t)\right]dt - \sum_{i=1}^2 H_i(T-t)\sigma_i(t)dW_i(t)
\end{aligned}$$

By Ito's formula, $d\ln\zeta_t = d\zeta_t/\zeta_t - \frac{1}{2}(d\zeta_t)^2/\zeta_t^2$. So

$$\frac{(d\zeta_t)^2}{\zeta_t^2} = (d\ln\zeta_t)^2 = \sum_{i=1}^2 H_i(T-t)^2\sigma_i^2(t)dt + 2\rho H_1(T-t)H_2(T-t)\sigma_1(t)\sigma_2(t)dt = -\frac{\partial}{\partial t}V(t,T).$$

This implies

$$\begin{aligned}
\frac{d\zeta_t}{\zeta_t} &= d\ln\zeta_t + \frac{1}{2}\frac{(d\zeta_t)^2}{\zeta_t^2} \\
&= \left[-\sum_{i=1}^2 x_i(t) + \sum_{i=1}^2 h_i(T-t)x_i(t) + \sum_{i=1}^2 H_i(T-t)\kappa_i x_i(t)\right]dt - \sum_{i=1}^2 H_i(T-t)\sigma_i(t)dW_i(t) \\
&= \sum_{i=1}^2 \left[-1 + h_i(T-t) + H_i(T-t)\kappa_i\right]x_i(t)dt - \sum_{i=1}^2 H_i(T-t)\sigma_i(t)dW_i(t) \\
&= -\sum_{i=1}^2 H_i(T-t)\sigma_i(t)dW_i(t),
\end{aligned}$$

which is as expected. Define $L_t = -\sum_{i=1}^2\int_0^t H_i(T-u)\sigma_i(u)dW_i(u)$. Then $\zeta_t = \mathcal{E}(L_t) := \exp\left\{L_t - \frac{1}{2}\langle L\rangle_t\right\}$. By Girsanov's Theorem,

$$W_1^T(t) = W_1(t) - \langle W_1, L\rangle_t = W_1(t) + \int_0^t H_1(T-u)\sigma_1(u)du + \rho\int_0^t H_2(T-u)\sigma_2(u)du$$

and

$$W_2^T(t) = W_2(t) - \langle W_2, L\rangle_t = W_2(t) + \rho\int_0^t H_1(T-u)\sigma_1(u)du + \int_0^t H_2(T-u)\sigma_2(u)du$$

are Brownian motion under Q^T.

Therefore, under the T-forward measure Q^T, we have

$$\boxed{\begin{cases} x_1(t) = e^{-\kappa_1(t-s)}x_1(s) - M_1^T(s,t) + \int_s^t e^{-\kappa_1(t-u)}\sigma_1(u)dW_1^T(u), \ x_1(0) = 0 \\ x_2(t) = e^{-\kappa_2(t-s)}x_2(s) - M_2^T(s,t) + \int_s^t e^{-\kappa_2(t-u)}\sigma_2(u)dW_2^T(u), \ x_2(0) = 0 \end{cases}} \tag{15}$$

where (W_1^T, W_2^T) is a two-dimensional Brownian motion under Q^T with instantaneous correlation ρ and

$$\begin{cases} M_1^T(s,t) = \int_s^t e^{-\kappa_1(t-u)}\sigma_1(u)[H_1(T-u)\sigma_1(u) + \rho H_2(T-u)\sigma_2(u)]du \\ M_2^T(s,t) = \int_s^t e^{-\kappa_2(t-u)}\sigma_2(u)[\rho H_1(T-u)\sigma_1(u) + H_2(T-u)\sigma_2(u)]du \end{cases}$$

Conditioning on $\mathcal{F}_s$, the random vector $(x_1(t), x_2(t))'$ is Gaussian with mean

$$\mu^T(s,t) = E_s^T[(x_1(t), x_2(t))'] = (h_1(t-s)x_1(s) - M_1^T(s,t), h_2(t-s)x_2(t) - M_2^T(s,t))'$$

and covariance matrix is the same as the covariance matrix under risk-neutral measure Q.

2.3 Dynamics of $\int_t^T x_1(u)du$ and $\int_t^T x_2(u)du$ under risk-neutral measure

For convenience of computation, we drop the subscript. Then we have

$$\begin{aligned}
\int_t^T x(u)du &= x(T)T - x(t)t - \int_t^T udx(u) = \int_t^T (T-u)dx(u) + (T-t)x(t) \\
&= \int_t^T (T-u)[-\kappa x(u)du + \sigma(u)dW(u)] + (T-t)x(t) \\
&= -\kappa \int_t^T (T-u)x(u)du + \int_t^T (T-u)\sigma(u)dW(u) + (T-t)x(t)
\end{aligned}$$

By equation (14), for $\kappa \neq 0$, we have

$$\begin{aligned}
-\kappa \int_t^T (T-u)x(u)du &= -\kappa \int_t^T (T-u)\left[e^{-\kappa(u-t)}x(t) + \int_t^u e^{-\kappa(u-s)}\sigma(s)dW(s)\right]du \\
&= -\kappa x(t)\int_t^T (T-u)e^{-\kappa(u-t)}du - \kappa \int_t^T (T-u)\int_t^u e^{-\kappa(u-s)}\sigma(s)dW(s)du \\
&= x(t)\int_t^T (T-u)de^{-\kappa(u-t)} - \kappa \int_t^T \int_t^u e^{\kappa s}\sigma(s)dW(s)d_u\left(\int_t^u (T-v)e^{-\kappa v}dv\right)
\end{aligned}$$

Therefore for $\kappa \neq 0$

$$\begin{aligned}
& -\kappa \int_t^T (T-u)x(u)du \\
= \; & -x(t)(T-t) + x(t)H(T-t) \\
& -\kappa\left[\int_t^T e^{\kappa s}\sigma(s)dW(s)\int_t^T (T-v)e^{-\kappa v}dv - \int_t^T\left(\int_t^u (T-v)e^{-\kappa v}dv\right)e^{\kappa u}\sigma(u)dW(u)\right],
\end{aligned}$$

where the last term can be further simplified to (note $\int_u^T (T-v)e^{-\kappa v}dv = \frac{(T-u)e^{-\kappa u}}{\kappa} - \frac{\int_u^T e^{-\kappa v}dv}{\kappa}$)

$$\begin{aligned}
-\kappa \int_t^T e^{\kappa u}\sigma(u)\left(\int_u^T (T-v)e^{-\kappa v}dv\right)dW(u) &= -\int_t^T e^{\kappa u}\sigma(u)\left((T-u)e^{-\kappa u} - \int_u^T e^{-\kappa v}dv\right)dW(u) \\
&= -\int_t^T \sigma(u)\left[(T-u) - \int_u^T e^{-\kappa(v-u)}dv\right]dW(u)
\end{aligned}$$

We can verify that when $\kappa = 0$, the above equality also holds. Therefore

$$\begin{aligned}
\int_t^T x(u)du &= -\kappa \int_t^T (T-u)x(u)du + \int_t^T (T-u)\sigma(u)dW(u) + (T-t)x(t) \\
&= -x(t)(T-t) + x(t)H(T-t) - \int_t^T \sigma(u)\left[(T-u) - \int_u^T e^{-\kappa(v-u)}dv\right] dW(u) \\
&\quad + \int_t^T (T-u)\sigma(u)dW(u) + (T-t)x(t) \\
&= x(t)H(T-t) + \int_t^T \sigma(u)\left(\int_u^T e^{-\kappa(v-u)}dv\right) dW(u) \\
&= x(t)H(T-t) + \int_t^T \sigma(u)H(T-u)dW(u)
\end{aligned}$$

That is, we have

$$\boxed{\begin{cases} \int_t^T x_1(u)du = x_1(t)H_1(T-t) + \int_t^T \sigma_1(u)H_1(T-u)dW_1(u) \\ \int_t^T x_2(u)du = x_2(t)H_2(T-t) + \int_t^T \sigma_2(u)H_2(T-u)dW_2(u) \end{cases}} \tag{16}$$

Then conditioning on $\mathcal{F}_t$, $(\int_t^T x_1(u)du, \int_t^T x_2(u)du)'$ is Gaussian with mean

$$\mu(t,T)_{integral} = E\left[\left(\int_t^T x_1(u)du, \int_t^T x_2(u)du\right)' \middle| \mathcal{F}_t\right] = (H_1(T-t)x_1(t), H_2(T-t)x_2(t))'$$

and covariance matrix

$$\Sigma(t,T)_{integral} = \begin{pmatrix} \int_t^T \sigma_1^2(u)H_1^2(T-u)du & \int_t^T \sigma_1(u)\sigma_2(u)H_1(T-t)H_2(T-t)\rho du \\ \int_t^T \sigma_1(u)\sigma_2(u)H_1(T-t)H_2(T-t)\rho du & \int_t^T \sigma_2^2(u)H_2^2(T-u)du \end{pmatrix}$$

2.4 Derivation of zero-coupon bond price

By formula (**??**) and formula (16) The price at time t of a zero-coupon bond maturing at time T and with unit face value is

$$\begin{aligned}
P(t,T) &= E^Q\left[e^{-\int_t^T r(u)du}\middle| \mathcal{F}_t\right] = e^{-\int_t^T \varphi(u)du} E^Q\left[e^{-\int_t^T (x_1(u)+x_2(u))du}\middle| \mathcal{F}_t\right] \\
&= \exp\left\{-\int_t^T \varphi(u)du - H_1(T-t)x_1(t) - H_2(T-t)x_2(t) + \frac{1}{2}V(t,T)\right\}
\end{aligned}$$

and

$$V(t,T) = \int_t^T \sigma_1^2(u)H_1^2(T-u)du + \int_t^T \sigma_2^2(u)H_2^2(T-u)du + 2\rho\int_t^T \sigma_1(u)\sigma_2(u)H_1(T-u)H_2(T-u)du.$$

A Summary of Girsanov's Theorem for continuous semimartingale

For sake of convenience, we record here a version of Girsanov's Theorem as presented in Revuz and Yor [3]. We will freely use jargons in the theory of continuous semimartingales.

Suppose $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$ is a filtered probability space that satisfies the usual hypotheses. Q is another probability measure such that $Q|_{\mathcal{F}_t}$ is absolutely continuous with respect to $P|_{\mathcal{F}_t}$. We call D_t the Radon-Nikodym derivative of Q with respect to P on $\mathcal{F}_t$. These random variables $(D_t)_{t\geq 0}$ form a $(\mathcal{F}_t, P)$-martingale and can be chosen in such a way that it has cadlag path a.s..

Theorem A.1 (Girsanov's Theorem). *If D is continuous, every continuous $(\mathcal{F}_t, P)$-semimartingale is a continuous $(\mathcal{F}_t, Q)$-semimartingale. More precisely, if M is a continuous $(\mathcal{F}_t, P)$-local martingale, then*

$$\widetilde{M} = M - D^{-1}.\langle M, D\rangle$$

is a continuous $(\mathcal{F}_t, Q)$-local martingale. Moreover, if N is another continuous P-local martingale,

$$\langle \widetilde{M}, \widetilde{N}\rangle = \langle \widetilde{M}, N\rangle = \langle M, N\rangle.$$

To apply Girsanov's Theorem more conveniently, we often use the following results.

Proposition A.1. *If D is a strictly positive continuous local martingale, there exists a unique continuous local martingale L such that*

$$D_t = \exp\left\{L_t - \frac{1}{2}\langle L, L\rangle_t\right\} = \mathcal{E}(L)_t;$$

L is given by the formula

$$L_t = \log D_0 + \int_0^t D_s^{-1} dD_s.$$

Theorem A.2. *If $Q = \mathcal{E}(L)\cdot P$ and M is a continuous P-local martingale, then*

$$\widetilde{M} = M - D^{-1}\cdot\langle M, D\rangle = M - \langle M, L\rangle$$

is a continuous Q-local martingale. Moreover, $P = \mathcal{E}(-L)^{-1}\cdot Q = \mathcal{E}(-\widetilde{L})\cdot Q$.

B Lognormal approximation of sum of lognormals

We consider the following problem: suppose $(Z_1, \cdots, Z_n)$ is an n-dimensional Gaussian random vector with mean μ and covariance matrix Σ. Find the lognormal approximation of $\sum_{i=1}^n w_i e^{Z_i}$ where $w_1, \cdots, w_n$ are constants.

The typical method is moment-matching. Recall for a Gaussian random vector $X \sim N(\mu, \Sigma)$, its characteristic function is

$$\phi(t) = E[e^{it'X}] = e^{it'\mu - \frac{1}{2}t'\Sigma t},$$

which implies $E[e^{t'X}] = e^{t'\mu + \frac{1}{2}t'\Sigma t}$. Therefore the first moment is

$$M := E\left[\sum_{i=1}^n w_i e^{Z_i}\right] = \sum_{i=1}^n w_i e^{\mu_i + \frac{1}{2}\Sigma_{ii}}.$$

The second moment is

$$V := E\left[\left(\sum_{i=1}^n w_i e^{Z_i}\right)^2\right] = \sum_{i,j=1}^n w_i w_j E[e^{Z_i + Z_j}] = \sum_{i,j=1}^n w_i w_j e^{\mu_i + \mu_j + \frac{1}{2}(\Sigma_{ii} + \Sigma_{jj}) + \Sigma_{ij}},$$

since $Z_i + Z_j \sim N(\mu_i + \mu_j, \Sigma_{ii} + \Sigma_{jj} + 2\Sigma_{ij})$.

Suppose we want to find a normal random variable $\xi \sim N(\hat{\mu}, \hat{\sigma}^2)$ such that

$$\begin{cases} E[e^{\xi}] = E[\sum_{i=1}^n w_i e^{Z_i}] \\ E[e^{2\xi}] = E[\left(\sum_{i=1}^n w_i e^{Z_i}\right)^2] \end{cases}$$

Then we need to solve the equation

$$\begin{cases} e^{\hat{\mu} + \frac{1}{2}\hat{\sigma}^2} = M \\ e^{2\hat{\mu} + 2\hat{\sigma}^2} = V \end{cases}$$

which gives

$$\begin{cases} \hat{\sigma}^2 = \ln \frac{V}{M^2} \\ \hat{\mu} = \ln M - \frac{1}{2}\hat{\sigma}^2 \end{cases}$$

C Characteristic function of a Gaussian random vector

Recall for a Gaussian random vector $X \sim N(\mu, \Sigma)$, its characteristic function is (see, for example, Anderson [1] pp.43, Theorem 2.6.1)

$$\phi(t) = E[e^{it'X}] = e^{it'\mu - \frac{1}{2}t'\Sigma t}.$$

References

[1] T. W. Anderson. *An introduction to multivariate statistical analysis*, 3rd edition. Wiley-Interscience, 2003. 15, 25

[2] Damiano Brigo and Fabio Mercurio. *Interest rate models - Theory and practice*. Springer, 2007. 5

[3] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, 3rd Edition. Springer, 2004. 23