Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 324 (2006) 30–38
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

The mean value theorem and analytic functions of a complex variable

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Received 9 August 2005
Available online 10 January 2006
Submitted by A.V. Isaev

Abstract

The mean value theorem for real-valued differentiable functions defined on an interval is one of the most fundamental results in Analysis. When it comes to complex-valued functions the theorem fails even if the function is differentiable throughout the complex plane. we illustrate this by means of examples and also present three results of a positive nature.

Keywords: Mean value theorem; Entire functions; Quadrinomials; Univalent functions

We start by recalling the mean value theorem [8, p. 93] of real analysis.

Theorem A. *Let f be a real continuous function on a closed interval $[a, b]$ which is differentiable in the open interval (a, b). Then there is a point $\xi \in (a, b)$ at which*

$$f(b) - f(a) = f'(\xi)(b - a).$$

This result extends to functions whose domains lie in the complex plane provided their *ranges are on the real line*. In fact, if Ω is a *convex* domain in the n-dimensional Euclidean space, and f is real-valued and differentiable on Ω, then

$$f(b) - f(a) = Df(\xi) \cdot (b - a) = \langle \operatorname{grad} f(\xi), b - a \rangle \quad (a \in \Omega,\ b \in \Omega),$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.11.049

where ξ lies on the straight line segment that connects a to b. For $n=1$, it reduces to Theorem A, and for $n=2$, it gives an extension of the mean value theorem to functions whose domains lie in the complex plane but whose ranges are on the real line. The situation changes if f is allowed to take non-real values. To see this, let us consider the entire function

$$f(z) := \exp\left(\mathrm{i}\frac{2z-(z_1+z_2)}{z_2-z_1}\pi\right), \tag{1}$$

where z_1 and z_2 are any two distinct points in the complex plane. Then, clearly

$$f(z_1)=\mathrm{e}^{-\mathrm{i}\pi}=-1, \qquad f(z_2)=\mathrm{e}^{\mathrm{i}\pi}=-1$$

and

$$f'(z)=\frac{2\mathrm{i}\pi}{z_2-z_1}\exp\left(\mathrm{i}\frac{2z-(z_1+z_2)}{z_2-z_1}\pi\right)\neq 0 \quad (z\in\mathbb{C}).$$

Thus,

$$f(z_2)-f(z_1)-f'(z)(z_2-z_1)=-f'(z)(z_2-z_1)\neq 0 \quad (z\in\mathbb{C}).$$

This example shows that the equation

$$f(z_2)-f(z_1)-f'(z)(z_2-z_1)=0 \tag{2}$$

may not have any solution at all.

We wish to find interesting classes of entire functions for which Eq. (2) does have a solution ζ, and besides, what, if anything, can be said about the location of the point ζ. As a first step we recall the equivalence between the mean value theorem and the following one known as Rolle's theorem [9, p. 178].

Theorem B. *Let g be a real continuous function on a closed interval $[a,b]$ and differentiable in the open interval (a,b). Furthermore, let $g(a)=g(b)$. Then there is a point $\xi\in(a,b)$ such that $g'(\xi)=0$.*

Proof (of the equivalence). It is clear that Theorem B must hold if Theorem A does. In order to show that Theorem A holds if Theorem B does, let f satisfy the conditions of Theorem A. Then

$$g(x) := \begin{vmatrix} f(x) & f(a) & f(b) \\ x & a & b \\ 1 & 1 & 1 \end{vmatrix} = (a-b)f(x)-(x-b)f(a)+(x-a)f(b)$$

is a real function that is continuous on the closed interval $[a,b]$, differentiable in the open interval (a,b), and $g(a)=g(b)=0$. Hence, by Theorem B, there exists a point $\xi\in(a,b)$ such that $g'(\xi)=0$, that is $f(b)-f(a)=f'(\xi)(b-a)$, which proves that Theorem B implies Theorem A. The two are therefore equivalent. □

Next, we recall that if f is an entire function and $M(r) := \max_{|z|=r}|f(z)|$ then

$$\rho := \limsup_{r\to\infty}\frac{\log\log M(r)}{\log r}$$

is said to be the order (see [1, Chapter 2]) of f. All constants are supposed to be of order 0. In the case where ρ is finite and positive, its type is defined to be

$$T := \limsup_{r\to\infty}\frac{\log M(r)}{r^\rho}.$$

Let $\rho \in (0, \infty)$ and $\tau \in [0, \infty)$ be given. An entire function f is said to be of growth (ρ, τ) if, for any $\varepsilon > 0$,

$$\left|f(z)\right| < k(\varepsilon)\mathrm{e}^{(\tau+\varepsilon)|z|} \quad (z \in \mathbb{C}),$$

where $k(\varepsilon)$ is a constant depending on ε. Functions of growth $(1, \tau)$ are said to be of exponential type τ. In particular, any entire function of order $\rho < 1$ is of exponential type 0 and so is any entire function of order 1 type 0. This means that an entire function f is of exponential type 0 if, for any $\varepsilon > 0$, there exists a constant $k(\varepsilon)$ depending on ε such that $|f(z)| < k(\varepsilon)\mathrm{e}^{\varepsilon|z|}$ for any z in the complex plane. Thus, entire functions of exponential type 0 form a much wider class than the class of all polynomials.

It is interesting to note that the function f given in (1), which takes the same value at two distinct points z_1 and z_2 but for which Eq. (2) does not have a solution, is of order 1 type $T := 2\pi/|z_1 - z_2| > 0$, i.e. f is not of exponential type 0. The following result of Clunie and Rahman [2], which generalizes an earlier result of Grace [3], can be seen as an analogue of Rolle's theorem.

Theorem C. *Let φ be an entire function of exponential type 0 such that $\varphi(-1) = \varphi(1)$. Then, each of the half-planes $\{\Re z \leqslant 0\}$ and $\{\Re z \geqslant 0\}$ contains a zero of φ.*

They also point out that for any $\tau > 0$ one can find an entire function φ_τ of exponential type τ for which Theorem C does not hold. For example,

$$\varphi_\tau(z) := \left(z - \frac{\mathrm{e}^{2\tau} + 1}{\mathrm{e}^{2\tau} - 1}\right)\mathrm{e}^{\tau z}$$

is an entire function of order 1 type τ such that $\varphi_\tau(1) = \varphi_\tau(-1)$, but φ_τ' has only one zero, namely

$$\frac{\mathrm{e}^{2\tau} + 1}{\mathrm{e}^{2\tau} - 1} - \frac{1}{\tau},$$

which is positive.

Let z_1 and z_2 be any two distinct points in $\mathbb{C}$. Furthermore, let g be an entire function of exponential type 0 such that $g(z_1) = g(z_2)$. Then

$$\varphi(z) := g\left(\frac{z_2 - z_1}{2}z + \frac{z_2 + z_1}{2}\right)$$

is an entire function to which Theorem C applies. We thus see that Theorem C may be re-stated as follows.

Theorem C′. *Let z_1 and z_2 be any two distinct points in the complex plane. Furthermore, let g be an entire function of exponential type 0 such that $g(z_1) = g(z_2)$. Then, each of the half-planes $|z - z_1| \geqslant |z - z_2|$ and $|z - z_1| \leqslant |z - z_2|$ contains a zero of g.*

In the same way as Rolle's theorem is seen to be equivalent to the mean value theorem, it can be seen that Theorem C′ is equivalent to the following result.

Theorem 1. *Let f be an entire function of exponential type 0. Then for $z_1 \neq z_2$, Eq. (2) has a solution in each of the half-planes $|z - z_1| \geqslant |z - z_2|$ and $|z - z_1| \leqslant |z - z_2|$.*

Remark 1. The roots of (2) in $|z - z_1| \geqslant |z - z_2|$ and in $|z - z_1| \leqslant |z - z_2|$, whose existence is guaranteed by the preceding result, may all lie outside *any* prescribed disk $|z - (z_1 + z_2)/2| < d$. To see this, let us consider the polynomial

$$p_n(z) := \left\{ \frac{2z - (z_1 + z_2)}{z_2 - z_1} - \mathrm{i} \cot \frac{\pi}{n} \right\}^n - \left(-1 - \mathrm{i} \cot \frac{\pi}{n} \right)^n.$$

It vanishes at the points z_1 and z_2 whereas its derivative has its only zero at

$$z = \frac{1}{2} \left\{ z_1 + z_2 + \mathrm{i}(z_2 - z_1) \cot \frac{\pi}{n} \right\}.$$

Thus the equation

$$\frac{p_n(z_2) - p_n(z_1)}{z_2 - z_1} - p_n'(z) = 0$$

has no solution in the disk $|z - (z_1 + z_2)/2| < d$ if

$$d < d_n := \frac{|z_2 - z_1|}{2} \cot \frac{\pi}{n}.$$

Note that $d_n \to \infty$ as $n \to \infty$.

It was proved by Heawood (see [4] or [5] or [10]) that *if z_1 and z_2 are any two zeros of an nth degree polynomial f, at least one zero of its derivative f' lies in the disk*

$$\left| z - \frac{z_1 + z_2}{2} \right| \leqslant \frac{|z_2 - z_1|}{2} \cot \frac{\pi}{n}. \tag{3}$$

Let f be an arbitrary polynomial of degree at most n, and let z_1, z_2 be an arbitrary pair of distinct points in the complex plane. Then

$$f(z) - \left\{ f(z_1) + \frac{f(z_2) - f(z_1)}{z_2 - z_1} (z - z_1) \right\}$$

is a polynomial to which the result of Heawood applies. Hence the following theorem holds.

Theorem D. *Let f be a polynomial of degree at most n. Furthermore, let z_1 and z_2 be any pair of distinct points in the complex plane. Then Eq.* (2) *must have a solution in the disk* (3).

It should now be clear that there is no "mean value theorem for complex valued functions" having the simplicity and generality which are two important characteristics of the mean value theorem of real analysis, namely Theorem A. So, we need to lower our expectations, and that is what we shall do.

It would be nice if, for any function f analytic in the open unit disk Δ_1 and continuous in $\overline{\Delta}_1$ and for any pair of distinct points z_1, z_2 in $\overline{\Delta}_1$, Eq. (2) had a solution in Δ_1. The examples discussed above show that this is not true even for functions analytic throughout the complex plane. If we restrict ourselves to the class $\mathcal{P}_n$ of all polynomials of degree at most n then we can specify a disk containing at least one solution of (2) for any $f \in \mathcal{P}_n$ and any pair of distinct points z_1, z_2 in $\overline{\Delta}_1$.

Theorem E. *If f is a polynomial of degree at most n then, for any two distinct points z_1 and z_2 of the closed unit disk, Eq.* (2) *must have a solution in* $|z| \leqslant \csc(\pi/n)$.

Proof. It is clearly enough to prove the result in the case where z_1 and z_2 both lie on the boundary of Δ_1, i.e. $z_1 = \mathrm{e}^{\mathrm{i}\theta_1}$, $\theta_1 \in \mathbb{R}$, and $z_2 = \mathrm{e}^{\mathrm{i}\theta_2}$, $\theta_2 \in \mathbb{R}$. Setting $\lambda := |z_1 + z_2|/2$ we conclude from Theorem D that Eq. (2) has a solution in the closed disk

$$|z| \leqslant \lambda + \frac{|z_2 - z_1|}{2} \cot\frac{\pi}{n} = \lambda + \sqrt{1-\lambda^2}\cot\frac{\pi}{n} \leqslant \csc\frac{\pi}{n}. \qquad \square$$

The polynomial

$$f(z) := \int_1^z \left(u + \mathrm{i}\mathrm{e}^{\pi\mathrm{i}/n}\csc\frac{\pi}{n}\right)^{n-1} \mathrm{d}u$$

vanishes at the points $z_1 = 1$ and $z_2 = -\mathrm{e}^{2\pi\mathrm{i}/n}$ whereas the only zero of $f'(z)$ lies at $z = -\mathrm{i}\mathrm{e}^{\pi\mathrm{i}/n}\csc(\pi/n)$. Hence the disk $|z| \leqslant \csc(\pi/n)$ in Theorem E cannot be replaced by one of a smaller radius.

It is remarkable that even for a polynomial $f(z) := a_0 + a_1 z + a_2 z^2 + a_3 z^3$ of degree 3, Eq. (2), where z_1 and z_2 lie in the closed unit disk, may not have a solution in $|z| < \csc(\pi/3) = 2/\sqrt{3}$. However, something of a positive nature can be said about quadrinomials $f(z) := a_0 + a_1 z + a_p z^p + a_q z^q$ in the case where $1 < p < q$ and $q - 1$ is not an integral multiple of $p - 1$. For this we shall apply the following result of Rahman and Waniurski (see [6, Theorems 1 and 2]; also see [7]) to first obtain a Rolle type theorem stated below as Theorem F′.

Theorem F. *Let $\psi(z) := z + b_p z^p + b_q z^q$, where $1 < p < q$ and $q - 1$ is not an integral multiple of $p - 1$, and let $\Delta_1 := \{z \in \mathbb{C}\colon |z| < 1\}$. Then ψ is univalent in Δ_1 if and only if $\psi'(z) \neq 0$ in Δ_1.*

Theorem F′. *Let $g(z) := a_0 + a_1 z + a_p z^p + a_q z^q$, where $1 < p < q$ and $q - 1$ is not an integral multiple of $p - 1$. In addition, let z_1 and z_2 be two distinct points in the open unit disk $\Delta_1 := \{z \in \mathbb{C}\colon |z| < 1\}$ such that $g(z_1) = g(z_2)$. Then $g'(z)$ must vanish somewhere in Δ_1.*

Proof. If $a_1 = 0$ then the result is trivial since in that case $g'(0) = 0$. So, let $a_1 \neq 0$, and suppose that $g(z_1) = g(z_2)$ for two distinct points z_1 and z_2 in the open unit disk Δ_1. Then g cannot be univalent in Δ_1; hence neither can $\psi(z) := z + b_p z^p + b_q z^q$, where $b_p := (a_p)/a_1$ and $b_q := (a_q)/a_1$. We may now apply Theorem F to conclude that $\psi'(z)$ must vanish somewhere in Δ_1. The same can then be said about $g'(z)$. $\square$

The argument used to show the equivalence between Rolle's theorem and the mean value theorem can be repeated once again to deduce the following result from Theorem F′.

Theorem 2. *Let $f(z) := a_0 + a_1 z + a_p z^p + a_q z^q$, where $1 < p < q$ and $q - 1$ is not an integral multiple of $p - 1$. Then, for any pair of points z_1 and z_2 in the open disk Δ_1, Eq. (2) has a solution in Δ_1.*

Remark 2. For $R \in (1, 2/\sqrt{3})$ let

$$f_R(z) := \mathrm{i}\sqrt{3} + 2\left(1 - \mathrm{i}\sqrt{3}\right)(Rz) - \left(3 - \mathrm{i}\sqrt{3}\right)(Rz)^2 + (Rz)^3.$$

This polynomial is of the form $a_0 + a_1 z + a_p z^p + a_q z^q$ with $p = 2$ and $q = 3$. It is easily checked that

$$f_R\left(\frac{1}{R}\right) = f_R\left(\frac{1 - \mathrm{i}\sqrt{3}}{2R}\right) = 0.$$

Thus,

$$z_1 := \frac{1}{R} \quad \text{and} \quad z_2 := \frac{1 - \mathrm{i}\sqrt{3}}{2R}$$

are two distinct points of the open unit disk such that $f_R(z_2) - f_R(z_1) = 0$. Since $f_R'(z)$ has only a (double) zero at $(\sqrt{3} - \mathrm{i})/(\sqrt{3}\, R)$, and no other zero, we see that the equation

$$f_R(z_2) - f_R(z_1) = (z_2 - z_1) f_R'(z)$$

has no solution in the open unit disk. This is explained by the fact that here $q - 1$ is an integral multiple of $p - 1$.

The following result can also be seen as a mean value theorem.

Theorem 3. *Let f be analytic in a domain Ω containing the closed unit disk, and suppose that $|f''(z)| \geqslant m'' > 0$ for $|z| \leqslant 1/2$. Furthermore, with*

$$\mathcal{M}_1(1, f) := \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right| \mathrm{d}\theta \quad \textit{and} \quad \mathcal{M}_1(1, f') := \frac{1}{2\pi} \int_0^{2\pi} \left|f'\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right| \mathrm{d}\theta,$$

let

$$\delta := \min\left\{\frac{1}{2}, \frac{1}{4} \frac{m''}{4\mathcal{M}_1(1, f) + (2 + \sqrt{2})\mathcal{M}_1(1, f')}\right\}. \tag{4}$$

Then, for any pair of distinct points z_1 and z_2 in $\Delta_\delta := \{z \in \mathbb{C}\colon |z| < \delta\}$, Eq. (2) *has one and only one solution in the disk*

$$\mathcal{D} := \left\{z\colon \left|z - \frac{z_1 + z_2}{2}\right| < \left|\frac{z_2 - z_1}{2}\right|\right\}. \tag{5}$$

Proof. Let z_1 and z_2 belong to Δ_δ. Set $a := (z_1 + z_2)/2$, and note that $|z_2 - z_1|/2 < \delta$. By a well-known formula,

$$f(z) - f(a) - \frac{1}{1!} f'(a)(z - a) = \frac{(z - a)^2}{2\pi \mathrm{i}} \int_{|\zeta|=1} \frac{f(\zeta)}{(\zeta - a)^2} \frac{1}{\zeta - z} \mathrm{d}\zeta \quad \big(|z| < 1\big). \tag{6}$$

By the definition of a,

$$(z_1 - a)^2 = (z_2 - a)^2 = \frac{(z_2 - z_1)^2}{4}.$$

Hence,

$$f(z_2) - f(z_1) - f'(a)(z_2 - z_1) = \frac{(z_2 - z_1)^2}{4} \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=1} \frac{f(\zeta)}{(\zeta - a)^2} \frac{z_2 - z_1}{(\zeta - z_1)(\zeta - z_2)} \mathrm{d}\zeta,$$

that is

$$\frac{f(z_2)-f(z_1)}{z_2-z_1}-f'(a)=\left(\frac{z_2-z_1}{2}\right)^2\frac{1}{2\pi\mathrm{i}}\int\limits_{|\zeta|=1}\frac{f(\zeta)}{(\zeta-a)^2}\frac{1}{(\zeta-z_1)(\zeta-z_2)}\,\mathrm{d}\zeta.$$

Besides, we may apply (6) to f' to conclude that

$$f'(z)-f'(a)-\frac{1}{1!}f''(a)(z-a)=\frac{(z-a)^2}{2\pi\mathrm{i}}\int\limits_{|\zeta|=1}\frac{f'(\zeta)}{(\zeta-a)^2}\frac{1}{\zeta-z}\,\mathrm{d}\zeta\qquad\big(|z|<1\big).$$

Thus, for any z in the open unit disk, we have

$$\begin{aligned}\frac{f(z_2)-f(z_1)}{z_2-z_1}-f'(z)={}&-f''(a)(z-a)-\frac{(z-a)^2}{2\pi\mathrm{i}}\int\limits_{|\zeta|=1}\frac{f'(\zeta)}{(\zeta-a)^2}\frac{1}{\zeta-z}\,\mathrm{d}\zeta\\&+\left(\frac{z_2-z_1}{2}\right)^2\frac{1}{2\pi\mathrm{i}}\int\limits_{|\zeta|=1}\frac{f(\zeta)}{(\zeta-a)^2}\frac{1}{(\zeta-z_1)(\zeta-z_2)}\,\mathrm{d}\zeta.\end{aligned}$$

This allows us to look at

$$\frac{f(z_2)-f(z_1)}{z_2-z_1}-f'(z)$$

as the sum of the two functions

$$f_1(z):=-f''(a)(z-a)\tag{7}$$

and

$$\begin{aligned}f_2(z):={}&-\frac{(z-a)^2}{2\pi\mathrm{i}}\int\limits_{|\zeta|=1}\frac{f'(\zeta)}{(\zeta-a)^2}\frac{1}{\zeta-z}\,\mathrm{d}\zeta\\&+\left(\frac{z_2-z_1}{2}\right)^2\frac{1}{2\pi\mathrm{i}}\int\limits_{|\zeta|=1}\frac{f(\zeta)}{(\zeta-a)^2}\frac{1}{(\zeta-z_1)(\zeta-z_2)}\,\mathrm{d}\zeta.\end{aligned}\tag{8}$$

Let $a:=|a|\mathrm{e}^{\mathrm{i}\alpha}$ and $t:=|z_2-z_1|/2$. The point a being the mid-point of the line segment joining z_1 and z_2, we may write $z_1=:a+t\mathrm{e}^{\mathrm{i}\varphi}$, $z_2=a-t\mathrm{e}^{\mathrm{i}\varphi}$. Since z_1 and z_2 lie in Δ_δ, we have

$$|a|^2+t^2+2|a|t\cos(\varphi-\alpha)\leqslant\delta^2\quad\text{and}\quad|a|^2+t^2-2|a|t\cos(\varphi-\alpha)\leqslant\delta^2,$$

from which it follows that

$$\frac{|z_2-z_1|}{2}=t<\sqrt{\delta^2-|a|^2}.$$

Using this estimate for t, we conclude that *if z lies on the boundary of the open disk $\mathcal{D}$ defined in* (5) then $|z|<|a|+\sqrt{\delta^2-|a|^2}\leqslant\sqrt{2}\delta$, and so

$$|\zeta-z|>1-\sqrt{2}\delta\geqslant\frac{\sqrt{2}-1}{\sqrt{2}}\quad\big(|\zeta|=1\big),$$

because $\delta\leqslant 1/2$. Hence, if z lies on the boundary of $\mathcal{D}$, then

$$\left|-\frac{(z-a)^2}{2\pi\mathrm{i}}\int\limits_{|\zeta|=1}\frac{f'(\zeta)}{(\zeta-a)^2}\frac{1}{\zeta-z}\,\mathrm{d}\zeta\right|<\big(2+\sqrt{2}\big)\mathcal{M}_1(1,f')\frac{1}{(1-\delta)^2}\left|\frac{z_2-z_1}{2}\right|^2.$$

Besides, we clearly have

$$\left|\left(\frac{z_2-z_1}{2}\right)^2\frac{1}{2\pi \mathrm{i}}\int\limits_{|\zeta|=1}\frac{f(\zeta)}{(\zeta-a)^2}\frac{1}{(\zeta-z_1)(\zeta-z_2)}\,\mathrm{d}\zeta\right|\leqslant \mathcal{M}_1(1,f)\frac{1}{(1-\delta)^4}\left|\frac{z_2-z_1}{2}\right|^2.$$

Since $|f''(a)|\geqslant m''$, we conclude that if f_1 and f_2 are as in (7) and (8), respectively, then

$$\left|f_1(z)\right|>\left|f_2(z)\right| \tag{9}$$

if

$$|z-a|=\left|\frac{z_2-z_1}{2}\right|<\frac{1}{4}\frac{m''}{4\mathcal{M}_1(1,f)+(2+\sqrt{2})\mathcal{M}_1(1,f')}.$$

However, in view of the definition of δ (see (4)), this latter condition is a priori fulfilled. By Rouché's theorem [11, pp. 116–117], the function

$$\frac{f(z_2)-f(z_1)}{z_2-z_1}-f'(z)=f_1(z)+f_2(z)$$

has, in $|z-a|<|z_2-z_1|/2$, the same number of zeros as the function $f_1(z)$ does, that is just one. □

Remark 3. How is it that the example given after Theorem A is not a counter-example for Theorem 3? Here is the reason. For $z_1=\varepsilon$ and $z_2=-z_1$ the function given in (1) reduces to $f(z):=\mathrm{e}^{-(\pi/\varepsilon)\mathrm{i}z}$. Clearly, then

$$\min_{|z|\leqslant 1/2}\left|f''(z)\right|=f''\left(-\frac{\mathrm{i}}{2}\right)=\left(\frac{\pi}{\varepsilon}\right)^2\mathrm{e}^{-\pi/(2\varepsilon)},$$

$$\begin{aligned}\mathcal{M}_1(1,f)&=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{(\pi/\varepsilon)\sin\theta}\,\mathrm{d}\theta>\frac{1}{2\pi}\int_0^{\pi}\mathrm{e}^{(\pi/\varepsilon)\sin\theta}\,\mathrm{d}\theta=\frac{1}{\pi}\int_0^{\pi/2}\mathrm{e}^{(\pi/\varepsilon)\sin\theta}\,\mathrm{d}\theta\\&>\frac{1}{\pi}\int_0^{\pi/2}\mathrm{e}^{2\theta/\varepsilon}\,\mathrm{d}\theta=\left(\frac{\varepsilon}{2\pi}\right)\left(\mathrm{e}^{\pi/\varepsilon}-1\right)>\frac{1}{2}\end{aligned}$$

since $\mathrm{e}^t-1>t$ for any $t>0$, and

$$\mathcal{M}_1(1,f')=\frac{\pi}{\varepsilon}\mathcal{M}_1(1,f)>\frac{\pi}{2\varepsilon}.$$

Hence, if δ is as in (4) then

$$\delta<\frac{m''}{8\mathcal{M}_1(1,f')}=\frac{\pi}{4\varepsilon}\frac{1}{\mathrm{e}^{\pi/2\varepsilon}}<\frac{\pi}{4\varepsilon}\frac{1}{\frac{1}{2}(\pi/2\varepsilon)^2}=\frac{2}{\pi}\varepsilon<\varepsilon.$$

In other words, the two points $z_1=\varepsilon$ and $z_2=-\varepsilon$ do not lie in the disk Δ_δ. This explains why the example given after the statement of Theorem A does not contradict Theorem 3.

References

[1] R.P. Boas, Entire Functions, Academic Press, New York, 1954.
[2] J.G. Clunie, Q.I. Rahman, Extension of a theorem of J.H. Grace to transcendental entire functions, Math. Proc. Cambridge Philos. Soc. 112 (1992) 565–573.
[3] J.H. Grace, The zeros of a polynomial, Proc. Cambridge Philos. Soc. 11 (1902) 352–357.
[4] P.J. Heawood, Geometrical relations between the roots of $f(x)=0$, $f'(x)=0$, Quart. J. Math. 38 (1907) 84–107.
[5] Q.I. Rahman, G. Schmeisser, Analytic Theory of Polynomials, London Math. Soc. Monogr. (N.S.), vol. 26, Clarendon Press, Oxford, 2002.
[6] Q.I. Rahman, J. Waniurski, Coefficient regions for univalent trinomials II, Ann. Univ. Mariae Curie-Skłodowska Sect. A 33 (1979) 189–219.
[7] Q.I. Rahman, J. Waniurski, Coefficient regions for univalent trinomials, Canad. J. Math. 32 (1980) 1–20.
[8] W. Rudin, Principles of Mathematical Analysis, second ed., McGraw–Hill, New York, 1964.
[9] M. Spivak, Calculus, second ed., Publish or Perish, Wilmington, DE, 1980.
[10] G. Szegö, Bemerkungen zu einem Satz von J.H. Grace über die Wurzeln algebraischer Gleichungen, Math. Z. 13 (1922) 28–55.
[11] E.C. Titchmarsh, The Theory of Functions, second ed., Oxford Univ. Press, London, 1939.