Recall the alternating series test:

**Theorem 1** (Alternating Series Test)**.** *If*

(1) $(b_n)_{n\geq 1}$ *is monotone decreasing, and*
(2) $\lim_{n\to\infty} b_n = 0$

*then the series*

$$\sum_{n=1}^{\infty} (-1)^n b_n$$

*converges.*

This result is "easily" proved by grouping terms and using the Cauchy criterion, or by considering even and odd partial sums separately and using the fact that bounded monotone sequences converge. It is a nice result, but it has two drawbacks: one, that the sign alternate strictly, and, two, the monoticity hypothesis. The monoticity, even when true, is frequently awkward to establish. We can not eliminate these hypotheses completely since the theorem is false without them, but we can modify them.

A sequence $(b_n)_{n\geq 1}$ is said to be *absolutely convergent* (bad terminology) or of *bounded variation* (better terminology) if

$$\sum_{n=1}^{\infty} |\, b_{n+1} - b_n \,| < +\infty.$$

**Exercise 1.** If $(b_n)_{n\geq 1}$ is of bounded variation then $\lim_{n\to\infty} b_n$ exists.

**Exercise 2.** If $(b_n)_{n\geq 1}$ is a bounded monotone sequence (increasing or decreasing) then $(b_n)_{n\geq 1}$ is of bounded variation.

Since we are dealing with series which may fail to be absolutely convergent, convergence will depend on delicate cancellation. Thus our arguments will proceed by grouping terms appropriately (just as we had to do in the proof of the alternating series test). The primary tool for these types of arguments is Abel's summation by parts:

Consider two sequences, $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$. Define partial sums $A_n$ by $A_0 = 0$ and, for $n \geq 1$,

$$A_n = \sum_{k=1}^{n} a_k.$$

Note

$$a_n = A_n - A_{n-1}, \quad \text{for } n \geq 1.$$

Thus for $n > m$

$$\begin{aligned}
\sum_{k=m+1}^{n} a_k b_k &= \sum_{k=m+1}^{n} (A_k - A_{k-1})\, b_k \\
&= \sum_{k=m+1}^{n} A_k b_k - \sum_{k=m}^{n-1} A_k b_{k+1} \\
&= A_n b_{n+1} - A_m b_{m+1} + \sum_{k=m+1}^{n} A_k \left(b_k - b_{k+1}\right).
\end{aligned}$$

*Bent Petersen File ref: 311abel.tex

This is the partial summation formula. Taking $m = 0$ we have

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1})$$

which leads immediately to a simple convergence result:

**Lemma 2.** *If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences of complex numbers,*

$$A_n = \sum_{k=1}^{n} a_k,$$

(1) *the series $\sum_{n=1}^{\infty} A_n (b_n - b_{n+1})$ converges, and*
(2) *the sequence $(A_n b_{n+1})_{n\geq 1}$ converges*

*then the series*

$$\sum_{n=1}^{\infty} a_n b_n$$

*converges.*

The trick to getting a useful result is to strengthen the hypotheses a bit, but not too much, so as to get conditions that can frequently be conveniently checked. The two main results are theorems of Abel, Dedekind and Dirichlet, dating from the 1860's.

**Theorem 3** (Abel–Dedekind)**.** *If the series $\sum_{n=1}^{\infty} a_n$ converges and the sequence $(b_n)_{n\geq 1}$ is of bounded variation then the series*

$$\sum_{n=1}^{\infty} a_n b_n$$

*converges.*

*Proof.* If $A_n = \sum_{k=1}^{n} a_k$ then the sequence $(A_n)_{n\geq 1}$ converges. Since $(b_n)_{n\geq 1}$ is of bounded variation, it also converges. Thus the sequence $(A_n b_{n+1})_{n\geq 1}$ converges. Now since $(A_n)_{n\geq 1}$ converges it is bounded and so there exists a constant $M > 0$ such that $|\, A_n \,| \leq M$ for each $n$. Then

$$|\, A_n (b_n - b_{n+1}) \,| \leq M \; |\, b_n - b_{n+1} \,|$$

for each $n$ implies the absolute convergence of $\sum_{n=1}^{\infty} A_n (b_n - b_{n+1})$. The theorem now follows from lemma 2. $\square$

The original theorem of Abel (from the 1820's) required that the sequence $(b_n)_{n\geq 1}$ be monotone and bounded (which is a stronger hypothesis than bounded variation).

If we weaken the hypothesis on the sequence $(A_n)_{n\geq 1}$ and strengthen the hypotheses on the sequence $(b_n)_{n\geq 1}$ suitably, we obtain a very useful result:

**Theorem 4** (Abel–Dedekind–Dirichlet)**.** *If the series $\sum_{n=1}^{\infty} a_n$ has bounded partial sums and the sequence $(b_n)_{n\geq 1}$ is of bounded variation and $\lim_{n\to\infty} b_n = 0$ then the series*

$$\sum_{n=1}^{\infty} a_n b_n$$

*converges.*

*Proof.* If $A_n = \sum_{k=1}^{n} a_k$ then by hypothesis there is a constant $K > 0$ such that $|\, A_n \,| \le K$ for each $n$. It follows that

$$|\, A_n b_{n+1} \,| \le K \;|\, b_{n+1} \,|$$

for each $n$, which implies

$$\lim_{n\to\infty} A_n b_{n+1} = 0.$$

In addition we have

$$|\, A_n \,(b_n - b_{n+1}) \,| \le K \;|\, b_n - b_{n+1} \,|$$

for each $n$, which implies the absolute convergence of $\sum_{n=1}^{\infty} A_n \,(b_n - b_{n+1})$. The theorem now follows from lemma 2. □

A bit weaker theorem, sometimes called the Abel–Dirichlet theorem, requires $(b_n)_{n\ge 1}$ be a monotone sequence converging to $0$.

If we take $a_n = (-1)^n$ in the Abel–Dedekind–Dirichlet theorem we obtain an alternating series test which does not require monoticity:

**Corollary 5** (Abel–Dirichlet–Dedekind alternating series test)**.** *If $(b_n)_{n\ge 1}$ is a sequence of complex numbers,*

(1) $\sum_{n=1}^{\infty} |\, b_n - b_{n+1} \,| < +\infty$*, and*
(2) $\lim_{n\to\infty} b_n = 0$

*then the series*

$$\sum_{n=1}^{\infty} (-1)^n b_n$$

*converges.*

Since a bounded monotone sequence has bounded variation the corollary is a generalization of the alternating series test.

While the corollary is frequently used in cases where $b_n \ge 0$ (so we do actually have alternating sign), this is not required. Thus we can replace $b_n$ with $(-1)^n b_n$ and obtain a result which brings out very clearly the dependence on cancellation:

**Corollary 6.** *If $(b_n)_{n\ge 1}$ is a sequence of complex numbers,*

(1) $\sum_{n=1}^{\infty} |\, b_n + b_{n+1} \,| < +\infty$*, and*
(2) $\lim_{n\to\infty} b_n = 0$

*then the series*

$$\sum_{n=1}^{\infty} b_n$$

*converges.*

*Example* 1. From the trigonometric identity

$$\sin(nx) = \frac{\cos((n - \frac{1}{2})x) - \cos((n + \frac{1}{2})x)}{2\sin(x/2)}$$

we obtain

$$\left| \sum_{k=1}^{n} \sin(kx) \right| \le \frac{1}{|\sin(x/2)\,|}.$$

Thus by the Abel–Dedekind–Dirichlet theorem

$$\sum_{n=1}^{\infty} b_n \sin(nx)$$

converges if $(b_n)_{n\geq 1}$ is a sequence with bounded variation and $\lim_{n\to\infty} b_n = 0$ (for example, if $b_n = \frac{1}{n}$).

*Example* 2. The sequence $\left(\left(1+\frac{1}{n}\right)^n\right)_{n\geq 1}$ is bounded and monotone, and so of bounded variation. Thus if $\sum_{n=1}^{\infty} a_n$ converges, the so does

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n a_n.$$

Note this result is not at all exciting if $\sum_{n=1}^{\infty} a_n$ happens to be absolutely convergent.

*Example* 3. The sequence $\left(\frac{1}{\log n}\right)_{n\geq 2}$ is a monotone sequence converging to $0$. Thus if $\sum_{n=1}^{\infty} a_n$ is a series with bounded partial sums then

$$\sum_{n=2}^{\infty} \frac{a_n}{\log n}$$

converges. Thus

$$\sum_{n=2}^{\infty} \frac{\sin(nx)}{\log n}$$

converges.

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