Solving Recurrence Relations

COS 341 Fall 2002, lectures 9-10

Linear homogeneous recurrence relations

Definition 1 *A* linear homogeneous recurrence relation of degree k with constant coefficients *is a recurrence relation of the form*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

A sequence satisfying a recurrence relation above uniquely defined by the recurrence relation and the k initial conditions:

$$a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}.$$

Theorem 1 *Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.*

Proof Sketch: First, we prove that, for any constants α_1, α_2, $\alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies the recurrence relation.

Second, we prove that every solution is of the form $\alpha_1 r_1^n + \alpha_2 r_2^n$. Suppose $\{a_n\}$ is a solution of the recurrence relation with initial conditions $a_0 = C_0$ and $a_1 = C_1$. Then we show that by picking suitable constants α_1, α_2, we can set the first two values of the sequence $\alpha_1 r_1^n + \alpha_2 r_2^n$ to be C_0 and C_1. Since the sequences $\{a_n\}$ and $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ satisfy the degree 2 recurrence and agree on the first two values, they must be identical. ■

Example: Find the solution to the recurrence relation $a_n = a_{n-1} + a_{n-2}$ with initial conditions $a_0 = 2$ and $a_1 = 7$.

Solution: The characteristic equation is $r^2 - r - 2 = 0$, i.e. $(r-2)(r+1) = 0$. The roots are 2 and -1. Thus the solution to the recurrence relation is of the form $\alpha_1 2^n + \alpha_2 (-1)^n$. Since this must satisfy the initial conditions, we get:

$$\begin{aligned} a_0 = 2 &= \alpha_1 + \alpha_2 \\ a_1 = 7 &= \alpha_1 \cdot 2 + \alpha_2(-1) \end{aligned}$$

Solving, we get $\alpha_1 = 3$ and $\alpha_2 = -1$. Thus, the solution to the recurrence relation is $a_n = 3 \cdot 2^n - (-1)^n$.

Theorem 2 *Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n \cdot r_0^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.*

Example: Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$, with initial conditions $a_0 = 1$ and $a_1 = 6$.

Solution: $a_n = 3^n + n3^n$ (steps omitted).

Theorem 3 *Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation*

$$r^k - c_1 r^{k-1} - \ldots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then, a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Theorem 4 *Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation*

$$r^k - c_1 r^{k-1} - \ldots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$ respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then, a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n &= (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ &\quad + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ &\quad + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Problem: Solve the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

Solution: $a_n = (1 + 3n - 2n^2)(-1)^n$ (steps omitted).

Linear nonhomoeneous recurrence relations with constant coefficients

Definition 2 *A* linear nonhomogeneous recurrence relation with constant coefficients *is a recurrence relation of the form*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k} + F(n)$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $F(n)$ is a function not identicaly zero depending only on n. The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k}$$

is called the associated homogeneous recurrence relation*.*

A *particular solution* of a recurrence relation is a sequence that satisfies the recurrence equation; however, it may or may not satisfy the initial conditions.

Theorem 5 *If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k} + F(n), \tag{1}$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $a_n^{(h)}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k}.$$

Proof Sketch: Since $\{a_n^{(p)}\}$ is a particular solution of (1),

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \ldots c_k a_{n-k}^{(p)} + F(n), \tag{2}$$

Let $b(n)$ be an arbitrary solution to the nonhomogeneous recurrence relation. Then,

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \ldots c_k b_{n-k} + F(n), \tag{3}$$

Subtracting, (2) from (3), we get:

$$(b_n - a_n^{(p)}) = c_1(b_{n-1} - a_{n-1}^{(p)}) + c_2(b_{n-2} - a_{n-2}^{(p)}) + \ldots c_k(b_{n-k} - a_{n-k}^{(p)}) + F(n)$$

Thus, $b_n - a_n^{(p)}$ is a solution to the associated homogeneous recurrence relation with constant coefficients. ■

The above theorem gives us a technique to solve nonhomogeneous recurrence relations using our tools to solve homogeneous recurrence relations. Given a non-homogeneous recurrence relation, we first *guess* a particular solution. Note that this satisfies the recurrence equation, but does not necessarily satisfy the initial conditions. Next, we use the fact the required solution to the recurrence relation is the sum of this particular solution and a solution to the associated homogeneous recurrence relation. We already know a general form for the solution to the homogeneous recurrence relation (from the previous theorems). This general form will have some unknown constants and their values can be determined from the fact the the sum of the particular solution and the homogeneous solution must satisfy the given initial conditions.

Example: Solve the recurrence relation $a_n = 3a_{n-1} + 2n$, with initial condition $a_1 = 3$.

Solution: $a_n = -n - \frac{3}{2} + \frac{11}{6}3^n$ (steps omitted).

Next, we give a systematic way to guess a particular solution for a large class of functions $F(n)$.

Theorem 6 *Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous linear recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots c_k a_{n-k} + F(n),$$

where $c_1, c_2, \dots, c_k$ are real numbers nd

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \dots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

The above theorem gives a recipe for picking a particular solution to a nonhomogeneous recurrence relation. The general form of the particular solution has several unknown constants. Their values can be determined by substituting this general form into the given nohomogeneous recurrence relation. This will yield a set of equations which can be solved to determine the values of the constants. (Substituting the general form yields a single equation. However, this equation says that a particular expression of the form $\sum_{i=1}^{t} \beta_i f_i(n)$ must be identically 0 for all values of $n \geq k$. Here β_i are linear expressions involving the unknown constants and $f_i(n)$ are functions of n. This then implies that $\beta_i = 0$ for all i, giving the required number of equations required to determine the unknown constants.)

Example: Solve the recurrence relation $a_n = a_{n-1} + n$, with initial condition $a_0 = 0$.

Solution: $a_n = \frac{1}{2}n^2 + \frac{1}{2}n$ (steps omitted).

Using generating functions to solve recurrence relations

We associate with the sequence $\{a_n\}$, the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Now, the recurrence relation for $\{a_n\}$ can be interpreted as an equation for $a(x)$. This allows us to get a formula for $a(x)$ from which a closed form expression for a_n can be derived.

Example: Find the generating function for the Fibonacci sequence and derive a closed form expression for the nth Fibonacci number.

Solution: Let $F(x) = \sum_{n=0\infty} f_n x^n$, be the generating function for the Fibonacci sequence. Since the Fibonacci sequence satisfies the recurrence $f_n = f_{n-1} + f_{n-2}$, we get an explicit form for $F(x)$ as follows:

$$\begin{aligned}
f_n &= f_{n-1} + f_{n-2} \quad n \geq 2 \\
f_n x^n &= f_{n-1} x^n + f_{n-2} x^{n-2} \quad n \geq 2 \\
\sum_{n=2}^{\infty} f_n x_n &= \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n \\
\sum_{n=2}^{\infty} f_n x_n &= x \sum_{n=1}^{\infty} f_n x^n + x^2 \sum_{n=0}^{\infty} f_n x^n \\
F(x) - f_0 - f_1 x &= x(F(x) - f_0) + x^2 F(x) \\
F(x)(1 - x - x^2) &= f_0 + x(f_1 - f_0) = x \\
F(x) &= \frac{x}{(1 - x - x^2)}
\end{aligned}$$

In order to get an closed form expression for f_n, we need to get a closed form expression for the coefficient of x^n in the expansion of the generating function. In order to do this, we use the technique of decomposition into partial fractions.

$$\frac{x}{(1 - x - x^2)} = \frac{A}{x - x_1} + \frac{B}{x - x_2}$$

where x_1 and x_2 are the roots of the polynomial $1 - x - x^2$. It is more convenient to express the generating function in the following form:

$$\frac{x}{(1 - x - x^2)} = \frac{a}{1 - r_1 x} + \frac{b}{1 - r_2 x}$$

where $r_1 = 1/x_1$ and $r_2 = 1/x_2$. It turns out that r_1 and r_2 are the roots of the characteristic equation (verify this). From this, we can get the closed form expression $f_n = a \cdot r_1^n + b \cdot r_2^n$. We can solve for a and b from the fact that the initial conditions must be satisfied and this will give us the result:

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^2 - \left(\frac{1 - \sqrt{5}}{2} \right)^2 \right].$$

The above technique can be generalized to get an expression for the solution of a general homogeneous recurrence relation with constant coefficients considered in Theorem 4.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots c_k a_{n-k}$$

Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \dots - c_k = 0$$

has t distinct roots $r_1, r_2, \dots, r_t$ with multiplicities $m_1, m_2, \dots, m_t$ respectively, so that $m_i \geq 1$ for $i = 1, 2, \dots, t$ and $m_1 + m_2 + \dots + m_t = k$.

Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function associated with the sequence $\{a_n\}$. From the recurrence relation, we can get an expression for $a(x)$ as follows:

$$\begin{aligned} a_n &= c_1 a_{n-1} + c_2 a_{n-2} + \dots c_k a_{n-k} & (4) \\ a(x) &= (c_1 x + c_2 x^2 + \dots + c_k x^k) a(x) + b_0 + b_1 x + \dots + b_{k-1} x^{k-1} & (5) \end{aligned}$$

Here b_i, $i = 0, \dots, k-1$ are constants where $b_i = a_i - \sum_{j=1}^{i} c_j a_{i-j}$. This gives the following expression for $a(x)$:

$$a(x) = \frac{b_0 + b_1 x + \dots + b_{k-1} x^{k-1}}{1 - c_1 x - c_2 x^2 - \dots - c_k x^k}$$

We express this using partial fractions. First, we factorize the denominator as follows:

$$1 - c_1 x - c_2 x^2 - \dots - c_k x^k = (1 - r_1 x)^{m_1} (1 - r_2 x)^{m_2} \cdots (1 - r_t x)^{m_t}$$

where $r_1, \dots, r_t$ are the roots of the characteristic equation with multiplicities $m_1, \dots, m_t$ respectively. (This is not obvious. Verify this !)

Now, we can write down the following expression for $a(x)$:

$$\begin{aligned} a(x) &= \frac{b_0 + b_1 x + \dots + b_{k-1} x^{k-1}}{(1 - r_1 x)^{m_1} (1 - r_2 x)^{m_2} \cdots (1 - r_t x)^{m_t}} \\ &= \frac{A_{1,0}}{(1 - r_1 x)} + \frac{A_{1,1}}{(1 - r_1 x)^2} + \dots + \frac{A_{1,m_1-1}}{(1 - r_1 x)^{m_1}} \\ &\quad + \frac{A_{2,0}}{(1 - r_2 x)} + \frac{A_{2,1}}{(1 - r_2 x)^2} + \dots + \frac{A_{2,m_2-1}}{(1 - r_2 x)^{m_2}} \\ &\quad + \cdots \\ &\quad + \frac{A_{t,0}}{(1 - r_t x)} + \frac{A_{t,1}}{(1 - r_t x)^2} + \dots + \frac{A_{t,m_t-1}}{(1 - r_t x)^{m_t}} \end{aligned}$$

where $A_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$. Computing the coefficient of x^n using the generalized binomial theorem, we get:

$$\begin{aligned} a_n &= (A_{1,0} + A_{1,1} \binom{n+1}{1} + \dots + A_{1,m_1-1} \binom{n+m_1-1}{m_1-1}) r_1^n \\ &\quad + (A_{2,0} + A_{2,1} \binom{n+1}{1} + \dots + A_{2,m_2-1} \binom{n+m_2-1}{m_2-1}) r_2^n \\ &\quad + \dots + (A_{t,0} + A_{t,1} \binom{n+1}{1} + \dots + A_{t,m_t-1} \binom{n+m_t-1}{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$.

Note that this solution is of a slightly different form than the solution claimed in Theorem 4, but the two forms are equivalent, i.e. one form can be converted to the other.

The generating function approach allows us to solve fairly general recurrence relations, as illustrated below.

Example: Find an explicit formula for the Catalan numbers defined by the recurrence relation

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

where $C_0 = 1$.

Solution: Let $C(x)$ be the generating function $\sum_{n=0}^{\infty} C_n x^n$. Then,

$$\begin{aligned}
C_n &= \sum_{k=0}^{n-1} C_k C_{n-1-k} \\
\text{Multiplying by } x^n\text{, we get:} & \\
C_n x^n &= x^n \sum_{k=0}^{n-1} C_k C_{n-1-k} \\
C_n x^n &= x \sum_{k=0}^{n-1} (C_k x^k) \cdot (C_{n-1-k} x^{n-1-k}) \\
\text{Summing this up from } n = 1 \text{ to } \infty\text{, we get:} & \\
\sum_{n=1}^{\infty} C_n x^n &= x \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} (C_k x^k) \cdot (C_{n-1-k} x^{n-1-k}) \\
\text{Note that } C(x) = \textstyle\sum_{n=0}^{\infty} C_n x^n. \text{ Hence,} C(x) - C_0 &= x C(x)^2 \\
x C(x)^2 - C(x) + 1 &= 0 \\
\text{Solving this quadratic equation, we get:} C(x) &= \frac{1 \pm \sqrt{1 - 4x}}{2x}
\end{aligned}$$

The generating function is uniquely defined by the recurrence relation and the initial condition $C_0 = 1$. Hence we should be able to rule out one of the two possibilities. If we choose the + sign in the expression for $C(x)$, $C(x) \to \infty$ for $x \to 0$. However for $x \to 0$, $C(x) \to C_0 = 1$. This rules out the + sign as a valid possibility. Thus the closed form expression for the generating function is

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

Using this, we can obtain a closed form expression for C_n, the nth Catalan number. We could certainly do this using the generalized binomial theorem. Here, we give an alternate calculation. We know from a previous homework exercise, that

$$\begin{aligned}
\frac{1}{\sqrt{1-4t}} &= \sum_{n=0}^{\infty} \binom{2n}{n} t^n \\
\text{Integrating from } t=0 \text{ to } t=x, \text{ we get:} & \\
\int_{t=0}^{x} \frac{1}{\sqrt{1-4t}} dt &= \sum_{n=0}^{\infty} \binom{2n}{n} \int_{t=0}^{x} t^n dt \\
\frac{1-\sqrt{1-4x}}{2x} &= \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n
\end{aligned}$$

Comparing the coefficient of x^n on both sides, we get that $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Solving functional recurrence relations

Recurrence relations often occur in the analysis of running times of algorithms. Such recurrence relations differ from the those we have seen so far in that they express a recurrence for the value of $F(n)$ where n is a real number. By making appropriate substitutions, many of these can be solved using the techniques we have learned for solving recurrence relations for sequences. The idea is to obtain a formula for $F(n)$ for n of a special form and then extend it to all value of n.

Example: Solve the recurrence:

$$F(n) = 2F(n/2) + n,$$

where $F(1) = 1$.

Solution 1: Consider $n = 2^k$.

$$F(2^k) = 2F(2^{k-1}) + 2^k$$

Let $a_k = F(2^k)$. Then,

$$a_k = 2a_{k-1} + 2^k, \qquad a_0 = 1$$

This can be solved using the techniques we have learned to get the solution $a_k = (k+1)2^k$. Hence $F(2^k) = (k+1)2^k$. To get a formula for $F(n)$, we substitute $k = \log_2 n$. This gives $F(n) = n(1 + \log_2 n)$. Finally, we verify that this solution satisfies the given recurrence relation:

$$\begin{aligned}
2F(n/2) + n &= 2\frac{n}{2}(1 + \log_2(n/2)) + n \\
&= n(1 + \log_2(n) - 1) + n \\
&= n(1 + \log_2 n) = F(n)
\end{aligned}$$

Solution 2:

$$
\begin{aligned}
F(n) &= 2F(n/2) + n \\
&= 2(2F(n/2) + n/2) + n = 4F(n/4) + 2n \\
&= 4(2F(n/8) + n/4) + 2n = 8F(n/8) + 3n \\
&= \cdots = 2^k F(n/2^k) + k \cdot n
\end{aligned}
$$

For $n = 2^k$, this gives $F(2^k) = 2^k F(1) + k \cdot 2^k = (k+1)2^k$. Now we continue as in the previous solution.