Mean Variance Analysis and CAPM

Yan Zeng

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Abstract

A summary of mean variance analysis in portfolio management and capital asset pricing model.

1. Mean-Variance Analysis and Modern Portfolio Theory

This section is based on (Fabozzi, et al., 2006, pp. 15-49).

- It appears that mean-variance portfolio optimization is utilized only at the more quantitative firms. Today, in many firms, portfolio management remains a purely judgmental process based on qualitative, not quantitative, assessments.
- The first quantitative efforts at most firms appear to be focused on providing risk measures to portfolio managers, where risk is defined as underperformance relative to a mandate.

1.1 The Benefits of Diversification

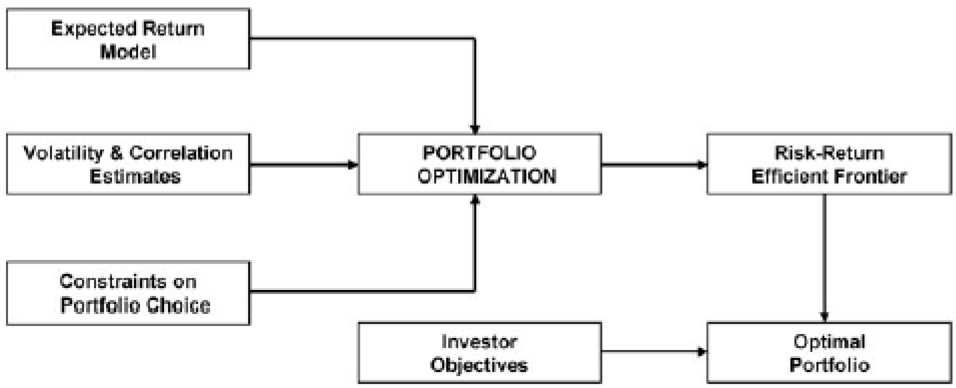
- Under certain so-called *mixing conditions* a Central Limit Theorem can be shown to hold for quite general random variables and processes. See for example, (Davidson, 1994).
- Asset returns are not normal but often exhibit fat tails; the variances of some asset returns are not bounded. If asset returns behave like certain so-called stable Paretian distributions, diversification may no longer be a meaningful economic activity.
- The major benefits of diversification can be obtained with as few as 10 to 20 individual equities in 1960s, but the same amount of diversification needs almost 200 individual equities today.
- The results of modern portfolio theory are consistent with the assumptions that either returns are jointly normally distributed, or that all investors only care about the mean and the variance of their portfolios.
  In practice, it is well known that asset returns are not normal and that many investors have preferences that go beyond that of the mean and the variance.
  Econophysics has developed methods for the accurate empirical analysis of the distribution of asset returns that show significant deviations from the normal distribution. The variances of some asset returns are not bounded, but rather that they are infinite. In specific cases where variances are unbounded and asset returns behave like certain stable Paretian distributions, diversification may no longer be possible.

1.2 Mean-Variance Analysis: Overview

- Asset return is the holding period return (HPR).

- Markowitz's mean-variance framework does not assume joint normality of security returns.
- The Modern Portfolio Theory investment process

EXHIBIT 2.2 The MPT Investment Process



Source: Exhibit 2 in Frank J. Fabozzi, Francis Gupta, and Harry M. Markowitz, "The Legacy of Modern Portfolio Theory," *Journal of Investing* 11 (Fall 2002), p. 8.

Figure 1: (Fabozzi, et al., 2006, p. 21).

- Feasible set, mean-variance efficient portfolios, efficient frontier, and global minimum variance portfolio (GMV).

EXHIBIT 2.1 Feasible and Markowitz Efficient Portfolios[a]

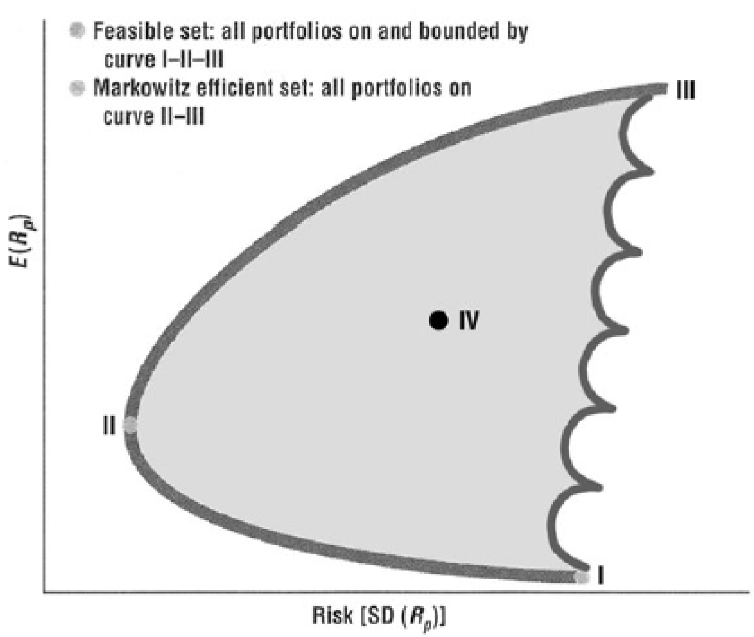
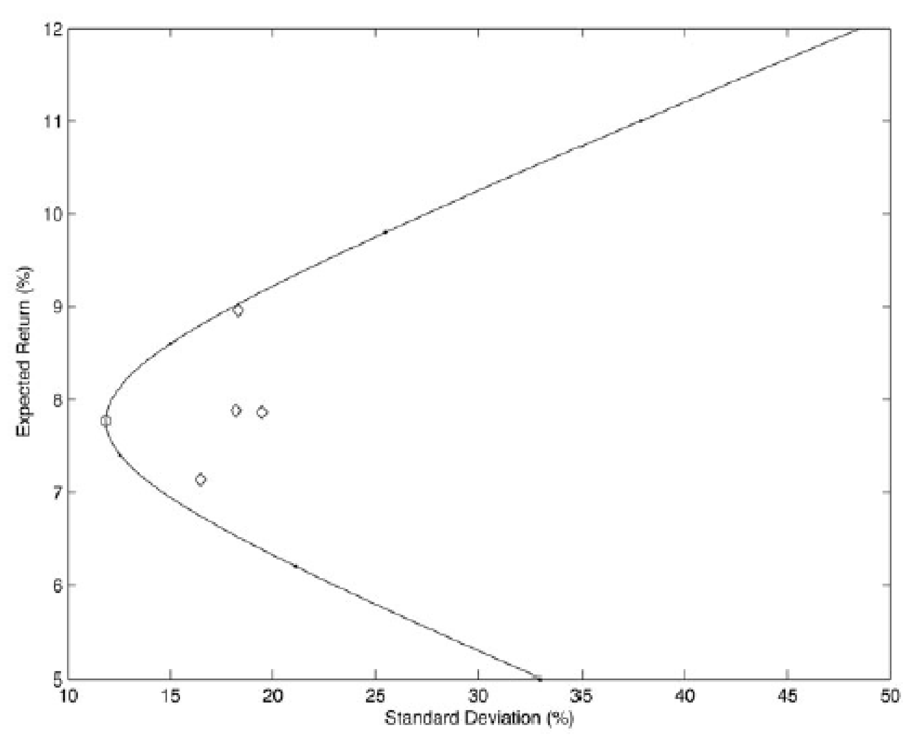


Figure 2: (Fabozzi, et al., 2006, p. 20).

1.3 Classical Framework for Mean-Variance Optimization

- Suppose the assets' return $\boldsymbol{R}$ have expected returns $\boldsymbol{\mu}$ and an $N \times N$ covariance matrix $\boldsymbol{\Sigma}$. Under the assumption that the assets are all risky, the covariance matrix $\boldsymbol{\Sigma}$ is invertible. To calculate the optimal weights $\mathbf{w}$ for a target mean return μ_0, the investor's problem is a constrained minimization problem in the sense that the investor must seek
$$\min_{\mathbf{w}} \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$$
subject to the constraints
$$\mu_0 = \mathbf{w}'\boldsymbol{\mu}, \mathbf{w}'\mathbf{1} = 1, \mathbf{1} = [1, 1, \dots, 1].$$
The efficient frontier with only risky assets has a parabolic shape in the expected return/standard deviation coordinate system.
- To find the global minimum variance portfolio, the constrained minimization problem is
$$\min_{\mathbf{w}} \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$$
subject to the constraints
$$\mathbf{w}'\mathbf{1} = 1, \mathbf{1} = [1, 1, \dots, 1].$$

EXHIBIT 2.5 The Mean-Variance Efficient Frontier of Country Equity Indices of Australia, Austria, Belgium, and Canada



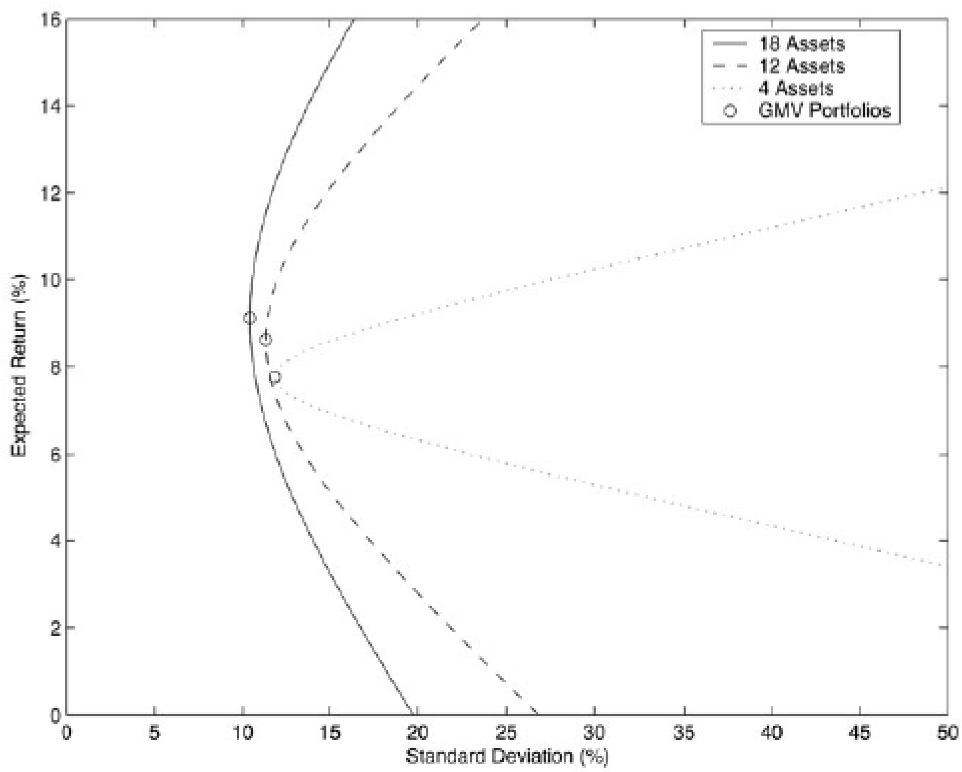
Note: Constructed from the data in Exhibit 2.3. The expected return and standard deviation combination of each country index is represented by a diamond-shaped mark. The GMV is represented by a circle.

Figure 3: (Fabozzi, et al., 2006, p. 27).

1.3.1 Increasing the Asset Universe

- The benefits of diversification are limited up to a point and we cannot expect to be able to completely eliminate portfolio risk.

EXHIBIT 2.7 The Efficient Frontier Widens as the Number of Low-Correlated Assets Increases



Note: The efficient frontiers have been constructed with 4, 12, and 18 countries (from the innermost to the outermost frontier) from the MSCI World Index.

Figure 4: (Fabozzi, et al., 2006, p. 30).

1.3.2 Adding Short-Selling Constraints

- Since we are restricting the opportunity set by constraining all the weights to be positive, the resulting efficient frontier is inside the unconstrained efficient frontier.

EXHIBIT 2.8 The Effect of Restricting Short Selling: Constrained versus Unconstrained Efficient Frontiers Constructed from 18 Countries from the MSCI World Index

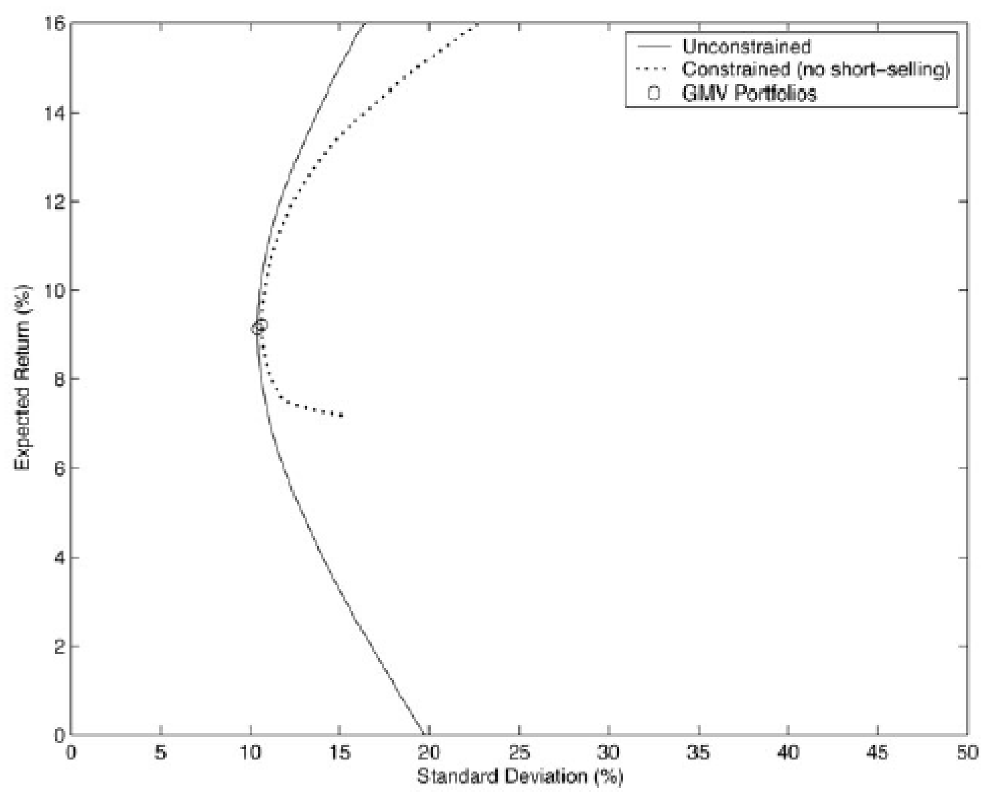


Figure 5: (Fabozzi, et al., 2006, p. 32).

1.3.3 Alternative Formulations of Classical Mean-Variance Optimization

Expected Return Maximization Formulation

- The mean-variance optimization problem minimizes the risk of the portfolio for a certain level of targeted expected return μ_0. We could also begin by choosing a certain level of targeted portfolio risk σ_0 and then maximize the expected return of the portfolio:

$$\max_{\mathbf{w}} \mathbf{w}'\boldsymbol{\mu}$$

subject to the constraints

$$\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} = \sigma_0^2, \mathbf{w}'\mathbf{1} = 1, \mathbf{1}' = [1, 1, \dots, 1].$$

The solution coincides with the upper half of the efficient frontier.

Risk Aversion Formulation

- We explicitly model the trade-off between risk and return in the objective function using a risk-aversion coefficient λ and solve the optimization problem:

$$\max_{\mathbf{w}} (\mathbf{w}'\boldsymbol{\mu} - \lambda \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w})$$

subject to

$$\mathbf{w}'\mathbf{1} = 1, \mathbf{1} = [1, 1, \dots, 1].$$

If we gradually increase λ from zero and for each instance solve the optimization problem, we end up calculating each portfolio along the efficient frontier. It is a common practice to calibrate λ such that a particular portfolio has the desired risk profile.

1.4 The Capital Market Line

- Assume that there is a risk-free asset, with a risk-free return denoted by R_f and that the investor is able to borrow and lend at this rate. The investor's objective is for a targeted level of expected portfolio return μ_0 to choose allocations by solving a quadratic optimization problem

$$\min_{\mathbf{w}_R} \mathbf{w}_R' \mathbf{\Sigma} \mathbf{w}_R$$

subject to the constraint

$$\mu_0 = \mathbf{w}_R' \boldsymbol{\mu} + (1 - \mathbf{w}_R' \mathbf{1}) R_f.^{1}$$

- The optimal weights of risky assets are solved to be

$$\boldsymbol{w}_R = C \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - R_f \mathbf{1}), C = \frac{\mu_0 - R_f}{(\boldsymbol{\mu} - R_f \mathbf{1})' \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - R_f \mathbf{1})}$$

This leads to the following qualitative results

- The efficient frontier becomes a straight line in the expected return/standard deviation coordinate system, since substitution gives a linear relationship between expected portfolio return and portfolio variance:

$$\text{portfolio variance } \sigma_p^2 = \mathbf{w}_R' \mathbf{\Sigma} \mathbf{w}_R = \frac{(\mu_0 - R_f)^2}{(\boldsymbol{\mu} - R_f \mathbf{1})' \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - R_f \mathbf{1})}$$

- With a risk-free asset, all minimum variance portfolios are a combination of the risk-free asset and *a given risky portfolio*, since for a targeted level of expected portfolio return μ_0, the minimum variance portfolio is ($\boldsymbol{X}$ denotes the column vector of risky assets and X_0 denotes the risk-free asset)

$$(\mu_0 - R_f) \boldsymbol{A}' \mathbf{1} \cdot \frac{\boldsymbol{A}' \boldsymbol{X}}{\boldsymbol{A}' \mathbf{1}} + [1 - (\mu_0 - R_f) \boldsymbol{A}' \mathbf{1}] X_0, \boldsymbol{A} = \frac{\boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - R_f \mathbf{1})}{(\boldsymbol{\mu} - R_f \mathbf{1})' \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - R_f \mathbf{1})}$$

Here the risky portfolio is $\frac{\boldsymbol{A}'}{\boldsymbol{A}'\mathbf{1}} \boldsymbol{X}$, called *the tangency portfolio*,[2] and the column vector $\boldsymbol{w}^{TGP}$ of weights for the tangency portfolio is

$$\boldsymbol{w}^{TGP} = \frac{\boldsymbol{A}}{\mathbf{1}' \boldsymbol{A}} = \frac{\boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - R_f \mathbf{1})}{\mathbf{1}' \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - R_f \mathbf{1})}$$

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[1] The constraint condition can be rewritten as $\mu_0 - R_f = \mathbf{w}_R' (\boldsymbol{\mu} - \mathbf{1}) R_f$. So by considering the return in excess of the risk-free rate, we have gone back to the minimization problem of finding the efficient frontier, only that we have dropped the constraint $\mathbf{w}_R' \mathbf{1} = 1$.

[2] The *tangency* condition means the line segment connecting $(0, R_f)$ and the risky portfolio on the risk-return plane is tangent to the efficient frontier of risky assets. This property is easily seen from geometric intuition if the solution is to satisfy the minimal variance requirement.

- Under certain assumptions the tangency portfolio must consist of all assets available to investors, and each asset must be held in proportion to its market value relative to the total market value of all assets.[3] Therefore, the tangency portfolio is often referred to as the "*market portfolio*," or simply the "*market*." Moreover,
- It is easy to verify that the market portfolio can be calculated directly from the *maximal Sharpe ratio optimization problem*:

$$\max_{\boldsymbol{w}} \frac{\boldsymbol{w}'\boldsymbol{\mu} - R_f}{\sqrt{\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}}}, \text{subject to } \boldsymbol{w}'\mathbf{1} = 1.$$

- The line from the risk-free rate that is tangent to the efficient frontier of risky assets is called the *Capital Market Line* (CML).

EXHIBIT 2.9 Capital Market Line and the Markowitz Efficient Frontier

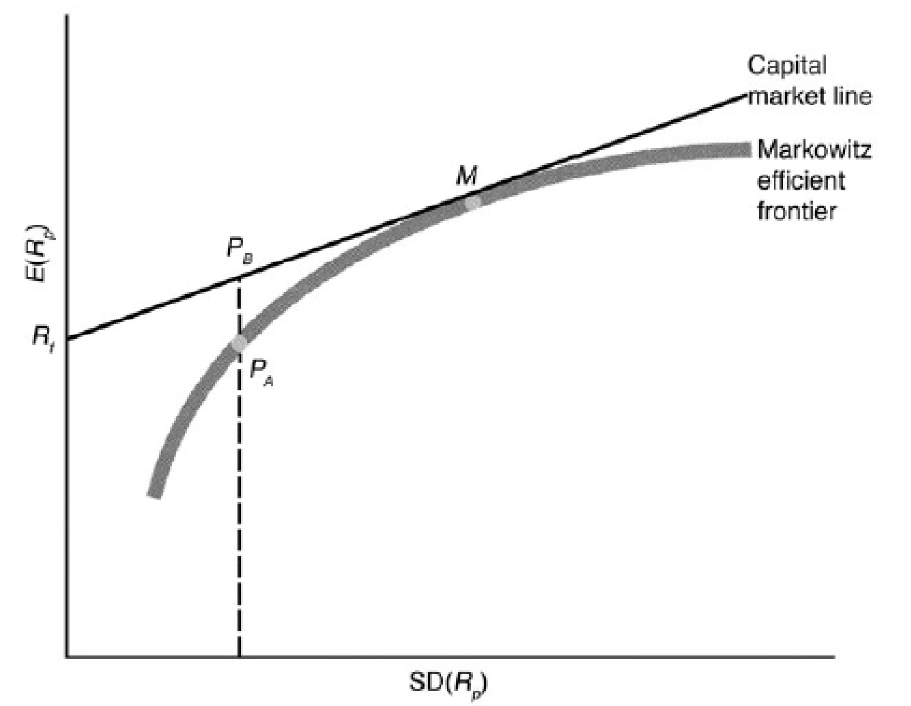


Figure 6: (Fabozzi, et al., 2006, p. 37).

- With the exception of the market portfolio, the minimum variance portfolios that are a combination of the market portfolio and the risk-free asset are superior to the portfolio on the Markowitz efficient frontier for the same level of risk.

[3] See (Fama, 1970) for details. An economic argument for the first property goes as follows (Sharpe, et al., 1998, p. 230): The risky portion of every investor's portfolio is just a multiple of the tangency portfolio. If every investor is purchasing the tangency portfolio and the tangency portfolio does not involve an investment in each security, then nobody is investing in those securities with zero proportions in the tangency portfolio. Consequently, the prices of these zero proportion securities must fall, thereby causing the expected returns of these securities to rise until the resulting tangency portfolio has a nonzero proportion associated with them.

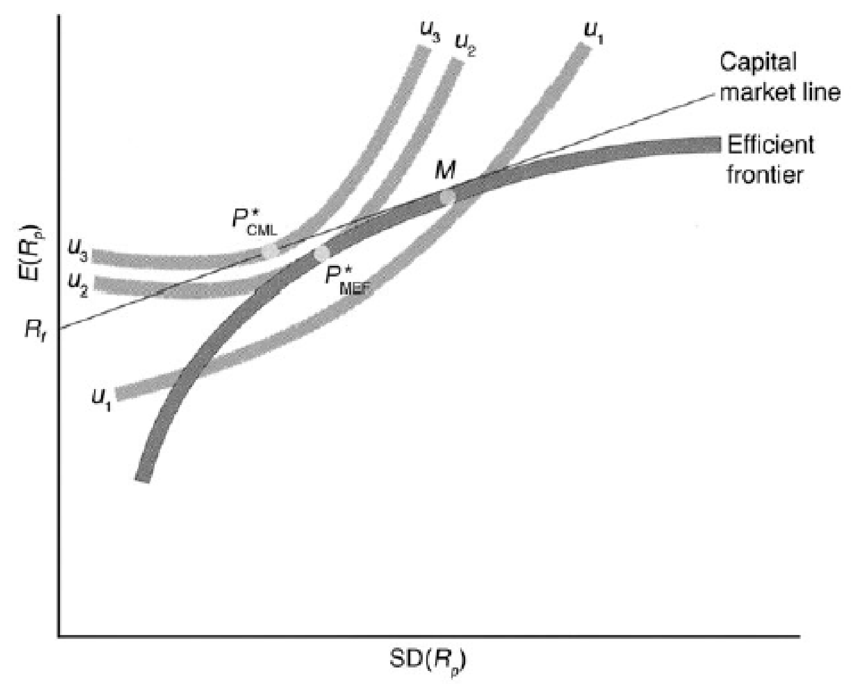
- With the introduction of the risk-free asset, we can now say that an investor will select a portfolio on the CML that represents a combination of borrowing or lending at the risk-free rate and the market portfolio. This important property is called *separation*.

- For an efficient portfolio with return R_p and standard deviation σ_p, the following equation gives the algebraic equation for the CML[4]

$$E(R_p) = R_f + \frac{E(R_M) - R_f}{\sigma_M}\sigma_p$$

- The slope of CML, $\frac{E(R_M)-R_f}{\sigma_M}$, is often referred to as the *risk premium* or the *equilibrium market price of risk*. It is the Sharpe ratio of the market portfolio.
- The investor will select the portfolio P_{CML}^* on the CML that is tangent to the highest indifference curve. Without the risk-free asset, an investor could only get to the indifference curve that is tangent to the Markowitz efficient frontier. This portfolio is denoted by P_{MEF}^*.

EXHIBIT 2.11 Optimal Portfolio and the Capital Market Line



u_1, u_2, u_3 = Indifference curves with $u_1 < u_2 < u_3$
M = Market portfolio
R_f = Risk-free rate
P^*_{CML} = Optimal portfolio on capital market line
P^*_{MEF} = Optimal portfolio on efficient frontier

Figure 7: (Fabozzi, et al., 2006, p. 43).

[4] This equation can be derived directly from the geometric interpretation of CML and agrees with what we obtained before: $\sigma_p^2 = \frac{(\mu_0 - R_f)^2}{(\boldsymbol{\mu} - R_f\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - R_f\mathbf{1})}$.

2. Capital Asset Pricing Model (CAPM) and SML

This section is based on (Sharpe, et al., 1998, pp. 250-252) and (Sigman, 2005).

The market portfolio and the Capital Market Line explain to us how individual investors should invest. The CAPM approach builds on this knowledge and allows the focus to change from how an individual should invest to what would happen to security prices if everyone invested in a similar manner.

More precisely, the portfolio model provides an algebraic condition on asset weights in mean-variance-efficient portfolios. The CAPM turns this algebraic statement into a testable prediction about the relation between risk and expected return by identifying a portfolio that must be efficient if asset prices are to clear the market of all assets.

2.1 Expected Version of the CAPM

Through an equilibrium argument and an argument illustrated by Figure 8, we conclude for any portfolio of assets[5]

$$E(R_p) = R_f + \beta_p\left[E(R_M) - R_f\right], where\ \beta_p = \frac{Cov(R_p, R_M)}{\sigma_M^2}$$

This equation is called the *security market line* (SML), and is one formulation of the Capital Asset Pricing Model.

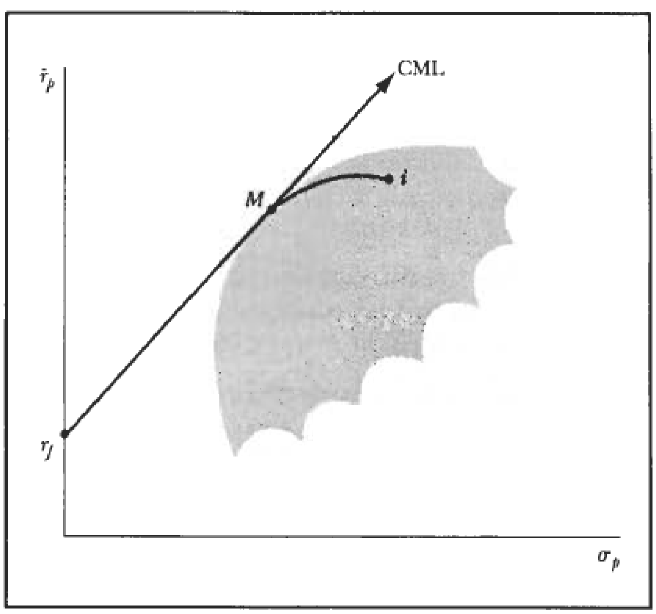


Figure 9.7
Deriving the Security Market Line

Figure 8 Deriving the Security Market Line, (Sharpe, et al., 1998, p. 250).

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[5] The order of proof is first for any risky asset, then for any portfolio of risky assets, and finally for any portfolio of risky assets and the risk-free asset. See (Sharpe, et al., 1998, pp. 250-252) for details.

Note a portfolio is always on the SML; it is on the CML if and only if it is an efficient portfolio (meaning a linear combination of the risk-free asset and the market portfolio).

2.2 Random Version of the CAPM

For any given portfolio, define the random error $\varepsilon_p = R_p - R_f - \beta_p[R_M - R_f]$. Then $E(\varepsilon_p) = 0$ by the expected version of the CAPM and $Cov(\varepsilon_p, R_M) = 0$ by direct computation.[6] This gives us the random version of the CAPM:

$$R_p = R_f + \beta_p[R_M - R_f] + \varepsilon_p, where\ E(\varepsilon_p) = 0\ and\ Cov(\varepsilon_p, R_M) = 0.$$

2.3 Decomposition of Portfolio's Risk

From the random version of the CAPM, we easily conclude the total risk of the portfolio can be decomposed into a *systematic risk* and a *nonsystematic risk*:

$$\sigma_p^2 = \beta_p^2\sigma_M^2 + \sigma_{\varepsilon_p}^2.$$

In particular, any efficient portfolio must be on the capital market line and satisfy

$$\sigma_p = \beta_p\sigma_M.$$

So any efficient portfolio only has systematic risk.

2.4 CAPM Pricing Formula

Consider an asset with price P at time $t = 0$ and a random payoff Q at time $t = 1$. Its return rate $r = \frac{Q}{P} - 1$. By the expected version of the CAPM

$$E(r) = \frac{E(Q)}{P} - 1 = R_f + [E(R_M) - R_f]\frac{Cov(r, R_M)}{\sigma_M^2} = R_f + [E(R_M) - R_f]\frac{Cov(Q, R_M)}{P\sigma_M^2}$$

Where the second = is due to CAPM and the third equality is obtained by plugging $r = \frac{Q}{P} - 1$ into $Cov(r, R_M)$. From the second equality, we have the first version of the CAPM pricing formula

$$\boxed{P = \frac{E(Q)}{1 + R_f + [E(R_M) - R_f]\frac{Cov(r, R_M)}{\sigma_M^2}}}$$

From $\frac{E(Q)}{P} - 1 = R_f + [E(R_M) - R_f]\frac{Cov(Q,R_M)}{P\sigma_M^2}$, we can obtain the second version of the CAPM pricing formula

[6] Once we recognize $\beta_p = \frac{Cov(R_p,R_M)}{\sigma_M^2}$ is the coefficient of the orthogonal projection of $(R_p - R_f)$ onto $(R_M - R_f)$, we can directly conclude from Hilbert space theory that the random error ε_p is uncorrelated to the market return R_M.

$$\boxed{P = \frac{E(Q) - \frac{Cov(Q, R_M)}{\sigma_M^2}[E(R_M) - R_f]}{1 + R_f}}$$

Appendix: Resources on the CAPM

Books

- (Bodie, et al., 2002) gives an introduction to the CAPM without giving any formal proof.
- (Chapados, 2011) briefly mentions the content of the CAPM without giving any formal proof.
- (Elton, et al., 2006) contains a rigorous proof of the standard version of the CAPM (in terms of expected return-beta relationship).
- (Fabozzi, et al., 2006) explains in detail the mean-variance analysis framework of Markowitz, and mentions the CAPM without giving any proof.
- (Fabozzi, et al., 2007) and (Fabozzi & Focardi, 2004) essentially give a presentation of the CAPM as same as (Fabozzi, et al., 2006).
- (Grinold & Kahn, 1999) gives a practical perspective on the CAPM, without giving formal proof.
- (Ingersoll, Jr., 1987) gives a fairly rigorous proof of the CAPM for general concave utility functions. The notation is old and terrible though.
- (Sharpe, et al., 1998, p. 250) Chapter 9 Appendix B gives a proof of the expected version of the CAPM.

Notes and Papers

- Karl Sigman's notes on the CAPM gives
  1) a proof of the CAPM similar to that of (Sharpe, et al., 1998, p. 250) Chapter 9 Appendix B;
  2) the CAPM in terms of expected return-beta relationship;
  3) the CAPM in terms of random return-beta relationship;
  4) the decomposition of portfolio risk into systematic risk and nonsystematic risk; the efficient portfolio only has systematic risk;
  5) two asset pricing formulas based on the CAPM.
- (Fama & French, 2004) explains in its first section (The Logic of the CAPM) the model of portfolio choice by Harry Markowitz and various versions of the CAPM (minimum variance condition, Sharpe-Lintner version, Black version, etc.). Proof of the CAPM is heuristic – based on Markowitz's model of portfolio choice, arguing that the tangency portfolio must be the market portfolio.

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