HJM Model and Interest Rate Derivatives Modeling

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Abstract

A quick reference on interest rate modeling based on Piza[2].

Contents

1 No-arbitrage condition in HJM model, forward rate model, and short rate model

This section is basically an expanded version of Piza[2], which gives a "top-down", theoretician's point of view.

We use the Heath-Jarrow-Morton (HJM) model as the master key to unlock the modeling of zero-coupon bond price dynamics. The focus will be a sufficient condition for no arbitrage property, explained in three different but equivalent languages – bond price dynamics, forward rate dynamics, and short rate dynamics.

1.1 No-arbitrage condition in HJM model

We assume that we have a family of zero-coupon bonds traded in the market. The price at time t of a zero-coupon bond with maturity T ($0 \leq t \leq T$) will be denoted by $P(t,T)$. We assume the bond price satisfies the following SDE:

$$dP(t,T) = P(t,T)\big[A(t,T)dt + B(t,T)dW_t\big],\ P(T,T) = 1,\ A(T,T) = B(T,T) = 0,$$

where W is a 1-dimensional standard Brownian motion. Also traded is an asset N with positive price, [1] which will be chosen as the numéraire, and satisfies the following SDE:

$$dN_t = N_t(\mu_t^N dt + \sigma_t^N dW_t), N_0 = 1.$$

By the Fundamental Theorem of Asset Pricing, a necessary and sufficient condition for the no arbitrage property (more precisely, no-free-lunch-with-vanishing-risk, NFLVR) is that we can find a probability measure Q such that $\bar{P}(t,T) := \frac{P(t,T)}{N_t}$ is a Q-local martingale. By martingale representation theorem for Brownian filtration, the corresponding density process Z necessarily has the following form

$$Z_t := \frac{dQ_t}{dP_t} = \exp\left\{\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right\},$$

where θ is an adapted process that satisfies certain integrability condition (e.g. $P\left(\int_0^T \theta_s^2 ds < \infty\right) = 1$). For the no-arbitrage condition in terms of $A(t,T)$, $B(t,T)$, μ_t^N and σ_t^N, we divide our calculation in the following steps.

Step 1: The SDE for discounted bond price $\bar{P}(t,T) = \frac{P(t,T)}{N_t}$. By Itô's formula, we have

$$d\left(\frac{1}{N_t}\right) = -\frac{dN_t}{N_t^2} + \frac{d\langle N\rangle_t}{N_t^3} = -\frac{1}{N_t}\Big[\big(\mu_t^N - (\sigma_t^N)^2\big)dt + \sigma_t^N dW_t\Big],$$

and

$$\begin{aligned} d\bar{P}(t,T) &= \frac{1}{N_t}dP(t,T) + P(t,T)d\left(\frac{1}{N_t}\right) + d\left\langle P(\cdot,T), \frac{1}{N(\cdot)}\right\rangle_t \\ &= \bar{P}(t,T)\big[A(t,T)dt + B(t,T)dW_t\big] - \bar{P}(t,T)\Big[\big(\mu_t^N - (\sigma_t^N)^2\big)dt + \sigma_t^N dW_t\Big] \\ &\quad -\bar{P}(t,T)\sigma_t^N B(t,T)dt. \end{aligned}$$

So the discounted bond price dynamics is

$$\begin{aligned} \frac{d\bar{P}(t,T)}{\bar{P}(t,T)} &= \big[A(t,T) - \mu_t^N + (\sigma_t^N)^2 - \sigma_t^N B(t,T)\big]dt + \big[B(t,T) - \sigma_t^N\big]dW_t \\ &= \big[B(t,T) - \sigma_t^N\big]\left[\frac{A(t,T) - \mu_t^N + (\sigma_t^N)^2 - \sigma_t^N B(t,T)}{B(t,T) - \sigma_t^N}dt + dW_t\right], \end{aligned}$$

[1] We require "tradable" here, but actually any positive adapted process can be chosen as an numéraire, as far as we can find a martingale measure associated with this process as a numéraire.

provided $B(t,T) - \sigma_t^N \neq 0$, $0 \le t \le T$.

Step 2: The no-arbitrage condition. If the discounted bond price $\bar{P}(t,T)$ is a local martingale under Q which is defined via

$$Z_t := \frac{dQ_t}{dP_t} = \exp\left\{ \int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds \right\},$$

we necessarily have

$$\frac{A(t,T) - \mu_t^N + (\sigma_t^N)^2 - \sigma_t^N B(t,T)}{B(t,T) - \sigma_t^N} = -\theta_t.$$

Proposition 1.1. *(NA condition in terms of drifts and volatilities of bond price and numéraire) The market $(N, \mathcal{P})$ of a family of zero-coupon bonds and a numéraire N has no arbitrage if the following expression*

$$\frac{A(t,T) - \mu_t^N + (\sigma_t^N)^2 - \sigma_t^N B(t,T)}{B(t,T) - \sigma_t^N}$$

is independent of T. In that case, define

$$\theta_t = -\frac{A(t,T) - \mu_t^N + (\sigma_t^N)^2 - \sigma_t^N B(t,T)}{B(t,T) - \sigma_t^N},$$

then the local martingale measure Q can be defined via the density process

$$Z_t := \frac{dQ_t}{dP_t} = \exp\left\{ \int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds \right\}.$$

Remark 1. *In particular, if N is the money market account with 0 volatility and short interest rate r_t, the no-arbitrage property holds under the original probability if and only if $A(t,T) \equiv r_t$.*

1.2 No-arbitrage condition in forward rate model

Denote by $f(t,T)$ the forward rate, then $P(t,T) = \exp\{-\int_t^T f(t,s)ds\}$. So we can translate the content of Proposition 1.1 into the language of forward rate. Indeed, assume $f(t,T)$ follows the following SDE:

$$df(t,T) = a(t,T)dt + b(t,T)dW_t.$$

We have the following relations (see Shiryaev [3, page 292] and Shreve [4, page 426])

$$A(t,T) = f(t,t) - \int_t^T a(t,s)ds + \frac{1}{2}\left(\int_t^T b(t,s)ds\right)^2, \; B(t,T) = -\int_t^T b(t,s)ds$$

and

$$a(t,T) = \frac{\partial B(t,T)}{\partial T}B(t,T) - \frac{\partial A(t,T)}{\partial T}, \; b(t,T) = -\frac{\partial B(t,T)}{\partial T}.$$

Then Proposition 1.1 is equivalent to the existence of a process θ_t, which is independent of T, such that

$$\begin{aligned} & f(t,t) - \int_t^T a(t,s)ds + \frac{1}{2}\left(\int_t^T b(t,s)ds\right)^2 - \mu_t^N + (\sigma_t^N)^2 + \sigma_t^N \int_t^T b(t,s)ds \\ = & \left[\int_t^T b(t,s)ds + \sigma_t^N\right]\theta_t. \end{aligned}$$

Differentiating with respect to T, the above equation becomes

$$-a(t,T) + \int_t^T b(t,s)ds \cdot b(t,T) + \sigma_t^N b(t,T) = b(t,T)\theta_t.$$

Proposition 1.2. *(NA condition in terms of drifts and volatilities of forward rate and numéraire) The market* $(N, \mathcal{P})$ *of a family of zero-coupon bonds and a numéraire* N *has no arbitrage if the following expression*

$$-\frac{a(t,T)}{b(t,T)} + \int_t^T b(t,s)ds + \sigma_t^N$$

is independent of T*. In that case, define*

$$\theta_t = -\frac{a(t,T)}{b(t,T)} + \int_t^T b(t,s)ds + \sigma_t^N,$$

then the local martingale measure Q *can be defined via the density process*

$$Z_t := \frac{dQ_t}{dP_t} = \exp\left\{\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right\}.$$

1.3 No-arbitrage condition in short rate model

Finally, we rephrase the no-arbitrage condition in the setting of short rate model. We assume the short rate r_t follows the following SDE:

$$dr_t = \alpha(t, r_t)dt + \beta(t, r_t)dW_t,$$

and we choose the money market account $B_t(r) = e^{\int_0^t r_s ds}$ as the numéraire. The bond prices $P(t,T)$ should evolve in such a way that there is an equivalent probability measure Q such that

$$P(t,T) = E^Q\left[\exp\left\{-\int_t^T r_s ds\right\}\bigg| \mathcal{F}_t\right]. \tag{1}$$

Formula (1) makes Q a martingale measure and hence, guarantees the no-arbitrage property.

We start our quest for Q with the observation that r_t is a Markov process (non-homogeneous or homogeneous). Therefore according to formula (1), $P(t,T)$ must be a function of t, r_t, and T. We let $F^T = F(t, r_t, T) = P(t,T)$. Then Itô's formula yields

$$\begin{aligned} dF^T &= \left(\frac{\partial F^T}{\partial t} + \alpha\frac{\partial F^T}{\partial r} + \frac{1}{2}\beta^2\frac{\partial^2 F^T}{\partial r^2}\right)dt + \beta\frac{\partial F^T}{\partial r}dW_t \\ &= F^T\left[A^T(t,r_t)dt + B^T(t,r_t)dW_t\right], \end{aligned}$$

where

$$A^T(t,r) = \frac{\frac{\partial F^T}{\partial t} + \alpha\frac{\partial F^T}{\partial r} + \frac{1}{2}\beta^2\frac{\partial^2 F^T}{\partial r^2}}{F^T},\ B^T(t,r) = \frac{\beta\frac{\partial F^T}{\partial r}}{F^T}.$$

Recall in Proposition 1, no-arbitrage property holds if

$$\frac{A(t,T) - \mu_t^N + (\sigma_t^N)^2 - \sigma_t^N B(t,T)}{B(t,T) - \sigma_t^N}$$

is independent of T. In the current setting of short rate model and money market account as the numéraire, we have $\mu_t^N = r_t$, $\sigma_t^N = 0$, $A^T(t,r_t) = A(t,T)$, and $B^T(t,r_t) = B(t,T)$. So we need

$$\frac{A^T(t,r_t) - r_t}{B^T(t,r_t)}$$

to be independent of T. If this is true, we can define $\theta(t,r) = -\frac{A^T(t,r)-r}{B^T(t,r)}$ and the equivalent martingale measure Q can be given via the density process

$$Z_t := \frac{dQ_t}{dP_t} = \exp\left\{\int_0^t \theta(s,r_s)dW_s - \frac{1}{2}\int_0^t \theta^2(s,r_s)ds\right\}.$$

This condition is equivalent to the function $F^T = F(t, r, T)$ $(0 \le t < T)$ satisfying the fundamental equation

$$\frac{\partial F}{\partial t} + (\alpha + \theta\beta)\frac{\partial F}{\partial r} + \frac{1}{2}\beta^2 \frac{\partial^2 F}{\partial r^2} = rF \tag{2}$$

with boundary condition $F(T, r, T) = 1$, $T > 0$, $r \ge 0$. This gives the necessary condition under which a market $(B(r), \mathcal{P})$ of the money market account and a family of zero-coupon bonds with $P(t,T) = F(t, r_t, T)$ is arbitrage-free.

Conversely, for any given (deterministic) function $\theta(t, r)$, if the fundamental equation (2) has a solution satisfying the boundary condition, we can define an equivalent probability measure under which the bond price discounted by money market account is a local martingale.

In summary, we have

Proposition 1.3. *(NA condition in the short rate model) The market $(B(r), \mathcal{P})$ of the money market account and a family of zero-coupon bonds has no arbitrage if we can find a (deterministic) function $\theta(t, r)$ such that the fundamental equation (2) has a solution satisfying the boundary condition $F(T, r, T) = 1$. In that case, the equivalent martingale measure Q can be given via the density process*

$$Z_t := \frac{dQ_t}{dP_t} = \exp\left\{\int_0^t \theta(s, r_s) dW_s - \frac{1}{2}\int_0^t \theta^2(s, r_s) ds\right\},$$

and the bond price $P(t,T) = F(t, r_t, T)$.

2 Examples

In this section, we study various examples. We are mostly concerned with explicit formulas for the following quantities: the forward rate $f(t,T)$, the short rate r_t, the numéraire N_t, the bond price $P(t,T)$, and the discounted bond price $\bar{P}(t,T) = \frac{P(t,T)}{N_t}$.

2.1 Specification of forward rate volatility in HJM

Assumption 1: *Suppose the original probability is already a martingale measure, i.e. bond prices $P(t,T)$ discounted by numéraire N are (local) martingales under original probability measure.* Under this assumption, the no-arbitrage condition in Proposition 1.1 becomes

$$A(t,T) - \mu_t^N + \left[\sigma_t^N - B(t,T)\right]\sigma_t^N = 0;$$

and the no-arbitrage condition in Proposition 1.2 becomes

$$\frac{a(t,T)}{b(t,T)} = \int_t^T b(t,s) ds + \sigma_t^N.$$

There are four variables in the first equation and three variables in the second equation. So for simplicity, we want to work under the forward rate framework. To simplify the term $\int_t^T b(t,s)ds$, we further assume **Assumption 2**: *$b(t,T) = H_T' \alpha_t$, where H and α are two deterministic functions with $H_0 = 0$.* Then the no-arbitrage condition in forward rate model becomes

$$\frac{a(t,T)}{b(t,T)} = (H_T - H_t)\alpha_t + \sigma_t^N.$$

Therefore

$$a(t,T) = H_T' \alpha_t [(H_T - H_t)\alpha_t + \sigma_t^N] = H_T' H_T \alpha_t^2 + H_T' \alpha_t(-H_t \alpha_t + \sigma_t^N).$$

The forward rate $f(t,T)$ is therefore

$$f(t,T) = f(0,T) + H_T' H_T \int_0^t \alpha_s^2 ds + H_T' \int_0^t \alpha_s(-H_s\alpha_s + \sigma_s^N) ds + H_T' X_t,$$

where $X_t = \int_0^t \alpha_s dW_s$. Consequently, the short rate $r_t = f(t,t)$ satisfies

$$r_t = f(0,t) + H_t' H_t \int_0^t \alpha_s^2 ds + H_t' \int_0^t \alpha_s(-H_s\alpha_s + \sigma_s^N) ds + H_t' X_t.$$

According to the formula $P(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}$, we have

$$\begin{aligned} P(t,T) &= \exp\left\{-\int_t^T f(0,s)ds - \frac{1}{2}(H_T^2 - H_t^2)\int_0^t \alpha_s^2 ds - (H_T - H_t)\right. \\ &\quad \left.\cdot\left[\int_0^t \alpha_s(-H_s\alpha_s + \sigma_s^N)ds + X_t\right]\right\}. \end{aligned}$$

Since $B(t,T) = -\int_t^T b(t,s)ds = -\alpha_t(H_T - H_t)$, from the equation

$$\frac{d\bar{P}(t,T)}{\bar{P}(t,T)} = \left[B(t,T) - \sigma_t^N\right]dW_t = -\left[(H_T - H_t)\alpha_t + \sigma_t^N\right]dW_t,$$

and assuming $N_0 = 1$, we have

$$\begin{aligned} \bar{P}(t,T) &= P(0,T)\mathcal{E}\left(\int_0^t \left[-\alpha_s(H_T - H_s) - \sigma_s^N\right]dW_s\right) \\ &= \exp\left\{-\int_0^T f(0,s)ds - \int_0^t \left[(H_T - H_s)\alpha_s + \sigma_s^N\right]dW_s - \frac{1}{2}\int_0^t \left[\alpha_s(H_T - H_s) + \sigma_s^N\right]^2 ds\right\} \\ &= \exp\left\{-\int_0^T f(0,s)ds - (H_T - H_t)X_t - \int_0^t H_s' X_s ds - \int_0^t \sigma_s^N dW_s\right. \\ &\quad \left.-\frac{1}{2}\int_0^t \left[\alpha_s(H_T - H_s) + \sigma_s^N\right]^2 ds\right\}. \end{aligned}$$

Hence, the numéraire N is equal to

$$\begin{aligned} N_t &= \frac{P(t,T)}{\bar{P}(t,T)} \\ &= \exp\left\{\int_0^t f(0,s)ds + \int_0^t H_s' X_s ds + \int_0^t \sigma_s^N dW_s - \frac{1}{2}(H_T^2 - H_t^2)\int_0^t \alpha_s^2 ds\right. \\ &\quad \left.-(H_T - H_t)\int_0^t \alpha_s(-H_s\alpha_s + \sigma_s^N)ds + \frac{1}{2}\int_0^t \left[(H_T - H_s)\alpha_s + \sigma_s^N\right]^2 ds\right\} \\ &= \exp\left\{\int_0^t f(0,s)ds + \int_0^t H_s' X_s ds + \int_0^t \sigma_s^N dW_s + \frac{1}{2}\int_0^t \left[(H_t - H_s)\alpha_s + \sigma_s^N\right]^2 ds\right\}. \end{aligned}$$

2.2 Linear Gaussian model (LGM) via HJM specification

In view of the class of models with the specification $b(t,T) = H_T'\alpha_t$, we note we can further simplify the expression of $a(t,T)$ if we
Assumption 3: *assume* $\sigma_t^N = H_t\alpha_t$. Then under Assumption 1-3,

$$a(t,T) = b(t,T)H_T\alpha_t = H_T' H_T \alpha_t^2$$

and

$$df(t,T) = a(t,T)dt + b(t,T)dW_t = H_T' H_T \alpha_t^2 dt + H_T' \alpha_t dW_t.$$

Define $\zeta_t = \int_0^t \alpha_s^2 ds$ and $X_t = \int_0^t \alpha_s dW_s$. We then have

$$\boxed{\begin{cases} f(t,T) = f(0,T) + H'_T H_T \zeta_t + H'_T X_t \\ r_t = f(0,t) + H'_t H_t \zeta_t + H'_t X_t \\ P(t,T) = \exp\left\{-\int_t^T f(0,s)ds - [H_T - H_t]X_t - \frac{1}{2}[H_T^2 - H_t^2]\zeta_t\right\} \\ \bar{P}(t,T) = \exp\left\{-\int_0^T f(0,s)ds - H_T X_t - \frac{1}{2}H_T^2\zeta_t\right\} \\ N_t = \frac{P(t,T)}{\bar{P}(t,T)} = \exp\left\{\int_0^t f(0,s)ds + H_t X_t + \frac{1}{2}H_t^2\zeta_t\right\} \end{cases}}$$

The model induced by Assumption 1-3 is called linear Gaussian model (LGM) by Patrick Hagan [1]. What is presented above is the 1-factor case. It can be extended to 2-factor case and makes connection with Hull-White model. For more details, see later.

2.3 Hull-White short rate model via HJM specification

We consider the HJM model and we assume the original probability is the martingale measure. The no-arbitrage condition

$$\frac{a(t,T)}{b(t,T)} = \int_t^T b(t,s)ds + \sigma_t^N$$

allows us to specify any two of the three variables, σ^N, $a(t,T)$, and $b(t,T)$. We therefore set $b(t,T) = H'_T\alpha_t$ and

Assumption $3'$: *choose the money market account as the numéraire so that* $\sigma_t^N \equiv 0$. Then the forward rate is fully specified by the SDE

$$df(t,T) = a(t,T)dt + b(t,T)dW_t$$

with $b(t,T) = H'_T\alpha_t$, $a(t,T) = H'_T(H_T - H_t)\alpha_t^2$. By the calculation in Section 2.1 (recall $\zeta_t = \int_0^t \alpha_s^2 ds$), we have

$$r_t = f(0,t) + H'_t H_t \zeta_t - H'_t \int_0^t H_s \zeta'_s ds + H'_t X_t.$$

Taking differentiation, we have (assuming $f(0,t)$ as a function of t is of finite variation)

$$\begin{aligned} dr_t &= \left[\frac{\partial f(0,t)}{\partial t} + H''_t H_t \zeta_t + (H'_t)^2\zeta_t + H'_t H_t \zeta'_t - H''_t \int_0^t H_s \zeta'_s ds - H'_t H_t \zeta'_t + H''_t X_t\right] dt + H'_t dX_t \\ &= \frac{H''_t}{H'_t}\left[\frac{H'_t}{H''_t}\frac{\partial f(0,t)}{\partial t} + H'_t H_t \zeta_t + \frac{(H'_t)^3}{H''_t}\zeta_t - H'_t \int_0^t H_s \zeta'_s ds + H'_t X_t\right] dt + H'_t dX_t \\ &= \frac{H''_t}{H'_t}\left[r_t - f(0,t) + \frac{H'_t}{H''_t}\frac{\partial f(0,t)}{\partial t} + \frac{(H'_t)^3}{H''_t}\zeta_t\right] dt + H'_t dX_t \\ &= \left\{-\frac{H''_t}{H'_t}\left[f(0,t) - \frac{H'_t}{H''_t}\frac{\partial f(0,t)}{\partial t} - \frac{(H'_t)^3}{H''_t}\zeta_t\right] + \frac{H''_t}{H'_t} r_t\right\} dt + H'_t \alpha_t dW_t. \end{aligned}$$

Define

$$\boxed{\begin{cases} \sigma_t^r = H'_t\alpha_t \\ \kappa_t = -\frac{H''_t}{H'_t} \\ \theta_t = -\frac{f(0,t)}{H'_t} + \frac{\partial f(0,t)}{\partial t} + (H'_t)^2\zeta_t. \end{cases}} \tag{3}$$

The SDE for r_t becomes

$$dr_t = (\theta_t - \kappa_t r_t)dt + \sigma_t^r dW_t,$$

which is the (1-factor) Hull-White short rate model. We can solve equation (3) for H_t and ζ_t and obtain

$$\boxed{\begin{cases} H_t = c\int_0^t e^{-\int_0^s \kappa_u du} ds + H_0 \\ \zeta_t = \frac{1}{c^2}\int_0^t (\sigma_s^r)^2 e^{\int_0^s 2\kappa_u du} ds. \end{cases}}$$

where c is a constant.

2.4 2-factor LGM model

So far, we have viewed the LGM model from the viewpoint of HJM, which is a kind of top-down, theoretician's perspective. We can also view LGM from a "bottom-up", empiricist's perspective. More precisely, given a filtration, we first choose an adapted positive process N_t as the numéraire. Then we *define* the prices of zero-coupon bonds via the risk-neutral pricing formula under the original probability:

$$P(t,T) = N_t E\left[\left.\frac{1}{N_T}\right|\mathcal{F}_t\right].$$

By the Fundamental Theorem of Asset Pricing, this will define a term structure of zero-coupon bond prices which is arbitrage-free. Note the market of traded assets is the bond market. The numéraire to be specified does not have to be a tradable asset.

In the one-factor LGM model, the numéraire is

$$N_t = \exp\left\{\int_0^t f(0,s)ds + H_t X_t + \frac{1}{2}H_t^2\zeta_t\right\} = \frac{1}{P(0,t)}\exp\left\{H_t X_t + \frac{1}{2}H_t^2\zeta_t\right\},$$

where H_t is a deterministic function of t, ζ_t is an increasing deterministic function of t with $\zeta_0 = 0$, and $X_t \sim N(0,\zeta_t)$. In the two-factor LGM model, we specify a two-factor state process $X_t = \begin{bmatrix} X_t^1 \\ X_t^2 \end{bmatrix}$, with

$$X_t^1 = \int_0^t \alpha_1(u)dW_u^1,\ X_t^2 = \int_0^t \alpha_2(u)dW_u^2,$$

where $\alpha_1(t)$ and $\alpha_2(t)$ are deterministic functions of t, and $W_t = \begin{bmatrix} W_t^1 \\ W_t^2 \end{bmatrix}$ is a two-dimensional Brownian motion with instantaneous correlation $dW_t^1 dW_t^2 = \rho(t)dt$. Define $\zeta_{11}(t) = \int_0^t \alpha_1^2(u)du$, $\zeta_{22}(t) = \int_0^t \alpha_2^2(u)du$, $\zeta_{12}(t) = \zeta_{21}(t) = \int_0^t \rho(u)\alpha_1(u)\alpha_2(u)du$, and

$$\zeta_t = \begin{bmatrix} \zeta_{11}(t) & \zeta_{12}(t) \\ \zeta_{21}(t) & \zeta_{22}(t) \end{bmatrix}.$$

Let H_t be a deterministic vector function of t with values in $\mathbb{R}^2$: $H_t = \begin{bmatrix} H_t^1 \\ H_t^2 \end{bmatrix}$. Define the numéraire by the formula

$$N_t = N(t,X_t) = \frac{1}{P(0,t)}\exp\left\{H_t^{tr}X_t + \frac{1}{2}H_t^{tr}\zeta_t H_t\right\}.$$

Suppose we have a contingent claim $F(T,X_T)$ with maturity T and dependent on the state variable at T. At a time $t < T$, its arbitrage-free price is given by the formula (we adopt the convention that $V(T,X_T) = F(T,X_T)$ and note $X_T - X_t$ is Gaussian)

$$\begin{aligned} V(t,X_t) &= N(t,X_t)E\left[\left.\frac{V(T,X_T)}{N(T,X_T)}\right|\mathcal{F}_t\right] \\ &= N(t,X_t)E\left[\left.\frac{V(T,X_T - X_t + X_t)}{N(T,X_T - X_t + X_t)}\right|\mathcal{F}_t\right] \\ &= \frac{N(t,X_t)}{2\pi\sqrt{\det|\Delta\zeta|}}\int_{\mathbb{R}^2}\frac{V(T,\xi+X_t)}{N(T,\xi+X_t)}\exp\left\{-\frac{1}{2}\xi^{tr}(\Delta\zeta)^{-1}\xi\right\}d\xi_1 d\xi_2 \\ &= \frac{N(t,X_t)}{2\pi\sqrt{|\det\Delta\zeta|}}\int_{\mathbb{R}^2}\frac{V(T,y)}{N(T,y)}\exp\left\{-\frac{1}{2}(y-X_t)^{tr}(\Delta\zeta)^{-1}(y-X_t)\right\}dy_1 dy_2, \end{aligned}$$

where $\Delta\zeta = \begin{bmatrix} \zeta_{11}(T) - \zeta_{11}(t) & \zeta_{12}(T) - \zeta_{12}(t) \\ \zeta_{21}(T) - \zeta_{21}(t) & \zeta_{22}(T) - \zeta_{22}(t) \end{bmatrix}$. In particular, the discounted time-t price of a zero-coupon bond maturing at time T is (note $\Delta\zeta$ is a symmetric matrix)

$$\begin{aligned}
& \bar{P}(t,T) \\
= \; & \frac{P(0,T)}{2\pi\sqrt{|\det \Delta\zeta|}} \int_{\mathbb{R}^2} \exp\left\{ -H_T^{tr} y - \frac{1}{2} H_T^{tr} \zeta_T H_T - \frac{1}{2}(y - X_t)^{tr} (\Delta\zeta)^{-1} (y - X_t) \right\} dy_1 dy_2 \\
= \; & \frac{P(0,T) e^{-H_T^{tr} X_t - \frac{1}{2} H_T^{tr} \zeta_t H_T}}{2\pi\sqrt{|\det \Delta\zeta|}} \times \\
& \int_{\mathbb{R}^2} \exp\left\{ -H_T^{tr}(y - X_t) - \frac{1}{2} H_T^{tr} \Delta\zeta H_T - \frac{1}{2}(y - X_t)^{tr} (\Delta\zeta)^{-1} (y - X_t) \right\} dy_1 dy_2 \\
= \; & \frac{P(0,T) e^{-H_T^{tr} X_t - \frac{1}{2} H_T^{tr} \zeta_t H_T}}{2\pi\sqrt{|\det \Delta\zeta|}} \int_{\mathbb{R}^2} \exp\left\{ -\frac{1}{2}(y - X_t + \Delta\zeta H_T)^{tr} (\Delta\zeta)^{-1} (y - X_t + \Delta\zeta H_T) \right\} dy_1 dy_2 \\
= \; & P(0,T) \exp\left\{ -H_T^{tr} X_t - \frac{1}{2} H_T^{tr} \zeta_t H_T \right\}.
\end{aligned}$$

Consequently, the time-t price of a zero-coupon bond maturing at time T is

$$P(t,T) = \bar{P}(t,T) N(t, X_t) = \frac{P(0,T)}{P(0,t)} \exp\left\{ -(H_T - H_t)^{tr} X_t - \frac{1}{2} H_T^{tr} \zeta_t H_T + \frac{1}{2} H_t^{tr} \zeta_t H_t \right\}.$$

Note the price of a zero-coupon bond can be represented by forward rate:

$$P(0,T) = e^{-\int_0^T f(0,s) ds}, \; P(t,T) = e^{-\int_t^T f(t,s) ds}.$$

So by taking differentiation with respect to T, we have

$$f(t,T) = f(0,T) + (H_T')^{tr} X_t + (H_T')^{tr} \zeta_t H_T.$$

In summary, we have the following formulas in the two-factor LGM model for forward rate, short rate, zero-coupon bond price, discounted zero-coupon bond price, and numéraire:

$$\begin{cases}
f(t,T) = f(0,T) + (H_T')^{tr} X_t + (H_T')^{tr} \zeta_t H_T \\
r_t = f(0,t) + (H_t')^{tr} X_t + (H_t')^{tr} \zeta_t H_t \\
P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left\{ -(H_T - H_t)^{tr} X_t - \frac{1}{2} H_T^{tr} \zeta_t H_T + \frac{1}{2} H_t^{tr} \zeta_t H_t \right\} \\
\bar{P}(t,T) = P(0,T) \exp\left\{ -H_T^{tr} X_t - \frac{1}{2} H_T^{tr} \zeta_t H_T \right\} \\
N(t, X_t) = \frac{1}{P(0,t)} \exp\left\{ H_t^{tr} X_t + \frac{1}{2} H_t^{tr} \zeta_t H_t \right\} \\
V(t, X_t) = \frac{N(t, X_t)}{2\pi\sqrt{|\det \Delta\zeta|}} \int_{\mathbb{R}^2} \frac{V(T,y)}{N(T,y)} \exp\left\{ -\frac{1}{2}(y - X_t)^{tr} (\Delta\zeta)^{-1} (y - X_t) \right\} dy_1 dy_2
\end{cases}$$

Sometimes, we would like to calculate the price $V(t, X_t)$ of a contingent claim directly instead of its discounted price, due to numerical instability introduced by large numéraire. For this purpose, recall for$\Delta\zeta =$

$\begin{bmatrix} \zeta_{11}(T)-\zeta_{11}(t) & \zeta_{12}(T)-\zeta_{12}(t) \\ \zeta_{21}(T)-\zeta_{21}(t) & \zeta_{22}(T)-\zeta_{22}(t) \end{bmatrix}$ and $N_t = N(t, X_t) = \frac{1}{P(0,t)} \exp\left\{H_t^{tr} X_t + \frac{1}{2} H_t^{tr} \zeta_t H_t\right\}$, we have

$$\begin{aligned}
& V(t, X_t) \\
= \; & \frac{N(t, X_t)}{2\pi\sqrt{|\det \Delta\zeta|}} \int_{\mathbb{R}^2} \frac{V(T,y)}{N(T,y)} \exp\left\{-\frac{1}{2}(y - X_t)^{tr}(\Delta\zeta)^{-1}(y - X_t)\right\} dy_1 dy_2 \\
= \; & \frac{P(0,T)}{P(0,t)} \int_{\mathbb{R}^2} \frac{V(T,y)\exp\left\{H_t^{tr}X_t + \frac{1}{2}H_t^{tr}\zeta_t H_t - H_T^{tr} y - \frac{1}{2}H_T^{tr}\zeta_T H_T - \frac{1}{2}(y - X_t)^{tr}(\Delta\zeta)^{-1}(y - X_t)\right\}}{2\pi\sqrt{|\det \Delta\zeta|}} dy \\
= \; & \frac{P(0,T)}{P(0,t)} \exp\left\{-(H_T - H_t)^{tr} X_t - \frac{1}{2} H_T^{tr}\zeta_t H_T + \frac{1}{2} H_t^{tr} \zeta_t H_t\right\} \times \\
& \int_{\mathbb{R}^2} \frac{V(T,y)\exp\left\{H_T^{tr} X_t - \frac{1}{2}H_T^{tr}\Delta\zeta H_T - H_T^{tr} y - \frac{1}{2}(y - X_t)^{tr}(\Delta\zeta)^{-1}(y - X_t)\right\}}{2\pi\sqrt{|\det \Delta\zeta|}} dy \\
= \; & P(t,T) \int_{\mathbb{R}^2} \frac{V(T,y)\exp\left\{-\frac{1}{2}(y - X_t + \Delta\zeta H_T)^{tr}(\Delta\zeta)^{-1}(y - X_t + \Delta\zeta H_T)\right\}}{2\pi\sqrt{|\det \Delta\zeta|}} dy.
\end{aligned}$$

In summary, for the pricing of a European contingent claim, we have

$$\boxed{V(t, X_t) = P(t,T) \int_{\mathbb{R}^2} \frac{V(T,y)\exp\left\{-\frac{1}{2}(y - X_t + \Delta\zeta H_T)^{tr}(\Delta\zeta)^{-1}(y - X_t + \Delta\zeta H_T)\right\}}{2\pi\sqrt{|\det \Delta\zeta|}} dy}$$

2.5 Connection between 2-factor LGM model and 2-factor Hull-White model

This section explains the connection between Hull-White model and the LGM model. It's basically a summary of Hagan [1], Section 2.

The two-factor Hull-White model takes the money market account as the numéraire, and assumes the short rate process has the following dynamics under risk-neutral measure:

$$r_t = \theta(t) + Y_t^1 + Y_t^2,$$

where

$$\begin{cases} dY_t^1 = -\kappa_1(t) Y_t^1 dt + \sigma_1(t) dW_t^1, & Y_0^1 = 0, \\ dY_t^2 = -\kappa_2(t) Y_t^2 dt + \sigma_2(t) dW_t^2, & Y_0^1 = 0, \\ dW_t^1 dW_t^2 = \rho(t) dt, \end{cases}$$

and the deterministic function $\theta(t)$ is used to make the model match today's discount curve.

The LGM model parameters are five functions of time: the symmetric covariance matrix ζ_t and the *response* or *mean reversion* function H_t. The Hull-White model has the two local volatility functions $\sigma_j(t)$, the two mean reversion functions $\kappa_j(t)$, the correlation between the factors $\rho(t)$, and $\theta(t)$ $(j = 1, 2)$.

The mean reversion functions H_t^j $(j = 1, 2)$ of LGM can be obtained from the mean reversion parameter of Hull-White:

$$\boxed{H_t^j = \int_0^t e^{-\int_0^s \kappa_j(u) du} ds + K_j, \; j = 1, 2,}$$

where K_j is some constant. Conversely, the mean reversion parameters of Hull-White can be obtained by

$$\boxed{\kappa_j(t) = -\frac{(H_t^j)''}{(H_t^j)'}, \; j = 1, 2.}$$

The symmetric covariance matrix ζ_t of LGM can be obtained from the local volatility parameters of Hull-White:

$$\boxed{\zeta_{11}(t) = \int_0^t \sigma_1^2(s) e^{2\int_0^s \kappa_1(u) du} ds, \; \zeta_{22}(t) = \int_0^t \sigma_2^2(s) e^{2\int_0^s \kappa_2(u) du} ds,}$$

$$\boxed{\zeta_{12}(t) = \zeta_{21}(t) = \int_0^t \rho(s)\sigma_1(s)\sigma_2(s)e^{\int_0^s [\kappa_1(u)+\kappa_2(u)]du}ds.}$$

Conversely, the HW parameters can be determined from ζ_t of LGM:

$$\boxed{\sigma_1(t) = (H_t^1)'\sqrt{\zeta_{11}'(t)},\ \sigma_2(t) = (H_t^2)'\sqrt{\zeta_{22}'(t)},\ \rho(t) = \frac{\zeta_{12}'(t)}{\sqrt{\zeta_{11}'(t)\zeta_{22}'(t)}}.}$$

For one-factor LGM and Hull-White models, the above relations become:

$$\boxed{\kappa(t) = -\frac{H''(t)}{H'(t)},\ \sigma(t) = H'(t)\sqrt{\zeta'(t)},\ H(t) = A\int_0^t e^{-\int_0^s \kappa(u)du}ds + B,\ \zeta(t) = \frac{1}{A^2}\int_0^t \sigma^2(s)e^{2\int_0^s \kappa(u)du}ds.}$$

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