# Fundamental Theorem of Asset Pricing in a Nutshell: With a View toward Numéraire Change

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**Abstract**

A survey of the Fundamental Theorem of Asset Pricing in mathematical finance.

## Contents

# 1 Introduction

This note serves as a summary of the numéraire change techniques as presented in Brigo and Mercurio [3] §2.2 and §2.3. The idea is to present the main results in a logically "natural" order so that we can easily remember them.

Roughly speaking, we are concerned with the following question: *Is the Fundamental Theorem of Asset Pricing (FTAP) invariant under numéraire change?*

The answer is *negative*. The key idea of this presentation is therefore the following: the version of FTAP as formulated in Delbaen and Schachermayer [8] starts from relaxing the no-arbitrage condition, so that it states its most general results in terms of local martingales or even $\sigma$-martingales. This is insufficient for the practical usage of risk-neutral pricing, as we really need martingale property, not *local* martingale property. We should instead start from the other way around: insist on martingale property and derive the right formulation of no-arbitrage.

In Section 2 and 4, we review the classical formulation of the first and second Fundamental Theorem of Asset Pricing (FTAP). Such a formulation is not invariant under numéraire change. In Section 3 and 5, we review the "right" formulation of Fundamental Theorem of Asset Pricing, insisting on martingale measures instead of local martingales measures. Such a formulation turns out to be invariant under numéraire change. In Section 6, we give concrete formulas for the case of Itô processes.

This note is based on a series of papers by Delbaen and Schachermayer ([8], [9], [10], [11], [12]), Schachermayer [24], Geman et al. [16], Yan et al. ([29], [22], [30], [28]), and Shiryaev [25], as well as the references therein.

# 2 Fundamental Theorem of Asset Pricing: classical formulation

We first summarize the state of the art before Delbaen and Schachermayer [8]. The case when the time set is finite is completely settled in Dalang et al. [5] and the use of simple or even elementary integrands as trading strategies is no restrction at all. For the case of discrete but infinite time sets, the problem is solved in Schachermayer [24]; the case of continuous and bounded processes in continuous time is solved in Delbaen [6]. In these two cases the theorems are stated in terms of simple integrands and limits of sequences and by using the concept of *no free lunch with bounded risk*.

To state the results of Delbaen and Schachermayer [8], we consider a probability space $(\Omega, \mathcal{F}, P)$ and a right-continuous filtration $\mathbb{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}$ $(T \leq \infty)$. In the given economy, $(K+1)$ non-dividend paying securities are traded continuously from time 0 until time $T$. Their prices are modeled by a $(K+1)$-dimensional adapted, positive semimartingale $S = \{(S_t^0, S_t^1, \cdots, S_t^K) : 0 \leq t \leq T\}$. We assume $S^0 \equiv 1$.[1]

**Definition 2.1.** *(Yan [29] p.661, p.662) A* **trading strategy** *is an $\mathbb{R}^{K+1}$-valued predictable process $\phi = \{\phi_t : 0 \leq t \leq T\}$ which is integrable w.r.t semimartingale $S$.*[2] *The* **value process** *associated with a strategy $\phi$ is defined by*

$$V_t(\phi) = \phi_t S_t = \sum_{k=0}^{K} \phi_t^k S_t^k, 0 \leq t \leq T,$$

*and the* **gains process** *associated with a strategy $\phi$ is defined by*

$$G_t(\phi) = (\phi \cdot S)_t := \int_0^t \phi_u dS_u = \sum_{k=0}^{K} \int_0^t \phi_u^k dS_u^k, 0 \leq t \leq T.$$

*A trading strategy $\phi$ is* $\mathbf{self-financing}$ *if*

$$V_t(\phi) = V_0(\phi) + G_t(\phi), 0 \leq t < T.$$

[1] This assumption hides the fact that we need a numéraire (i.e. a positive discounter), since 1 is used as the numéraire.

[2] We need here the notion of integration w.r.t. a vector-valued semimartingale, which is is defined globally, not componentwisely (see Jacod [19]). This is because the notion of componentwise stochastic integral is insufficient for stating FTAP in the most general setting (see Shiryaeve [25] p.635, Shiryaev and Cherny [26]). However, when $\phi$ is locally bounded, componentwise integration is sufficient for stating FTAP. See Cherny [4] for more details.

**Definition 2.2.** *(Delbaen and Schachermayer [8] Definition 2.7) A trading strategy $\phi$ is* **admissible** *if $G_t(\phi)$ is bounded from below, i.e. there is a constant $M$ such that $G_t(\phi) \geq -M$ a.s. for all $t \geq 0$.*

**Definition 2.3.** *(Delbaen and Schachermayer [10] or Shiryaev [25] p.650, VII §2a.2 Definition 1) We say that the vector of price processes $S$ satisfies the condition of* **no arbitrage** *(NA) at time $T$ if for each self-financing strategy $\phi$, we have*

$$P(G_T(\phi) \geq 0) = 1 \Rightarrow P(G_T(\phi) = 0) = 1.$$

The above concept of no arbitrage is already sufficient for stating FTAP in the discrete time case, but for continuous time case, we need the following concept:

**Definition 2.4.** *(Delbaen and Schachermayer [8] Definition 2.8 or Shiryaev [25] p.650, VII §2a.2 Definition 3) Let*

$$K = \{G_T(\phi) | \phi \text{ admissible and } G_\infty(\phi) = \lim_{t\to\infty} G_t(\phi) \text{ exists a.s. if } T = \infty\}$$

*and*

$$C = \{g \in L^\infty(\Omega, \mathcal{F}_T, P) |\ g \leq f \text{ for some } f \in K\}.$$

*We say that $S$ satisfies the condition of* **no free lunch with vanishing risk** *(NFLVR) for admissible strategies, if*

$$\bar{C} \cap L^\infty_+(\Omega, \mathcal{F}_T, P) = \{0\},$$

*where $\bar{C}$ denotes the closure of $C$ with respect to the norm topology of $L^\infty(\Omega, \mathcal{F}_T, P)$.*

To understand intuitively the NFLVR condition, we note $S$ allows for a free lunch with vanishing risk, if there is $f \in L^\infty_+(\Omega, \mathcal{F}_T, P) \setminus \{0\}$, a sequence $(G_T(\phi_n))_{n=0}^\infty \subset K$, where $(\phi^n)_{n=0}^\infty$ is a sequence of admissible integrands, and $(g_n)_{n=0}^\infty \subset L^\infty(\Omega, \mathcal{F}_T, P)$ satisfying $g_n \leq G_T(\phi_n)$, such that

$$\lim_{n\to\infty} \|f - g_n\|_{L^\infty(\Omega, \mathcal{F}_T, P)} = 0.$$

In particular the negative parts $(G_T^-(\phi_n))_{n=0}^\infty$ and $(g_n^-)_{n=0}^\infty$ tend to zero uniformly, which explains the term "vanishing risk".

The last piece of our vocabulary for stating FTAP is the following one.

**Definition 2.5.** *(Shiryaev [25] p.652, VII §2b.1 and p.656, VII §2c.2) An* **equivalent martingale measure** *(EMM) is a probability measure equivalent to $P$ and under which $S$ is a martingale. An* **equivalent local martingale measure** *(ELMM) is a probability measure equivalent to $P$ and under which $S$ is a local martingale. An* **equivalent $\sigma$-martingale measure** *(E$\sigma$MM) is a probability measure equivalent to $P$ and under which $S$ is a $\sigma$-martingale, i.e. $S = S_0 + H \cdot M$ with $M$ a martingale and $H$ a positive predictable process integrable w.r.t $M$.*

Now we are ready to state a list of results on the classical formulation of Fundamental Theorem of Asset Pricing:

**Theorem 2.1.** *(Shiryaev [25] p.655, VII §2c, Theorem 1, 2, and Corollary) Let $S$ be defined as above.*

*a) If $S$ is bounded, then*

$$NFLVR \Leftrightarrow EMM.$$

*b) If $S$ is locally bounded, then*

$$NFLVR \Leftrightarrow ELMM.$$

*c) If $S$ is a general semimartingale, then*

$$NFLVR \Leftrightarrow E\sigma MM.$$

*For a clearer insight into the connection between the above results and the corresponding results in the discrete-time case (see Theorem 2.2), we reformulate the theorem as follows.*

*In general semimartingale models $S_t = (1, S_t^1, \cdots, S_t^K)_{0 \le t \le T}$, $T < \infty$, we have*

$$EMM \Rightarrow ELMM \Rightarrow E\sigma MM \Leftrightarrow NFLVR.$$

*When $S$ is moreover locally bounded, we have*

$$EMM \Rightarrow ELMM \Leftrightarrow E\sigma MM \Leftrightarrow NFLVR.$$

*When $S$ is further assumed to be bounded, we have*

$$EMM \Leftrightarrow ELMM \Leftrightarrow E\sigma MM \Leftrightarrow NFLVR.$$

As a comparison, we recall FTAP in the discrete-time case.

**Theorem 2.2.** *(Dalang et al. [5]. Also see Delbaen and Schachermayer [10] Theorem 15) In the discrete- and finite-time case (i.e. $t = 0, 1, \cdots, T < \infty$) , we have*

$$EMM \Leftrightarrow ELMM \Leftrightarrow E\sigma MM \Leftrightarrow NA.$$

**Remark 1.** *For Theorem 2.2 to hold, we need no additional assumptions on trading strategies beside predictability. We also comment that $T < \infty$ is essential for Theorem 2.2; otherwise a counter example exists (see Shiryaev [25] p.415, V §2b.3). For the case of $T = \infty$, the NA condition needs to be modified to "no free lunch with bounded risk" (see Schachermayer [24]).*

In the above statement of FTAP, we have set $S^0 \equiv 1$. In practice, we usually do not have an asset whose price is identically 1. So FTAP as stated in Theorem 2.1 and 2.2 is really a mathematical simplification: instead of a general semimartingale $S = (S^0, S^1, \cdots, S^K)$, we considered the discounted process:

$$\frac{1}{S_0}\left(S^0, S^1, \cdots, S^K\right) = \left(1, \frac{S^1}{S^0}, \cdots, \frac{S^N}{S^0}\right).$$

Therefore we implicitly used $S^0$ as a numéraire, and EMM or ELMM should be understood as w.r.t a numéraire: discounted by this numéraire, $S$ is a martingale or local martingale. So measure and numéraire appear in a dual pair. Similarly, the notion of *admissibility* should be understood as "in given numéraire". That is, we require $G_t(\phi)$ denominated in the numéraire is bounded from below. Formally, we have

**Definition 2.6.** *A* **numéraire** *is any strictly positive semimartingale.*

**Definition 2.7.** *An* **equivalent martingale measure** $Q^N$ **associated with the numéraire** $N$ *is a probability measure equivalent to $P$ such that $S/N$ is a martingale under $Q^N$.*

**Definition 2.8.** *A trading strategy $\phi$ is* **admissible under the numéraire** $N$ *if there is a constant $M$ such that $G_t(\phi)/N_t \ge -M$ a.s. for all $t \ge 0$.*

To reconcile any potential conceptual conflicts, we need the following

**Lemma 2.1.** *$\phi$ is a self-financing strategy if and only if for any numéraire $N$, we have*

$$d\left(\frac{V_t(\phi)}{N_t}\right) = \sum_{k=0}^{K} \phi_t^k d\left(\frac{S_t^k}{N_t}\right).$$

*Proof.* Sufficiency is obvious as we can take $N_t \equiv 1$. For necessity, we note by integration-by-part formula

$$\begin{aligned}
& d\left(\frac{V_t(\phi)}{N_t}\right) \\
= \; & \frac{dV_t(\phi)}{N_{t-}} + V_{t-}(\phi) d\left(\frac{1}{N_t}\right) + d[V(\phi), 1/N]_t = \sum_{k=0}^{K} \frac{\phi_t^k dS_t^k}{N_{t-}} + \sum_{k=0}^{K} \phi_t^k S_{t-}^k d\left(\frac{1}{N_t}\right) + \sum_{k=0}^{K} \phi_t^k d[S^k, 1/N]_t \\
= \; & \sum_{k=0}^{K} \phi_t^k \left\{ \frac{dS_t^k}{N_{t-}} + S_{t-}^k d\left(\frac{1}{N_t}\right) + d[S^k, 1/N]_t \right\} = \sum_{k=0}^{K} \phi_t^k d\left(\frac{S_t^k}{N_t}\right),
\end{aligned}$$

where the second "=" has used the observation that

$$V_{t-}(\phi) = V_t(\phi) - \Delta V_t(\phi) = \sum_{k=0}^{K} \phi_t^k S_t^k - \sum_{k=0}^{K} \phi_t^k \Delta S_t^k = \sum_{k=0}^{K} \phi_t^k S_{t-}^k.$$

□

As we change numéraire, a question naturally arises: *Is Fundamental Theorem of Asset Pricing invariant under a numéraire change?* This question can be more precisely stated as follows:[3]

1) Under numéraire change, is NFLVR preserved?
2) Under numéraire change, is the existence of EMM or ELMM preserved?
3) Under numéraire change, if the existence of EMM or ELMM is preserved, is the uniqueness of such a measure also preserved?
4) How are the equivalent martingale measures related to each other? For example, can we represent the Radon-Nikodym derivatives in terms of the numéraires?
5) If we change the numéraire $N$ to another numéraire $U$, does the risk-neutral pricing formula still holds? That is, for an $\mathcal{F}_T$-measurable random variable $\xi$ satisfying suitable integrability conditions, do we have $(t < T)$

$$N_t E^N \left[ \frac{\xi}{N_T} \middle| \mathcal{F}_t \right] = U_t E^U \left[ \frac{\xi}{U_T} \middle| \mathcal{F}_t \right]?$$

For the sake of risk-neutral pricing, we shall focus on EMM. The following example justifies our choice: *NFLVR and existence of ELMM are not preserved under numéraire change.*

**Example 1.** *(Delbaen [7] or Delbaen and Schachermayer [9], Corollary 5) Let $R$ be the Bessel(3) process starting from* 1*, i.e. $R_t = \|B_t\|$ where $B$ is a 3-dimensional Brownian motion starting at some point $x_0 \in \mathbb{R}^3 \setminus \{0\}$ with $\|x\| = 1$ and $\|\cdot\|$ is the Euclidean norm. Then $R$ hits origin with probability* 0 *and there exists a 1-dimensional Brownian motion $W$ such that $R$ satisfies the SDE*

$$dR_t = dW_t + \frac{dt}{R_t},$$

*where $dW_t = \sum_{i=1}^{3} \frac{B_i(t)}{R_t} dB_i(t)$.*

*For the asset pair $(\frac{1}{R_t}, 1)$ over a time horizon $[0, T]$ $(T < \infty)$, $(\frac{1}{R_t}, 1)$ is a pair of local martingales under the original probability $P$, as it's easy to verify $d\left(\frac{1}{R_t}\right) = -\frac{dW_t}{R_t^2}$. Since $\frac{1}{R_t}$ is locally bounded, by Theorem 2.1 b), the system $(\frac{1}{R_t}, 1)$ satisfies NFLVR.*

*Suppose we now take $\frac{1}{R}$ as the numéraire. Discounted by this numéraire, the asset system becomes $(1, R_t)$. We show $R_t$ cannot be a local martingale under a probability measure $P'$ equivalent to $P$. Indeed, assume such a probability measure $P'$ exists. Define $M_t = E^P\left[\frac{dP'}{dP}|\mathcal{F}_t\right]$. By Lemma 3.1 and localization, we can conclude $MR$ is a local martingale under $P$. But $\frac{1}{R}$ is the only local martingale $X$ such that $X_0 = 1$ and such that $XR$ is a local martingale. So $M_t = \frac{1}{R_t}$ is a strict $P$-local martingale, not a martingale. This contradiction shows ELMM does not exist, and hence NFLVR property is not preserved under numéraire change.*

The above example gives negative answer to Question 1) and 2) for ELMM. However, for EMM, things are much better, as we shall see in Section 3.

[3]Here, EMM and ELMM are always understood as being associated with a given numéraire.

# 3 Fundamental Theorem of Asset Pricing: numéraire-free formulation

We stick to the notation used so far, but we don't assume $S^0 \equiv 1$. For convenience, we assume $T < \infty$ and at least one of $S^i$'s is bounded away from 0. The latter condition is to facilitate the "right" formulation of no-arbitrage condition (see the remark after Definition 3.2).

We first prove a useful lemma.

**Lemma 3.1.** *On a filtered probability space $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, \mathcal{F}, P)$ $(T < \infty)$, suppose $P'$ is a probability measure equivalent to $P$. For an adapted process $X$ to be a $P'$-martingale, it is necessary and sufficient that $MX$ is a $P$-martingale, where $M_t := E^P\left[\frac{dP'}{dP}\middle|\mathcal{F}_t\right]$.*

*Proof.* We first verify the integrability property. For any $t \in [0,T]$,

$$E^P[|X_t|] = E^{P'}\left[\frac{dP}{dP'}|X_t|\right] = E^{P'}\left[E\left[\frac{dP}{dP'}\middle|\mathcal{F}_t\right]|X_t|\right] = E^{P'}[M_t|X_t|].$$

So $X_t$ is integrable under $P$ if and only if $M_tX_t$ is integrable under $P'$, $\forall t \in [0,T]$.

We then verify the martingale property. By Proposition A.1, $E^{P'}[X_T|\mathcal{F}_t] = M_t^{-1}E^P[X_TM_T|\mathcal{F}_t]$. So

$$X_t = E^{P'}[X_T|\mathcal{F}_t] \Leftrightarrow X_tM_t = E^P[X_TM_T|\mathcal{F}_t].$$

This proves the lemma. □

## 3.1 Existence and uniqueness of EMM are invariant under numéraire change

As we promised, the new formulation of FTAP will start from the martingale property instead of the no-arbitrage condition.

**Definition 3.1.** *(Yan [29] Definition 2.1) The market is said to be* **fair** *if there exists a numéraire $N$ and an associated EMM $Q^N$, such that $\widetilde{S} = \left(\frac{S^0}{N}, \frac{S^1}{N}, \cdots, \frac{S^K}{N}\right)$ becomes a $Q^N$-martingale.*

To justify Definition 3.1, we essentially need to answer Question 2) for EMM.

**Proposition 3.1.** *(Yan [29] Theorem 2.2) Fairness of the market is independent of the choice of numéraire. More precisely, if a numéraire $N$ has an associated EMM $Q^N$, then any other numéraire $U$ also has an associated EMM $Q^U$, and the mapping*

$$Q^N \longmapsto Q^U \text{ with } \frac{dQ^U}{dQ^N} = \frac{U_T/U_0}{N_T/N_0},$$

*establishes a one-to-one correspondence between the set of EMM's associated with numéraire $N$ and the set of EMM's associated with numéraire $Q$.*

*Proof.* By Lemma 3.1, $S/U$ is a martingale under some measure $Q^U$ if and only if $MS/U$ is a martingale under $Q^N$, where

$$M_t := E^N\left[\frac{dQ^U}{dQ^N}\middle|\mathcal{F}_t\right].$$

We already know $S/N$ is a $Q^N$-martingale. Since $S/N = S/U \cdot U/N$, we would like to set $M_t = \frac{U_t}{N_t}$. Noting the normalization requirement, we should set

$$M_t = \frac{U_t/U_0}{N_t/N_0}.$$

Then it's easy to verify that a probability measure $Q^U$ defined by such the density process $M$ is the EMM associated with $U$. It's also easy to verify the above mapping is one-to-one and onto. □

**Remark 2.** *Proposition 3.1 also answers Question 4) and gives Question 2) and 3) positive answers for the case of EMM. As to Question 5), we shall see later on that they are intimately linked with market completeness.*

### 3.2 Numéraire-free formulation of FTAP

Now that we are formulating FTAP in terms of EMM, we should expect the class of trading strategies be enlarged so that the "no arbitrage" condition becomes more stringent. Indeed, we ned the following concept.

**Definition 3.2.** *(Yan [29] Definition 2.3) A trading strategy $\phi$ is said to be* **allowable***, if there exists a positive constant $C$ such that the gains process $G.(\phi)$ is bounded from below by $-C\sum_{i=0}^K S^i$, i.e.*

$$G_t(\phi) = \int_0^t \phi_u dS_u \geq -C\sum_{i=0}^K S_t^i,\ t \geq 0.$$

**Remark 3.** *In essence, we are taking $\sum_{i=0}^K S^i$ as the universal numéraire. And as far as one of $S^i$'s is bounded away from 0, allowability is indeed a generalization of admissibility.*

In analogy with NFLVR for admissible strategies, we have the following notion of NFLVR for allowable strategies.

**Definition 3.3.** *(Yan [29] Remark after Theorem 3.2) Let*

$$K_1 = \{G_T(\phi)|\phi \text{ allowable and } G_\infty(\phi) = \lim_{t\to\infty} G_t(\phi) \text{ exists a.s. if } T = \infty\}$$

*and*

$$C_1 = \{g \in L^\infty(\Omega, \mathcal{F}_T, P)|\ g \leq f \text{ for some } f \in K_1\}.$$

*We say that $S$ satisfies the condition of* **no free lunch with vanishing risk for allowable strategies***, if*

$$\bar{C}_1 \cap L_+^\infty(\Omega, \mathcal{F}_T, P) = \{0\},$$

*where $\bar{C}$ denotes the closure of $C$ with respect to the norm topology of $L^\infty(\Omega, \mathcal{F}_T, P)$.*

Then the numéraire-free formulation of FTAP is given as follows. It is numéraire free because of Proposition 3.1.

**Theorem 3.1.** *(Yan [29] Theorem 3.2)*

*Fair market (existence of EMM for some numéraire) $\Leftrightarrow$ $NFLVR$ for allowable strategies.*

## 4 Second Fundamental Theorem of Asset Pricing: classical formulation

Beside pricing, we are also concerned with hedging. Indeed, pricing without a hedging strategy is really empty talk. We would like to know when a contingent claim, be it European or American, is "attainable" by a "reasonable" trading strategy. This leads to the Second Fundamental Theorem of Asset Pricing. We now move back to the setting of Section 2: $T \leq \infty$ and $S^0 \equiv 1$.

**Definition 4.1.** *(Shiryaev [25] p.661, VII §2d) We say that a semimartingale model $S = (S^0, \cdots, S^K)$ is* **complete** *(or* **T-complete***) if each non-negative bounded $\mathcal{F}_T$-measurable contingent claim $\xi$ is* **replicable** *(or* **attainable***), i.e., there exists an admissible self-financing strategy $\phi$ such that $V_T(\phi) = \xi$.*

Recall we assume $S^0 \equiv 1$ and thus the above definition implicitly uses 1 as numéraire, which is needed for the general definition of admissibility. But we observe by Lemma 2.1 that the notion of attainability is invariant under numéraire change.

By the *Second Fundamental Theorem of Asset Pricing* (Shiryaev [25] p.481, V §4) an arbitrage-free model with discrete time $n \leq N < \infty$ and finitely many assets is complete if and only if the set of equivalent martingale measures consists of a single element. For continuous time case, we have the following sufficient condition

**Theorem 4.1.** *(Shiryaev [25] p.660, VII §2d)*

*Uniqueness of EMM $\Rightarrow$ Each (European) integrable contingent claim is attainable.*

If the above result looks too sleek to give us any intuition, we recall that in the setting of binomial model, a price process becomes acceptable to both buyers and sellers if and only if the buyer's best "bid" meets the seller's best "ask" (see Appendix B). Using this specific example as a guidance, we review here the financial application of the optional decomposition theorem as developed in Kramkov [21], as well as its generalization by Föllmer and Kabanov [14].

Denote by $\mathbf{M}^e_{loc}(S)$ the set of all probability measures equivalent to the original probability and under which $S$ (discounted by the implicit numéraire 1) is a local martingale. That is, $\mathbf{M}^e_{loc}(S)$ is the set of ELMM's for $S$.

**Theorem 4.2.** *(Optional Decomposition Theorem, Föllmer and Kabanov [14] Theorem 1) Assume $\mathbf{M}^e_{loc}(S) \neq \emptyset$. Let $V$ be a right-continuous adapted process. Then $V$ is a local supermartingale with respect to any $Q \in \mathbf{M}^e_{loc}(S)$ if and only if there exists an increasing right-continuous adapted process $C$ with $C_0 = 0$ and an $S$-integrable predictable process $\phi$, such that $V = V_0 + \int_0^{\cdot} \phi_u dS_u - C$ for all $Q \in \mathbf{M}^e_{loc}(S)$.*

**Remark 4.** *Kramkov [21] proved the above result for locally bounded $S$ and nonnegative $V$. Föllmer and Kabanov [14] extends the result to general semimartingale $S$ and general $V$.*

To see the financial interpretation of Theorem 4.2, we have

**Definition 4.2.** *(Kramkov [21] §3)A* **wealth and consumption portfolio** *is a triple $\prod = (v, \phi, C)$, where $v$ is the* **initial wealth of the portfolio***, $\phi$ is an $S$-integrable predictable process, and $C$ is an increasing right-continuous adapted* **process of consumption** *with $C_0 = 0$. The* **capital process** *$V(\phi)$ of portfolio $\prod$ equals*

$$V_t(\phi) = v + \int_0^t \phi_u dS_u - C_t, \ t \geq 0.$$

*The portfolio $\prod$ is called* **self-financing** *if $C \equiv 0$. $\prod$ is called* **admissible** *if $V$ is bounded from below.*

We can translate Theorem 4.2 into the following version:

**Theorem 4.3.** *(Financial interpretation of Optional Decomposition Theorem, Kramkov [21] Theorem 3.1) Assume $\mathbf{M}^e_{loc}(S) \neq \emptyset$. Let $V$ be a right-continuous adapted process. Then*
*(i) $V$ is the capital process of a wealth and consumption portfolio if and only if $V$ is a local supermartingale with respect to all $Q \in \mathbf{M}^e_{loc}(S)$.*
*(ii) $V$ is the capital process of a self-financing portfolio if and only if $V$ is a local martingale with respect to all $Q \in \mathbf{M}^e_{loc}(S)$.*

The ultimate purpose for defining a wealth and consumption portfolio is to hedge a contingent claim. Therefore, we have

**Definition 4.3.** *A wealth and consumption portfolio $\prod$ with capital process $V$ is call the* **hedging portfolio** *for a nonnegative $\mathcal{F}_T$-measurable contingent claim $\xi$ if $V_T \geq \xi$. $\prod$ is called a* **perfect hedge** *if $V_T = \xi$. A hedging portfolio $\hat{\prod}$ with capital process $\hat{V}$ is call the* **minimal hedge** *for $\xi$ if $\hat{V}_t \leq V_t$ a.s. for all $t \geq 0$ and hedging portfolio $\prod$ with capital process $V$.*

The supermartingale property of the capital process $V$ of hedging portfolio $\prod$ implies

$$V_t \geq \text{ess sup}_{Q \in \mathbf{M}^e_{loc}(S)} E^Q[V_T|\mathcal{F}_t] \geq \text{ess sup}_{Q \in \mathbf{M}^e_{loc}(S)} E^Q[\xi|\mathcal{F}_t], \ t \geq 0.$$

The following theorem states that the lower bound in the above inequalities is achieved and is equal to the capital process of the minimal hedge.

**Theorem 4.4.** *(Hedging of European contingent claim, Kramkov [21] Theorem 3.2) Let $\xi$ be a nonnegative $\mathcal{F}_T$-measurable contingent claim such that $\sup_{Q\in\mathbf{M}^e_{loc}(S)} E^Q[\xi] < \infty$. Then*
*1) the minimal admissible hedge $\hat{\prod} = (\hat{v}, \hat{\phi}, \hat{C})$ exists and its capital process $\hat{V}$ equals*

$$\hat{V}_t = \hat{v} + \int_0^t \hat{\phi}_u dS_u - \hat{C}_t = ess\ sup_{Q\in\mathbf{M}^e_{loc}(S)} E^Q[\xi|\mathcal{F}_t].$$

*2) $\hat{\prod}$ is a perfect hedge and if $\tau$ is a finite stopping time and $\xi$ is $\mathcal{F}_\tau$-measurable, then $\hat{V}_\infty = \xi$.*
*3) the minimal hedge $\hat{\prod}$ is self-financing if and only if there is a measure $\hat{Q} \in \mathbf{M}^e_{loc}(S)$ such that $E^{\hat{Q}}[\xi] = \sup_{Q\in\mathbf{M}^e_{loc}(S)} E^Q[\xi] < \infty$. In this case, $\hat{V}$ is a $\hat{Q}$-uniformly integrable martingale.*

**Remark 5.** *For a general $\mathcal{F} = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$-measurable claim $\xi$ this equality does not necessarily hold. See Kramkov [21] Example 3.1 for a counter example.*

**Remark 6.** *Various versions of the above theorem are proved by El Karoui and Quenez [13], Jacka [18], Ansel and Stricker [1], Delbaen and Schachermayer [9], and Kramkov [21].*

A similar result also exists for (super-)hedging American contingent claims. Let $\xi = (\xi_t)_{t\geq 0}$ be an adapted positive process. We interpret $\xi$ as the reward process of an American type option.

**Definition 4.4.** *The wealth and consumption portfolio $\prod = (v, \phi, C)$ with capital process $V$ is called a* **hedging portfolio** *for $\xi$ if*

$$V_t \geq \xi_t,\ t \geq 0.$$

*The portfolio $\tilde{\prod} = (\tilde{v}, \tilde{\phi}, \tilde{C})$ with capital process $\tilde{V}$ is termed the* **minimal hedging portfolio** *if*

$$V_t \geq \tilde{V}_t \geq f_t$$

*for any $t \geq 0$ and hedging portfolio $\prod$ with capital process $V$.*

Denote by $\mathcal{M}_t$ the set of stopping times with values in $[t, \infty)$. The following theorem can be considered as a generalization of the results obtained by Bensoussan [2] and Karatzas [20] in the setting of incomplete markets.

**Theorem 4.5.** *(Superhedging of American contingent claim, Kramkov [21] Theorem 3.3) Let $\xi = (\xi_t)_{t\geq 0}$ be an adapted positive process such that*

$$\sup_{\tau\in\mathcal{M}_0} \sup_{Q\in\mathbf{M}^e_{loc}(S)} E^Q[\xi_\tau] < \infty.$$

*Then the minimal hedging portfolio $\tilde{\prod} = (\tilde{v}, \tilde{\phi}, \tilde{C})$ exists, and its capital process at time $t \geq 0$ equals*

$$\tilde{V}_t = \tilde{v} + \int_0^t \tilde{\phi}_u dS_u - \tilde{C} = ess\ sup_{Q\in\mathbf{M}^e_{loc}(S),\ \tau\in\mathcal{M}_t} E^Q[\xi_\tau|\mathcal{F}_t].$$

# 5 Second Fundamental Theorem of Asset Pricing: numéraire -free formulation

In this section, we develop a numéraire-free version of Second Fundamental Theorem of Asset Pricing. We drop the assumption that $S^0 \equiv 1$ and assume at least one of $S^i$'s is bounded away from 0 (see Section 3 for rationale).

The foundation of everything is a version of Kramkov's optional decomposition theorem in the equivalent martingale measure setting. Denote by $\mathbf{M}^e(S; N)$ the set of all equivalent martingale measures for $S$ associated with the numéraire $N$.

**Theorem 5.1.** *(Xia and Yan [28] Theorem 4.1) Assume* $\mathbf{M}^e(S;1) \neq \emptyset$*. Let* $V$ *be a right-continuous adapted process. Then* $V$ *is a local supermartingale with respect to any* $Q \in \mathbf{M}^e(S;1)$ *if and only if there exists an increasing right-continuous adapted process* $C$ *with* $C_0 = 0$ *and an* $S$*-integrable predictable process* $\phi$*, such that* $V = V_0 + \int_0^{\cdot} \phi_u dS_u - C$ *for all* $Q \in \mathbf{M}^e(S;1)$*.*

To answer Question 5), we need a stronger definition of replicating strategies.

**Definition 5.1.** *A trading strategy* $\phi$ *is said to be* **regular** *if it is allowable and there is a numéraire* $N$ *such that* $V(\phi)/N$ *is a* $Q^N$*-martingale for some* $Q^N \in \mathbf{M}^e(S;N)$*. A trading strategy* $\phi$ *is said to be* **strongly regular** *if it is allowable and there is a numéraire* $N$ *such that* $V(\phi)/N$ *is a* $Q^N$*-martingale for all* $Q^N \in \mathbf{M}^e(S;N)$*.*

By Proposition 3.1 and Proposition A.1, we can easily see

**Proposition 5.1.** *The (strong) regularity of a trading strategy is invariant under numéraire change. That is, if* $\phi$ *is (strongly) regular under numéraire* $N$*, then it is also (strongly) regular under any other numéraire* $U$*. Furthermore, if* $\phi$ *and* $\phi'$ *are regular strategies such that* $V_T(\phi) = V_T(\phi')$*. Then* $V_t(\phi) = V_t(\phi')$ *for all* $t \in [0,T]$*.*

Based on Theorem 5.1, we can easily deduce the following characterization of attainable claims, which is a numéraire-free version of the well-known one in the literature.

**Theorem 5.2.** *(Xia and Yan [28] Theorem 6.3) Let* $\xi$ *be a nonnegative* $\mathcal{F}_T$*-measurable contingent claim such that* $\sup_{Q^N \in \mathbf{M}^e(S;N)} E^N\left[\frac{\xi}{N_T}\right] < \infty$ *for some numéraire* $N$*. Then the following conditions are equivalent:*
*1) There is some* $Q_0^N \in \mathbf{M}^e(S;N)$ *such that*

$$E^{Q_0^N}\left[\frac{\xi}{N_T}\right] = \sup_{Q^N \in \mathbf{M}^e(S;N)} E^N\left[\frac{\xi}{N_T}\right].$$

*2) For any numéraire* $U$*, there exists* $Q_0^U \in \mathbf{M}^e(S;U)$ *such that*

$$E^{Q_0^U}\left[\frac{\xi}{U_T}\right] = \sup_{Q^U \in \mathbf{M}^e(S;N)} E^U\left[\frac{\xi}{U_T}\right].$$

*3)* $\xi$ *is attainable by a regular trading strategy.*

**Remark 7.** *Regularity of hedging strategy ensures*

$$N_t E^{Q^N}\left[\frac{\xi}{N_T}\middle|\mathcal{F}_t\right] = U_t E^{Q^U}\left[\frac{\xi}{U_T}\middle|\mathcal{F}_t\right]$$

*when* $Q^N$ *and* $Q^U$ *are related by the equation*

$$\left.\frac{dQ^U}{dQ^N}\right|_{\mathcal{F}_t} = \frac{U_t/U_0}{N_t/N_0}.$$

*Thus, the process ess* $\sup_{Q^N \in \mathbf{M}^e(S;N)} N_t E^N\left[\frac{\xi}{N_T}\middle|\mathcal{F}_t\right]$ *is invariant under numéraire change and gives the capital process of the minimal hedge.*

**Theorem 5.3.** *(Xia and Yan [28] Theorem 6.4) Let* $\xi$ *be a nonnegative* $\mathcal{F}_T$*-measurable contingent claim. Then the following conditions are equivalent:*
*1) There is a numéraire* $N$ *such that for all* $Q^N \in \mathbf{M}^e(S;N)$*,* $E^{Q^N}\left[\frac{\xi}{N_T}\right]$ *are the same constant.*
*2) For any given numéraire* $U$*,* $E^{Q^U}\left[\frac{\xi}{U_T}\right]$ *are the same constant for all* $Q^U \in \mathbf{M}^e(S;U)$*.*
*3)* $\xi$ *is attainable by a strongly regular strategy.*

**Remark 8.** *The equation in Question 5),* $N_t E^N \left[ \frac{\xi}{N_T} \middle| \mathcal{F}_t \right] = U_t E^U \left[ \frac{\xi}{U_T} \middle| \mathcal{F}_t \right]$ *, has two meanings. First, it means such an equality holds when* $U$ *and* $N$ *are different numéraire and* $Q^N$ *and* $Q^U$ *are related by*

$$\left.\frac{dQ^U}{dQ^N}\right|_{\mathcal{F}_t} = \frac{U_t/U_0}{N_t/N_0}.$$

*Second, it means when* $U$ *and* $N$ *are the same numéraire but* $Q^N$ *and* $Q^U$ *are two different elements of* $\mathbf{M}^e(S;N)$*. Therefore, by Theorem 5.2 and Theorem 5.3, Question 5) has a positive answer if and only if* $\xi$ *is attainable by a strongly regular strategy.*

Since regularity is easier to verify than strong regularity, the following result looks desirable:

**Theorem 5.4.** *(Xia and Yan [28] Theorem 6.5) Let* $\phi$ *be a regular strategy such that* $0 \leq V_T(\phi) \leq c \sum_{j=0}^K S_T^j$ *a.s. for some constant c, then* $\phi$ *is strongly regular.*

Finally, we have the following result, which gives a precise formulation of the folklore that "market is complete if and only if EMM is unique".

**Theorem 5.5.** *(Xia and Yan [28] Theorem 6.7) The following conditions are equivalent:*
*1)* $\mathbf{M}^e(S;1)$ *is a singleton.*
*2) Any contingent claim* $\xi$ *satisfying* $\sup_{Q^N \in \mathbf{M}^e(S;N)} E^{Q^N} \left[ \frac{\xi}{N_T} \right] < \infty$ *for some numéraire* $N$ *is attainable by a regular trading strategy.*

# 6 The Itô process model

We study the Itô process model to see a more specific form of the Fundamental Theorem of Asset Pricing and the change of numéraire technique. More specifically, under the assumption that the asset processes $S^0$, $S^1$, $\cdots$, $S^K$ are Itô processes, we ask the following four questions

a) What conditions should we impose on the drift and volatility parameters so that the market is arbitrage-free?

b) What conditions should we impose on the drift and volatility parameters so that the market is complete?

c) If we change the numéraire, what are the SDE's for $S$ under the new numéraire?

d) If we change to a new numéraire, what does the associated equivalent martingale measure look like?

## 6.1 Model set-up

We follow the development of Yan [29], Section 5. We fix a finite time horizon $T$. Let $B = \begin{bmatrix} B^1 \\ \cdots \\ B^d \end{bmatrix}$ be a d-dimensional standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ the natural filtration of $(B_t)_{0 \leq t \leq T}$. We consider a financial market which consists of $(K+1)$ assets $S = (S^0, S^1, \cdots, S^K)$. The price process $(S_t^i)_{0 \leq t \leq T}$ of each asset $i$ is assumed to be a strictly positive Itô process. Since its logarithm is also an Itô process, we can represent $S^i$ as

$$dS_t^i = S_t^i \left[ \sigma_t^i dB_t + \mu_t^i dt \right]$$

where $\mu = (\mu^0, \cdots, \mu^K)$ is the *vector of expected rate of return* and $\sigma^i$ is a $1 \times d$ vector.

## 6.2 Conditions for absence of arbitrage

To formulate the condition in a concise and elegant form, we need to introduce a numéraire and write the stochastic differential equations for the asset process $S$ discounted by this numéraire. We specify asset $S^0$ as the numéraire and set $\gamma_t := (S_t^0)^{-1}$. By Itô's formula, we have

$$d\gamma_t = -\gamma_t[\sigma_t^0 dB_t + (\mu_t^0 - |\sigma_t^0|^2)dt],$$

where $|\sigma_t^0|$ stands for the Euclidean norm of $\sigma_t^0$. Then the discounted process $\widetilde{S}_t^i = S_t^i \gamma_t$ $(i = 1, \cdots, K)$ satisfies the SDE

$$d\widetilde{S}_t^i = \widetilde{S}_t^i[a_t^i dB_t + b_t^i dt],$$

where

$$a_t^i = \sigma_t^i - \sigma_t^0,\ b_t^i = \mu_t^i - \mu_t^0 + |\sigma_t^0|^2 - \sigma_t^i \cdot \sigma_t^0$$

with $\cdot$ denoting the Euclidean inner product. In particular, if asset $S^0$ is a bank account with interest rate process $r$, then

$$a_t^i = \sigma_t^i,\ b_t^i = \mu_t^i - r_t.$$

Using the above notations, we can formulate the condition for absence of arbitrage as follows:

**Proposition 6.1.** *If the market has an equivalent martingale measure associated with $S^0$, the linear equation*

$$a(t)\psi(t) = \begin{bmatrix} \sigma_t^1 - \sigma_t^0 \\ \sigma_t^2 - \sigma_t^0 \\ \cdots \\ \sigma_t^K - \sigma_t^0 \end{bmatrix} \psi(t) = \begin{bmatrix} \mu_t^1 - \mu_t^0 + |\sigma_t^0|^2 - \sigma_t^1 \cdot \sigma_t^0 \\ \mu_t^2 - \mu_t^0 + |\sigma_t^0|^2 - \sigma_t^2 \cdot \sigma_t^0 \\ \cdots \\ \mu_t^K - \mu_t^0 + |\sigma_t^0|^2 - \sigma_t^K \cdot \sigma_t^0 \end{bmatrix} = b(t),\ dt \times dP - a.e.,\ a.s.,\ on [0,T] \times \Omega \qquad (1)$$

*has a solution $\psi \in (\mathcal{L}^2)^d$ (recall $a_t$ is a $K \times d$ matrix, so $\psi$ is a $d \times 1$ matrix, i.e. a column vector), where $\mathcal{L}^2$ stands for the set of all adapted process $\phi$ with $\int_0^T \phi_u^2 du < \infty$. Conversely, if*

$$E\left[\exp\left\{\frac{1}{2}\int_0^T |a_t^i|^2 dt\right\}\right] < \infty,\ 1 \le i \le K \qquad (2)$$

*and equation (1) has a solution $\psi \in (\mathcal{L}^2)^d$ satisfying*

$$E\left[\exp\left\{\frac{1}{2}\int_0^T |\psi_t|^2 dt\right\}\right] < \infty, \qquad (3)$$

*then the probability measure $Q$ with Radon-Nikodym derivative*

$$\frac{dQ}{dP} = \mathcal{E}(-\psi.B)_T = \exp\left\{-\int_0^T \psi_t^T dB_t - \frac{1}{2}\int_0^T |\psi_t|^2 dt\right\}$$

*is an equivalent martingale measure, where $\psi^T$ stands for the transpose of $\psi$.*

*Proof.* Suppose the market has an equivalent martingale measure $Q$ associated with $S^0$. Put

$$M_t = E\left[\frac{dQ}{dP}\middle| \mathcal{F}_t\right].$$

Then $(M_t)_{0 \le t \le T}$ is a $P$-martingale. It's easy to see $M$ must be an exponential of something, i.e. there is some process $\psi_t$ such that $dM_t = M_t(-\psi_t^T)dB_t$ or equivalently,

$$M_t = \mathcal{E}(-\psi.B)_t = \exp\left\{-\int_0^t \psi_u^T dB_u - \frac{1}{2}\int_0^t |\psi_u|^2 du\right\}.$$

Indeed, by the martingale representation theorem for Brownian motion, there exists $\phi \in (\mathcal{L}^2)^d$ such that $dM_t = \phi_t^T dB_t$. Then we can set $\psi_t = -\frac{\phi_t}{M_t}$.

By Girsanov's Theorem (see, for example, Revuz and York [23], Chapter VIII, §1, Theorem 1.12), $B_t^* = B_t + \int_0^t \psi_u du$ is a Brownian motion under $Q$. Moreover, by a theorem of Fujisaki et al [15], $(B_t^*)_{0\le t\le T}$ has also the martingale representation property w.r.t. $\mathbb{F}$ under $Q$. Thus there exists some $\sigma^* \in (\mathcal{L}^2)^{K\times d}$ such that

$$d\begin{bmatrix}\widetilde{S}_t^1\\ \widetilde{S}_t^2\\ \cdots\\ \widetilde{S}_t^K\end{bmatrix} = \sigma_t^* dB_t^* = \sigma_t^*(dB_t + \psi_t dt) = \mathrm{diag}\{\widetilde{S}_t^1, \widetilde{S}_t^2, \cdots, \widetilde{S}_t^K\}(a_t dB_t + b_t dt).$$

According to the uniqueness of the representation of Itô process $(\widetilde{S})_{0\le t\le T}$ and the invariance of the stochastic integral under a change of probability, we have

$$\sigma_t^* = \mathrm{diag}\{\widetilde{S}_t^1, \widetilde{S}_t^2, \cdots, \widetilde{S}_t^K\}a_t,\ \sigma_t^*\psi_t = \mathrm{diag}\{\widetilde{S}_t^1, \widetilde{S}_t^2, \cdots, \widetilde{S}_t^K\}b_t.$$

Therefore we must have $a_t\psi_t = b_t$, $dt \times dP$-a.e., a.s. So $\psi$ is a solution of equation (1).

Now we assume that $a$ satisfies condition (2), and $\psi$ is a solution of equation (1) satisfying condition (3). By Novikov's criterion (see, for example, Revuz and Yor [23], Chapter VIII, §1, Proposition 1.15), $\mathcal{E}(-\psi.B)$ is a P-martingale. So we can define a probability measure $Q$ such that $\frac{dQ}{dP} = \mathcal{E}(-\psi.B)_T$. In order to prove that $\widetilde{S}$ is a $Q$-martingale, it suffices to prove that the product $\mathcal{E}(-\psi.B)\widetilde{S}$ is a $P$-martingale. Indeed, for $i = 1, 2, \cdots, K$,

$$\begin{aligned}\mathcal{E}(-\psi.B)_t\widetilde{S}_t^i &= \mathcal{E}(-\psi.B)_t\widetilde{S}_0^i \exp\left\{\int_0^t (a_u^i dB_u + b_u^i du) - \frac{1}{2}\int_0^t |a_u^i|^2 du\right\}\\ &= \widetilde{S}_0^i \exp\left\{\int_0^t [(a_u^i - \psi_u^T)dB_u + a_u^i\psi_u du] - \frac{1}{2}\int_0^t |(a_u^i|^2 + |\psi_u|^2)du\right\}\\ &= \widetilde{S}_0^i \exp\left\{\int_0^t (a_u^i - \psi_u^T)dB_u - \frac{1}{2}\int_0^t |(a_u^i - \psi_u^T|^2)du\right\}.\end{aligned}$$

Once again by Novikov's criterion, $\mathcal{E}(-\psi.B)_t\widetilde{S}_t^i$ is a $P$-martingale. □

### 6.3 Conditions for market completeness

**Proposition 6.2.** *Assume that $K \ge d$, $a_t = \begin{bmatrix}\sigma_t^1 - \sigma_t^0\\ \sigma_t^2 - \sigma_t^0\\ \cdots\\ \sigma_t^K - \sigma_t^0\end{bmatrix}$ satisfies (2) and $a_t^T a_t$ are non-degenerated for a.e. $t$, where $a_t^T$ stands for the transpose of $a_t$. Put $\psi_t = (a_t^T a_t)^{-1} a_t^T b_t$. If $\psi$ satisfies (1) and (3), then there exists a unique equivalent martingale measure $P^*$ for the market. Moreover, we have*

$$E\left[\frac{dP^*}{dP}\middle|\mathcal{F}_t\right] = \exp\left\{-\int_0^t \psi_u^T dB_u - \frac{1}{2}\int_0^t |\psi_u|^2 du\right\},\ 0 \le t \le T.$$

*Proof.* By Proposition 6.1, there exists an equivalent martingale measure $Q$. Suppose $Q'$ is another equivalent martingale measure. Then we can find $\theta \in (\mathcal{L}^2)^d$ such that

$$\frac{dQ'}{dP} = \mathcal{E}(-\theta.B)_T.$$

By Proposition 6.1, we have $a_t\theta_t = b_t$. Consequently, applying $(a_t^T a_t)^{-1} a_t^T$ to both sides of this equation, we get $\theta_t = \psi_t$. Hence $Q = Q'$ and the uniqueness is proved. □

### 6.4 SDE's under the new numéraire

**Proposition 6.3.** *Suppose there is a unique equivalent martingale measure $Q$ associated with $S^0$, then under $Q$, the SDE for each $S^i$ and $\widetilde{S}^i$ $(i=1,2,\cdots,K)$ becomes*

$$dS_t^i = S_t^i \left[\sigma_t^i dB_t^* + \left(\mu_t^i - \int_0^t \psi_u du\right) dt\right],$$

$$d\widetilde{S}_t^i = \widetilde{S}_t^i \left[a_t^i dB_t^* + \left(b_t^i - \int_0^t \psi_u du\right) dt\right] = \widetilde{S}_t^i \left[(\sigma_t^i - \sigma_t^0) dB_t^* + \left(\mu_t^i - \mu_t^0 + |\sigma_t^0|^2 - \sigma_t^i \cdot \sigma_t^0 - \int_0^t \psi_u du\right) dt\right],$$

*where $\psi_t$ is obtained by solving the equation $a_t\psi_t = b_t$, i.e.*

$$\begin{bmatrix} \sigma_t^1 - \sigma_t^0 \\ \sigma_t^2 - \sigma_t^0 \\ \cdots \\ \sigma_t^K - \sigma_t^0 \end{bmatrix} \psi(t) = \begin{bmatrix} \mu_t^1 - \mu_t^0 + |\sigma_t^0|^2 - \sigma_t^1 \cdot \sigma_t^0 \\ \mu_t^2 - \mu_t^0 + |\sigma_t^0|^2 - \sigma_t^2 \cdot \sigma_t^0 \\ \cdots \\ \mu_t^K - \mu_t^0 + |\sigma_t^0|^2 - \sigma_t^K \cdot \sigma_t^0 \end{bmatrix},$$

*and $B_t^* = B_t + \int_0^t \psi_u du$ is a Brownian motion under $Q$. When $a_t^T a_t$ is non-degenerated for a.e. $t$, one particular solution of $\psi_t$ is $(a_t^T a_t)^{-1} a_t^T b_t$,*

*Proof.* This is easily seen by combining Proposition 6.1 and 6.2, and noting that $B_t + \int_0^t \psi_u du$ is a Brownian motion under $Q$ (see the proof of Proposition 6.1). □

# A Bayes rule for conditional expectation

**Proposition A.1.** *Suppose $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ is a measurable space equipped with a filtration. Let $P$ and $Q$ be two equivalent probability measures on this filtered measurable space. Suppose $\eta$ is an $\mathcal{F}_T$-measurable random variable integrable under $Q$. Then for $t \leq T$,*

$$E^Q[\eta|\mathcal{F}_t] = M_t^{-1} E^P[\eta M_T|\mathcal{F}_t],$$

*where $M_t = E^P\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right]$.*

*Proof.* First of all, we note $\eta M_T$ is integrable under $P$: $E^P[\eta M_T] = E^Q[\eta]$. For any $A \in \mathcal{F}_t$, we have

$$E^Q[E^Q[\eta|\mathcal{F}_t]1_A] = E^Q[\eta 1_A] = E^P[\eta M_T 1_A] = E^P[E^P[\eta M_T|\mathcal{F}_t]1_A] = E^Q[M_t^{-1}E^P[\eta M_T|\mathcal{F}_t]1_A].$$

By the arbitrariness of $A$, we conclude $E^Q[\eta|\mathcal{F}_t] = M_t^{-1}E^P[\eta M_T|\mathcal{F}_t]$. □

# B Pricing of American contingent claims in the binomial model

In this section, we summarize the pricing of American contingent claims in the binomial model, as presented in Shreve [27]. We use the notation in [27] throughout this section. All the theorems quoted in this section are also from [27].

From the buyer's perspective: At time n, if the derivative security has not been exercised, then the buyer can choose a policy $\tau$ with $\tau \in \mathcal{S}_n$. The valuation formula for cash flow (Theorem 2.4.8) gives a fair price for the derivative security exercised according to $\tau$:

$$V_n(\tau) = \sum_{k=n}^{N} \widetilde{E}_n\left[1_{\{\tau=k\}} \frac{1}{(1+r)^{k-n}} G_k\right] = \widetilde{E}_n\left[1_{\{\tau\leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau\right].$$

The buyer wants to consider all the possible $\tau$'s, so that he can find the least upper bound of security value, which will be the maximum price of the derivative security acceptable to him. This is the price given by Definition 4.4.1: $V_n = \max_{\tau\in\mathcal{S}_n} \widetilde{E}_n[1_{\{\tau\leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau]$.

From the seller's perspective: A price process $(V_n)_{0\leq n\leq N}$ is acceptable to him if and only if at time n, he can construct a portfolio at cost $V_n$ so that (i) $V_n \geq G_n$ (so that the portfolio can cover the payoff should the buyer decide to exercise the option) and (ii) he needs no further investing into the portfolio as time goes by. Formally, the seller can find $(\Delta_n)_{0\leq n\leq N}$ and $(C_n)_{0\leq n\leq N}$ so that $C_n \geq 0$ and $V_{n+1} = \Delta_n S_{n+1} + (1+r)(V_n - C_n - \Delta_n S_n)$. Since $(\frac{S_n}{(1+r)^n})_{0\leq n\leq N}$ is a martingale under the risk-neutral measure $\widetilde{P}$, we conclude

$$\widetilde{E}_n\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right] - \frac{V_n}{(1+r)^n} = -\frac{C_n}{(1+r)^n} \leq 0,$$

i.e. $(\frac{V_n}{(1+r)^n})_{0\leq n\leq N}$ is a supermartingale. This inspired us to check if the converse is also true. This is exactly the content of Theorem 4.4.4. So $(V_n)_{0\leq n\leq N}$ is the value process of a portfolio that needs no further investing if and only if $\left(\frac{V_n}{(1+r)^n}\right)_{0\leq n\leq N}$ is a supermartingale under $\widetilde{P}$ (note this is independent of the requirement $V_n \geq G_n$). In summary, a price process $(V_n)_{0\leq n\leq N}$ is acceptable to the seller if and only if (i) $V_n \geq G_n$; (ii) $\left(\frac{V_n}{(1+r)^n}\right)_{0\leq n\leq N}$ is a supermartingale under $\widetilde{P}$.

Theorem 4.4.2 shows the buyer's upper bound is the seller's lower bound. So it gives the price acceptable to both. Theorem 4.4.3 gives a specific algorithm for calculating the price, Theorem 4.4.4 establishes the one-to-one correspondence between super-replication and supermartingale property, and finally, Theorem 4.4.5 shows how to decide on the optimal exercise policy.

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